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EIGENVECTORS AND FIXED POINTS OF NON-LINEAR OPERATORS

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Let X be a real infinite-dimensional Banach space and ψ a measure of noncompactness on X. Let Ω be a bounded open subset of X and $A : \overline{\Omega} \to X$ a ψ-condensing operator, which has no fixed points on $\partial\Omega$. Then the fixed point index, $\mathrm{ind}(A,\Omega)$, of A on Ω is defined (see, for example, ([1] and [18]). In particular, if A is a compact operator $\mathrm{ind}(A,\Omega)$ agrees with the classical Leray-Schauder degree of $I - A$ on Ω relative to the point 0, $\deg(I - A,\Omega,0)$. The main aim of this note is to investigate boundary conditions, under which the fixed point index of strict-ψ-contractive or ψ-condensing operators $A : \overline{\Omega} \to X$ is equal to zero. Correspondingly, results on eigenvectors and nonzero fixed points of k-ψ-contractive and ψ-condensing operators are obtained. In particular we generalize the Birkhoff-Kellog theorem [4] and Guo's domain compression and expansion theorem [17]. The note is based mainly on the results contained in [7] and [8].

1. Preliminaries and notation.

Throughout X is a real infinite-dimensional Banach space. We denote by $B_r(X) = \{x \in X : \|x\| \leq r\}$ the closed ball centered in 0 of radius $r > 0$, we write

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briefly $B(X)$ instead of $B_1(X)$. For a set M in X we denote by $\mathrm{int}M$, $\overline{M}$ and ∂M the interior, the closure and the boundary of M, respectively. All the operators considered in what follows are supposed to be continuous.

We recall that for a bounded set M in X: the *Kuratowski measure of noncompactness* $\alpha(M)$ is the infimum of all $\varepsilon > 0$ such that M admits a finite covering by sets of diameter at most ε; the *lattice measure of noncompactness* $\beta(M)$ is the supremum of all $\varepsilon > 0$ such that M contains a sequence $\{x_n\}$ with $||x_m - x_n|| \geq \varepsilon$, for $m \neq n$; the *Hausdorff measure of noncompactness* $\gamma(M)$ is the infimum of all $\varepsilon > 0$ such that M admits a finite ε-net in X.
We refer to [3] for all details. In the following ψ will stand for either α, β or γ.

An operator $F : dom(F) \subseteq X \to X$ is called a *k-ψ-contraction* if there is $k \geq 0$ such that $\psi(FM) \leq k\psi(M)$ for each bounded $M \subseteq dom(F)$, in particular F is called a *strict-ψ-contraction* if it is a k-ψ-contraction for some $k < 1$. The operator F is called ψ-condensing if $\psi(FM) < \psi(M)$ for each bounded $M \subseteq dom(F)$ which is not relatively compact. Clearly every strict-ψ-contractive operator is ψ-condensing.

Throughout Ω is a bounded open subset containing the origin 0 of the space X .

In [16], D. Guo has proved the following

Theorem 1.1. Let $A : \overline{\Omega} \to X$ be a compact operator. Suppose that
$(i)\, \inf_{x\in\partial\Omega} ||A(x)|| > 0$ and (ii) $A(x) \neq \lambda x$ for $x \in \partial\Omega$ and $0 < \lambda \leq 1$.
Then the Leray-Schauder degree $\deg(I - A, \Omega, 0) = 0$

Assuming that $A : \overline{\Omega} \to X$ is a strict-ψ-contraction, condition (i) of Theorem 1.1 is no more sufficient to yield $\mathrm{ind}(I - A, \Omega, 0) = 0$, as the following example shows.

Example 1.2. ([5]) Let $A : B(X) \to X$ be defined by $A = -kI$ where I is the identity operator and $k < 1$. Then $\inf_{x\in\partial B(X)} \|A(x)\| > 0$, but on the other hand

$$\mathrm{ind}(I - A, \mathrm{int}B(X), 0) = \mathrm{ind}((1 + k)I, \mathrm{int}B(X), 0) = 1.$$

The following generalizations of Theorem 1.1 have been obtained for strict-α-contractions.

Theorem 1.1. ([21]) *Let $A : B_r(X) \to X$ be a k-α-contraction ($k < 1$). Suppose that*
(i) $\inf_{x\in\partial B_r(X)} ||Ax|| > kr$ and (ii) $A(x) \neq \lambda x$ for $x \in \partial B_r(X)$ and $0 < \lambda \leq 1$.
Then $\mathrm{ind}(A, \mathrm{int}B_r(X)) = 0$.

Theorem 1.2. ([22]) *Let $A : \overline{\Omega} \to X$ be a k-α-contraction ($k < 1$). Suppose that $(i)\, \inf_{x\in\partial\Omega} \|Ax\| > k\Big(\sup_{x\in\partial\Omega} \|x\| + \alpha(\partial\Omega)\Big)$ and (ii) $A(x) \neq \lambda x$ for $x \in \partial\Omega$ and* $0 < \lambda \leq 1$. *Then* $\mathrm{ind}(A, B_r(X)) = 0$.

We generalize Theorem 1.1 to strict-ψ-contractive (and analogously to ψ-condensing) operators under a condition which arises in a natural way from the geometry of the space X. Precisely, If $A:\overline{\Omega}\to X$ ia a strict-ψ-contraction, we replace condition (i) of Theorem 1.1 by the following Birkhoff-Kellogg type condition

$$\inf_{x\in\partial\Omega}\|Ax\|>kk_{\psi}\sup_{x\in\partial\Omega}\|x\| \tag{1}$$

which depends on the *Wośko constant* k_{ψ} of the space X and is optimal when $k_{\psi}=1$.

2. The characteristics k_{ψ} and $c_{\rho,\beta}$

It is well known that in any infinite dimensional Banach space X there is always a retraction R from $B(X)$ onto $\partial B(X)$ (for details and references see [15]). Then the quantitative characteristic

$$k_{\psi}=\inf\{k\geq 1:\exists \text{ a } k\text{-}\psi\text{-contractive retraction } R:B(X)\to\partial B(X)\}$$

has been introduced by introduced by Wośko in [20]. We point out however that the problem was first studied in [13, 14].

The estimate of k_{ψ} is of interest in problems of nonlinear analysis (see, for example, [2, 7, 12]). Concerning general results, in [19] it was proved that $k_{\psi}\leq 6$ for any infinite dimensional Banach space X, reaching the value 4 or 3 depending on the geometry of the space. Moreover it has been proved that $k_{\gamma}=1$ in some Banach spaces of continuous functions ([9–11, 20]) and in some classical Banach spaces of measurable functions ([6]). In [2] it is proved that $k_{\psi}=1$ in Banach spaces whose norm is monotone with respect to some basis.

Though it has been shown that $k_{\psi}=1$ in some Banach spaces, the problem whether or not this is true in any Banach space X is open. We observe that most of the evaluations of k_{ψ} have required individual constructions in each space X. However a standard way to construct a retraction from $B(X)$ onto $S(X)$ is that of normalizing a map which coincide with the identity on $S(X)$ and maps $B(X)$ out of a ball $B_r(X)$ of radius $r<1$ (or possibly maps $B(X)$ into $X\setminus 0$). In this connection it is of some interest for any $0<\rho\leq\beta$ to define the geometrical characteristic

$$c_{\psi}(\rho,\beta,X):=\inf_{G_{\rho,\beta}\in S_{\rho,\beta}}\psi(G_{\rho,\beta}),$$

where $S_{\rho,\beta}$ denotes the set of all continuous maps $G_{\rho,\beta}:B_{\beta}(X)\to X$ such that $G_{\rho,\beta}\,x=x$ for all $x\in S_{\beta}(X)$, and $\|G_{\rho,\beta}\,x\|\geq\rho$ for all $x\in B_{\beta}(X)$.
We briefly write $c_{\rho,\beta}$ instead of $c_{\psi}(\rho,\beta,X)$. The map $\rho\to c_{\rho,\beta}$ is nondecreasing and right-continuous. Moreover for $0<\rho\leq\beta$ we have $1\leq c_{\rho,\beta}\leq k_{\psi}(X)$ and $c_{\beta,\beta}=k_{\psi}(X)$ in any infinite-dimensional Banach space X.

Using such a parameter we can give a formulation of Guo's theorem for strict-ψ-contractive operators under an hypothesis that looks weaker than (1) in Banach spaces X in which the known estimate of k_ψ is greater than 1.

3. Results

The following theorems generalizes Guo's result (Theorem 1.1) to strict-ψ-contractive and ψ-condensing operators, respectively.

Theorem 3.1. Let $A:\overline{\Omega}\to X$ be a k-ψ-contraction ($k<1$), satisfying

$$\inf_{x\in\partial\Omega}\|Ax\| > kk_\psi \sup_{x\in\partial\Omega}\|x\|. \tag{2}$$

Assume that one of the following conditions holds:

(a) $kk_\psi < 1$ and $Ax \neq \lambda x$ for $x\in\partial\Omega$ and $kk_\psi < \lambda \leq 1$;

(b) $kk_\psi \geq 1$.

Then $\text{ind}(A,\Omega)=0$.

Theorem 3.2. Let $A:\overline{\Omega}\to X$ be a ψ-condensing mapping, suppose that

$$\inf_{x\in\partial\Omega}||Ax|| > k_\psi \sup_{x\in\partial\Omega}||x||. \tag{3}$$

Then $\text{ind}(A,\Omega)=0$.

We obtain the existence of positive and negative eigenvalues with corresponding eigenvectors on the boundary for k-ψ-contractive operators (for any $k\geq 0$), generalizing the Birkhoff-Kellogg theorem ([4]).

Corollary 3.1. *Let $A:\overline{\Omega}\to X$ be a k-ψ-contraction (for any $k>0$). Suppose that*

$$\inf_{x\in\partial\Omega}||Ax|| > kk_\psi \sup_{x\in\partial\Omega}||x||.$$

Then there exist $\lambda > kk_\psi$ and $x_\lambda\in\partial\Omega$ such that $\lambda x_\lambda = Ax_\lambda$, and also there exist $\mu < -k_\psi k$ and $x_\mu\in\partial\Omega$ such that $\mu x_\mu = Ax_\mu$.

The next two corollaries extend Guo's domain compression and expansion fixed point theorems [17].

Corollary 3.2. *Let Ω_1 and Ω_2 be bounded open sets in X, such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, and let $A : \overline{\Omega}_2 \to X$ be a strict-ψ-contraction. Suppose that one of the following conditions holds:*
(a) $kk_\psi < 1$ *and one of the following is satisfied*

$$\begin{cases} \inf_{x\in\partial\Omega_1} \|Ax\| > & kk_\psi \sup_{x\in\partial\Omega_1} \|x\| \\ Ax \neq \lambda x & x \in \partial\Omega_1,\ \ kk_\psi < \lambda < 1 \\ Ax \neq \nu x & x \in \partial\Omega_2,\ \ \nu > 1 \end{cases}$$

or

$$\begin{cases} \inf_{x\in\partial\Omega_2} \|Ax\| > & kk_\psi \sup_{x\in\partial\Omega_2} \|x\| \\ Ax \neq \lambda x & x \in \partial\Omega_2,\ \ kk_\psi < \lambda < 1 \\ Ax \neq \nu x & x \in \partial\Omega_1,\ \ \nu > 1 \end{cases}$$

(b) $kk_\psi \geq 1$ *and one of the following is satisfied*

$$\begin{cases} \inf_{x\in\partial\Omega_1} \|Ax\| > & kk_\psi \sup_{x\in\partial\Omega_1} \|x\| \\ Ax \neq \nu x & x \in \partial\Omega_2,\ \ \nu > 1 \end{cases}$$

or

$$\begin{cases} \inf_{x\in\partial\Omega_2} \|Ax\| > & kk_\psi \sup_{x\in\partial\Omega_2} \|x\| \\ Ax \neq \nu x & x \in \partial\Omega_1,\ \ \nu > 1 \end{cases}$$

Then A has at least a fixed point on $\overline{\Omega}_2 \setminus \Omega_1$.

Corollary 3.3. *Let Ω_1 and Ω_2 be bounded open sets in X, such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $A : \overline{\Omega}_2 \to X$ be a ψ-condensing mapping. Suppose that one of the following conditions holds*

$$\begin{cases} \inf_{x\in\partial\Omega_1} \|Ax\| > & k_\psi \sup_{x\in\partial\Omega_1} \|x\| \\ Ax \neq \nu x & x \in \partial\Omega_2,\ \ \nu > 1 \end{cases}$$

or

$$\begin{cases} \inf_{x\in\partial\Omega_2} \|Ax\| > & k_\psi \sup_{x\in\partial\Omega_2} \|x\| \\ Ax \neq \nu x & x \in \partial\Omega_1,\ \ \nu > 1. \end{cases}$$

Then A has at least a fixed point in $\overline{\Omega}_2 \setminus \Omega_1$.

Finally, we restate Theorem 3.1 using the parameter $c_{\alpha,\beta}$, all the other results can be reformulated similarly (see [8]).

Theorem 3.3. Let $A : \overline{\Omega} \to X$ with $\psi(A) = k < 1$. Let $\alpha = \inf_{x\in\partial\Omega} \|Ax\|$ and $\beta = \sup_{x\in\partial\Omega} \|x\|$. Assume that one of the following conditions holds:

(i) $kc_{\alpha,\beta} < 1$ and

$$\inf_{x\in\partial\Omega} \|Ax\| > kc_{\alpha,\beta} \sup_{x\in\partial\Omega} \|x\|.$$

In addition, $Ax \neq \lambda x$ for $x \in \partial\Omega$ and $kc_{\alpha,\beta} < \lambda \leq 1$

(ii) $kc_{\alpha,\beta} \geq 1$ and there is an α' such that

$$\inf_{x\in\partial\Omega} \|Ax\| \geq \alpha' > kc_{\alpha',\beta} \sup_{x\in\partial\Omega} \|x\|.$$

Then $\mathrm{ind}(A,\Omega) = 0$.

The results of this note, including their proofs, are contained in [7] and [8].

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