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MANY IDENTITIES FROM ONE

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We specialise an ingeniously contrived identity to obtain a plethora of startling combinatorial identities. Limiting cases include well known evaluations of $\zeta(2)$, $\zeta(4)$ and $\zeta(6)$.

Let x, y, u and v be formal parameters and $\{a_i\}_{i\geq 0}$, $\{b_j\}_{j\geq 0}$ be two sequences taken from a commutative ring. Define the a-rising factorial for $n \geq 0$ by

$$(1) \qquad (x;a)_0 = 1, \quad (x;a)_n = \prod_{k=1}^{n}(x+a_{k-1}) \quad (n>0).$$

Then we have the following algebraic identity.

Theorem. *Suppose that* $0 \leq m \leq n$ *and* r *be natural numbers. There holds*

$$(2) \qquad \sum_{k=m}^{n}\Big\{(x+a_k)(y+b_{r+k})z-(u+a_k)(v+b_{r+k})\Big\}\frac{(x;a)_k(y;b)_{r+k}}{(u;a)_{k+1}(v;b)_{r+k+1}}\,z^k$$
$$=\frac{(x;a)_{n+1}(y;b)_{r+n+1}}{(u;a)_{n+1}(v;b)_{r+n+1}}\,z^{n+1}-\frac{(x;a)_m(y;b)_{r+m}}{(u;a)_m(v;b)_{r+m}}\,z^m.$$

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Proof. Notice that the summand on the left of (2) can be split as

$$\frac{(x;a)_{k+1}(y;b)_{r+k+1}}{(u;a)_{k+1}(v;b)_{r+k+1}}\,z^{k+1}-\frac{(x;a)_k(y;b)_{r+k}}{(u;a)_k(v;b)_{r+k}}\,z^k.$$

Then the theorem follows immediately from the diagonal cancellation. □

In what follows, we shall demonstrate a number of interesting identities from (2) which could be served as the reason for the present paper's title.

First of all, take $a_k = b_k = -k$, $u = -x$, $v = -y$, $m = +1$ and $r = 0$ in (2). Then $z = \pm 1$ correspond to the following identities.

Proposition 1.

$$(3a)\qquad \sum_{k=1}^{n} k\frac{\binom{x}{k}\binom{y}{k}}{\binom{x+k}{k}\binom{y+k}{k}} = \frac{xy}{2(x+y)}\left\{1-\frac{\binom{x}{n+1}\binom{y}{n+1}}{\binom{x+n}{n+1}\binom{y+n}{n+1}}\right\},$$

$$(3b)\qquad \sum_{k=1}^{n}(-1)^k(xy+k^2)\frac{\binom{x}{k}\binom{y}{k}}{\binom{x+k}{k}\binom{y+k}{k}} = \frac{xy}{2}\left\{-1+(-1)^n\frac{\binom{x}{n+1}\binom{y}{n+1}}{\binom{x+n}{n+1}\binom{y+n}{n+1}}\right\}.$$

Taking $y = v = 0$ and $z = b_k = 1$ in (2) yields

Proposition 2.

$$(4)\qquad \sum_{k=m}^{n}(x-u)\frac{(x;a)_k}{(u;a)_{k+1}} = \frac{(x;a)_{n+1}}{(u;a)_{n+1}}-\frac{(x;a)_m}{(u;a)_m}.$$

When $m = 0$, this reduces to [2]

Corollary 3.

$$(5)\qquad 1+\sum_{k=1}^{n}\prod_{i=1}^{k}\frac{x+a_{i-1}}{u+a_i} = \frac{u+a_0}{u-x}\left\{1-\prod_{j=0}^{n}\frac{x+a_j}{u+a_j}\right\}.$$

This identity contains two formulae of Gould (see [3], Eq.(2.1–2.2) and (4.1–4.2)) as special cases:

$$(6a)\qquad \sum_{k=0}^{n}(-1)^k\frac{\binom{x}{k}}{\binom{u+k}{k}} = \frac{u}{u+x}\left\{1-\frac{\binom{x}{n+1}}{\binom{-u}{n+1}}\right\},$$

$$(6b)\qquad \sum_{k=0}^{n}\frac{\binom{x}{k}}{\binom{u}{k}} = \frac{u+1}{u-x+1}\left\{1-\frac{\binom{x}{n+1}}{\binom{u+1}{n+1}}\right\}.$$

Next, let $\{\omega_k\}_{1\le k\le p}$ be the p-th roots of unity. The substitution of $x \to -x^p$, $u \to -u^p$, and $a \to a^p$ in (4) gives that

Proposition 4.

$$\sum_{k=m}^{n}\prod_{j=1}^{p}\frac{(-\omega_j x;\, a)_k}{(-\omega_j u;\, a)_{k+1}} = \tag{7}$$

$$= \frac{1}{x^p - u^p}\left\{\prod_{j=1}^{p}\frac{(-\omega_j x;\, a)_m}{(-\omega_j u;\, a)_m} - \prod_{j=1}^{p}\frac{(-\omega_j x;\, a)_{n+1}}{(-\omega_j u;\, a)_{n+1}}\right\}.$$

When $m = 0$, this reduces to an identity due to Chu [2]. In the latter case, setting $a_k = y - k$ would generate the following

Proposition 5.

$$\sum_{k=0}^{n}\prod_{j=1}^{p}\frac{\binom{y-\omega_j x}{k}}{\binom{y-\omega_j u-1}{k}} = \frac{y^p - u^p}{x^p - u^p}\left\{1 - \prod_{j=1}^{p}\frac{\binom{y-\omega_j x}{n+1}}{\binom{y-\omega_j u}{n+1}}\right\}. \tag{8}$$

For $y = -1$ and $y = u = 0$, Eq.(8) corresponding to two interesting special formulas [4] displayed below:

Corollary 6.

$$\sum_{k=0}^{n}\prod_{j=1}^{p}\frac{\binom{k-\omega_j x}{k}}{\binom{k-\omega_j u+1}{k}} = \frac{1-u^p}{x^p - u^p}\left\{1 - \prod_{j=1}^{p}\frac{\binom{n-\omega_j x+1}{n+1}}{\binom{n-\omega_j u+1}{n+1}}\right\}, \tag{9a}$$

$$\sum_{k=m}^{n}(-1)^{kp}\prod_{j=1}^{p}\binom{\omega_j x}{k} = \prod_{j=1}^{p}\binom{n-\omega_j x}{n}. \tag{9b}$$

In fact, the last formulae is the limit, as $y \to 0$, of

$$\sum_{k=0}^{n}(-1)^{kp}\prod_{j=1}^{p}\frac{\binom{y-\omega_j x}{k}}{\binom{k-y}{k}} = \frac{y^p}{x^p}\left\{1 - \prod_{j=1}^{p}\frac{\binom{y-\omega_j x}{n+1}}{\binom{y}{n+1}}\right\}$$

which is a reformulation of Eq.(8) under $u = 0$.

For $p = 2r$, an even integer, we have

$$\{\omega_j\}_{1\le j\le 2r} = \{\omega_j\}_{1\le j\le r}\bigcup\{-\omega_j\}_{1\le j\le r}.$$

From the infinite product

$$\frac{\sin \pi t}{\pi t} = \prod_{n=1}^{\infty}\left(1 - \frac{t^2}{n^2}\right), \tag{10}$$

we can derive limiting forms of (9a–9b):

Proposition 7.

$$(11a)\qquad \sum_{k=0}^{\infty}\prod_{j=1}^{r}\frac{\binom{k-\omega_j x}{k}\binom{k+\omega_j x}{k}}{\binom{k-\omega_j u+1}{k}\binom{k+\omega_j u+1}{k}}=\frac{1-u^{2r}}{x^{2r}-u^{2r}}\left\{1-\frac{u^r}{x^r}\prod_{j=1}^{r}\frac{\sin\pi\omega_j x}{\sin\pi\omega_j u}\right\},$$

$$(11b)\qquad \sum_{k=0}^{\infty}\prod_{j=1}^{r}\binom{\omega_j x}{k}\binom{-\omega_j x}{k}=\prod_{j=1}^{r}\frac{\sin\pi\omega_j x}{\pi\omega_j x}.$$

These contain the following simple but interesting examples:

Corollary 8.

$$(12a)\qquad \sum_{k=0}^{\infty}\frac{1}{\binom{u-2}{k}\binom{-u-2}{k}}=(1-u^{-2})\left\{1-\frac{\pi u}{\sin\pi u}\right\},$$

$$(12b)\qquad \sum_{k=0}^{\infty}\binom{x}{k}\binom{-x}{k}=\frac{\sin\pi x}{\pi x}.$$

When $r=1,2,3$, taking the limits as $u\to 0$ and $x\to 0$ successively in (11a), we obtain the celebrated Eulerian formulae:

$$(13a)\qquad \sum_{n>0}n^{-2}=\pi^2/6,$$

$$(13b)\qquad \sum_{n>0}n^{-4}=\pi^4/90,$$

$$(13c)\qquad \sum_{n>0}n^{-6}=\pi^6/945.$$

Finally, put $a_k=b_k=-k$, $x\to x-t$, $y\to -x-t$, $u\to u-t$, $v\to -u-t$ and $z=1$ in (2). We get the following nice result.

Proposition 9.

$$(14)\qquad \sum_{k=0}^{n}\frac{\binom{x-t}{k}\binom{-x-t}{r+k}}{\binom{t-u+k}{k}\binom{t+u+r+k}{r+k}}=$$

$$=\frac{u^2-t^2}{(u-x)(u+x+r)}\left\{\frac{\binom{-x-t}{r}}{\binom{-u-t}{r}}-\frac{\binom{x-t}{n+1}\binom{-x-t}{n+r+1}}{\binom{u-t}{n+1}\binom{-u-t}{n+r+1}}\right\}.$$

After a trival manipulation in the cases of $u = t = 0$ and $x = t - 1 = 0$, we have the respective consequences:

Corollary 10.

$$\sum_{k=0}^{n} \binom{x}{k}\binom{-x}{r+k} = (-1)^r \frac{x}{x+r}\binom{x-1}{n}\binom{-x-1}{r+n}, \tag{15a}$$

$$\sum_{k=0}^{n} \frac{1}{\binom{u-2}{k}\binom{-u-2}{r+k}} = \tag{15b}$$

$$= (-1)^r \frac{u^2-1}{u(u+r)} \left\{ \frac{1}{\binom{-u-1}{r}} - \frac{1}{\binom{u-1}{n+1}\binom{-u-1}{r+n+1}} \right\}.$$

In view of (10), their limiting forms are produced as follows:

Corollary 11.

$$\sum_{k=0}^{\infty} \binom{x}{k}\binom{-x}{r+k} = \frac{\sin \pi x}{\pi(x+r)}, \tag{16a}$$

$$\sum_{k=0}^{\infty} \frac{1}{\binom{u-2}{k}\binom{-u-2}{r+k}} = \frac{u^2-1}{u(u+r)} \left\{ \frac{1}{\binom{u+r}{r}} - \frac{\pi u}{\sin \pi u} \right\}. \tag{16b}$$

For $r = 0$, these reduce to (12a) and (12b) respectively.

Before closing this short paper, I would like to sketch an application of the theorem to basic hypergeometric series.

Recall that the q-factorial is defined by

$$[x; q]_0 = 1, \qquad [x; q]_n = \prod_{k=1}^{n} (1 - xq^{k-1}) \quad (n > 0). \tag{17}$$

Then the replacement $a_k = q^{-k}t$ in (7) gives rise to

Proposition 12.

$$\sum_{k=0}^{n} q^{kp} \prod_{j=1}^{p} \frac{[\omega_j x/t; q]_k}{[q\omega_j u/t; q]_k} = \frac{t^p - u^p}{x^p - u^p} \left\{ 1 - \prod_{j=1}^{p} \frac{[\omega_j x/t; q]_{n+1}}{[\omega_j u/t; q]_{n+1}} \right\}. \tag{18}$$

When $t = p = 1$, its limiting form reduces to an elegant result [1]:

Corollary 13.

$$\sum_{k=0}^{\infty} q^k \frac{[x;q]_k}{[qu;q]_k} = \frac{u-1}{u-x}\left\{1 - \frac{[x;q]_\infty}{[u;q]_\infty}\right\}. \tag{19}$$

From the examples listed above, it can be noted that identity (2) in the theorem contains a plentitude of combinatorial identities. The writer believes that it should do more significant ones as its special cases. For the interested readers, there is no harm to try.

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