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# ORDINARY DIFFERENTIAL EQUATIONS IN AFFINE GEOMETRY

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The method of *qualitative analysis* is used, as applied to a class of fourth order, nonlinear ordinary differential equations, in order to classify, both locally and globally, two classes of hypersurfaces of decomposable type in affine geometry: those with constant unimodular affine mean curvature $L$, and those with constant Riemannian scalar curvature $R$. This allows to provide a large number of new examples of hypersurfaces in affine geometry.

## Introduction.

The use of differential equations in questions concerning classificatory problems in differential geometry has a long standing history. For example, the classification of locally strongly convex, complete affine hyperspheres, whose solution was achieved, historically, through the contributions of several mathematicians: Blaschke [1], in the first place, and then Jörgens [9], Calabi ([2],[3]), Schneider ([12],[13]), Pogorelov [10], Cheng-Yau [4], Sasaki [11] and myself [6]. Another instance of such a use was the previous paper [7], where the author, by means of *qualitative analysis*, got both the local and global classification of all hypersurfaces of decomposable type with constant general affine mean curvature. In this sense, the present article constitutes a follow-up to that paper. The present objective, however, is much wider because it will

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allow to treat, in a unified way, the cases of various families of hypersurfaces in *unimodular* affine geometry. The differential equations involved here are of the following kinds:

$$\frac{y^{iv}}{(y'')^r} - s\,\frac{(y''')^2}{(y'')^{r+1}} = K = \text{constant},$$

where the constant values of $r$ and $s$ have a dimensional meaning, which shall vary according to the problem to be considered. In fact, one of the main reasons for treating the above equation with variable values for those two constants is its possible future use in order to solve other, diverse geometrical problems.

By performing qualitative analysis to the above differential equation we shall be able, during the course of the present exposition, to classify those hypersurfaces in affine space, of decomposable type and with constant (vanishing and nonvanishing) *unimodular* affine mean curvature; and also those with constant Riemannian scalar curvature. Moreover, since these scalar invariants are related between them by the Pick invariant (in the so-called Higher Dimensional Affine Theorema Egregium), it is worth considering, too, the third order equation that represents the latter, also for hypersurfaces of the mentioned type. This has been previously done in [8] and we shall summarize here those results, both for the sake of completeness and since they are used in Section 8 ahead.

In the classification of locally strongly convex hypersurfaces, mentioned above, two different notions of completeness were used:

(1) Unimodular affine metric completeness, i.e. completeness with regards to the Levi-Civita connection determined by the first fundamental form $I_{ua}$.

(2) Geometrical completeness; meaning that $X(M)$ is complete if, ad only if, it is a closed set with respect to the ambient space topology, induced by the vector space structure of $E$. Some authors prefer to refer to this as Euclidian completeness, i.e. with respect to the (also) Riemannian structure induced on $X(M)$ from a Euclidian metric assumed to be further defined on the ambient vector space $E$.

The qualitative analysis practiced in this article includes behavior at limit points. Thus, it will be quite atraightforward to decide which of the hypersurfaces presented here are geometrically complete.

This article is organized as follows: in Section 1 we obtain the *characterizing* ordinary differential equation of the various subclasses to be considered. In Section 2, we state two theorems, whose proofs were previously exposed in [8], concerning the local, as well as global, classification of those hypersurfaces, in the above class, with constant Pick invariant. The reduction of the above fourth order ordinary differential equation: first to second order and, immediately after, to first order occupies Section 3, where, besides, the four cases of *current geometrical interest* are presented. In Section 4, we perform qualitative analysis to the inverse function representing the solution to the reduced, second order equation; while the complete integration and analysis of the direct function is exposed in Section 5. In Section 6 we present the full classification of hypersurfaces of decomposable type with constant unimodular affine mean curvature; while Section 7 is dedicated to those with constant unimodular affine scalar curvature. Finally, in Section 8, we consider the case, for hypersurfaces within the above class, where the three aforementioned scalar invariants are simultaneously constant.

## 1. Characterizing Differential Equations.

We consider in this section hypersurfaces of decomposable type, i.e. those which are expressible in the form of Monge's with respect to a suitable affine system of coordinates $(t^1, t^2, \dots t^n, t^{n+1})$ for the vector space $E$, by an equation $X(t^1, \dots, t^n) = (t^1, \dots, t^n, f(t^1, \dots, t^n))$, with $(t^1, \dots, t^n)$ varying in an open, connected subset of $R^n$, and the map $f$ assumed to be enough differentiable; and where besides $f$ can be decomposed into a sum of $n$ terms, each of them depending on only one of the independent variables $t^1, \dots, t^n$ : $f(t^1, \dots, t^n) = f^1(t^1) + f^2(t^2) + \cdots + f^n(t^n)$. This family was treated previously in [7], where, under the viewpoint of *general affine geometry*, we classified those hypersurfaces in the class with constant **general** affine mean curvature. Later in [8], the viewpoint was that of *unimodular affine geometry*, and the classification regarded to those with constant Pick invariant. We shall use here most of the calculations developed in those articles. Thus, for example, the unimodular affine *principal curvatures* $k_1, k_2, \dots, k_n$ are the eigenvalues of the third fundamental form $III_{ga} := L_{ij}\sigma^i\sigma^j$, with respect to the **unimodular** affine first fundamental form $I_{ua} := g_{ij}\sigma^i\sigma^j$, i.e. determined as the roots of the equation $\det(L_{ij} + k\, g_{ij}) = 0$, and their normalized elementary symmetric functions $L_1, L_2, \dots, L_n$ are the unimodular affine *curvature functions*. Hence, the unimodular affine mean curvature is obtained by averaging the contraction of the third fundamental form with respect to the first (unimodular affine) fundamental form, i.e.

$$L := L_1 = -\frac{1}{n}\sum_{i,j} g^{ij} L_{ij} = -\frac{1}{n}\sum_k L^k_k. \tag{1.1}$$

We can also calculate the *scalar curvature*. In fact, from Gauss' equation it follows that:

$$\sum g^{jk} g^{iq} R_{jikq} = (1-n) \sum g^{jq} L_{jq} + \sum g^{iq} A^s_{ij} A^j_{sq}. \tag{1.2}$$

Hence, from the last equation we obtain that

$$R = L + J, \tag{1.3}$$

this being known as the Higher Dimensional Affine Theorema Egregium, where the "normalized" scalar curvature is defined by

$$R := \frac{1}{n(n-1)} \sum g^{jk} g^{iq} R_{jikq}, \tag{1.4}$$

and the Pick invariant $J$ can be expressed, in terms of the components of the first two fundamental forms by

$$J := \frac{1}{n(n-1)} \sum g^{iq} A^s_{ik} A^k_{sq}. \tag{1.5}$$

It follows that, for hypersurfaces of decomposable type, the scalar components of the third fundamental form can be written

$$L_{kk} = -\frac{1}{(n+2)^2} \left\{ (n+2) \frac{(f^k)^{iv}}{(f^k)''} - (2n+3) \frac{[(f^k)''']^2}{[(f^k)'']^2} \right\}, \tag{1.6}$$

$$L_{jk} = -\frac{1}{(n+2)^2} \frac{(f^j)'''}{(f^j)''} \frac{(f^k)'''}{(f^k)''}, \text{ if } j \neq k; \tag{1.7}$$

and since the first fundamental form is given by $I_{ua} = F^{-1/(n+2)}(\sum f_{ij} dt^i dt^j)$, with

$$f_{ij} = \begin{cases} (f^i)'' & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases} \tag{1.8}$$

$$F = |(f^1)'' \cdot (f^2)'' \cdots (f^n)''|, \tag{1.9}$$

we obtain that the **unimodular** affine mean curvature is computable as

$$(1.10)\qquad L = L_1 = -\frac{1}{n}F^{\frac{1}{n+2}}\sum_{k=1}^{n}\frac{L_{kk}}{(f^k)''} =$$
$$= \frac{1}{n}\frac{1}{(n+2)^2}F^{\frac{1}{n+2}}\sum_{k=1}^{n}\left\{(n+2)\frac{(f^k)^{iv}}{[(f^k)'']^2} - (2n+3)\frac{[(f^k)''']^2}{[(f^k)'']^3}\right\}.$$

**Lemma 1.1.** *Let $X : M^n \to E^{n+1}$ be a nondegenerate hypersurface of decomposable type. Then, its unimodular affine mean curvature vanishes identically if, and only if, each of its components $f^k$ satisfies the ordinary differential equation*

$$(1.11)\qquad (n+2)\frac{(f^k)^{iv}}{[(f^k)'']^2} - (2n+3)\frac{[(f^k)''']^2}{[(f^k)'']^3} = C_k = \text{ constant},$$

*with the additional condition $\sum C_k = 0$.*

*Proof.* Immediate from (1.10), since we have that $F \neq 0$, by the nondegeneracy assumed for $X$.

By an argument quite similar to that used in [8], (Lemma 2.2) it also follows

**Lemma 1.2.** *Let $X : M^n \to E^{n+1}$ be a nondegenerate hypersurface of decomposable type. Then, its unimodular affine mean curvature is identically equal to a non-vanishing constant if, and only if, one of its components $f^{k_0}$ satisfies the ordinary differential equation*

$$(1.12)\qquad (n+2)\frac{(f^{k_0})^{iv}}{[(f^{k_0})'']^2} - (2n+3)\frac{[(f^{k_0})''']^2}{[(f^{k_0})'']^3} = C_{k_0}[(f^{k_0})'']^{-\frac{1}{n+2}},$$

*where $C_{k_0}$ is a nonvanishing constant; while all of the remaining ones are of parabolic type.*

Finally, and by the same token, we record the characterization of hypersurfaces of decomposable type and constant scalar curvature. This invariant can be obtained either from the higher dimensional Affine Theorema Egregium, represented by equation (1.3), or by direct calculation.

**Lemma 1.3.** *Let $X : M^n \to E^{n+1}$ be a nondegenerate hypersurface of decomposable type. Then, its unimodular affine scalar curvature vanishes* *identically if, and only if, each of its components* $f^k$ *satisfies the ordinary differential equation*

$$(n+2)\frac{(f^k)^{iv}}{[(f^k)'']^2} - \frac{7n+10}{4}\frac{[(f^k)''']^2}{[(f^k)'']^3} = C_k = constant, \tag{1.13}$$

*with the additional condition* $\sum C_k = 0$.

**Lemma 1.4.** *Let* $X : M^N \to E^{n+1}$ *be a nondegenerate hypersurface of decomposable type. Then, its unimodular affine scalar curvature is identically equal to a nonvanishing constant if, and only if, one of its components* $f^{k_0}$ *satisfies the ordinary differential equation*

$$(n+2)\frac{(f^{k_0})^{iv}}{[(f^{k_0})'']^2} - \frac{7n+10}{4}\frac{[(f^{k_0})''']^2}{[(f^{k_0})'']^3} = C_{k_0}[(f^{k_0})'']^{-\frac{1}{n+2}}, \tag{1.14}$$

*where* $C_{k_0}$ *is a nonvanishing constant; while all of the remaining ones are of parabolic type.*

## 2. Decomposable Hypersurfaces with constant Pick Invariant.

In this section we state, without proofs, two theorems regarding the classification of those hypersurfaces of decomposable type with constant Pick Invariant. See [8] for full details.

**Theorem 2.1.** *Let* $X : M^n \to E^{n+1}$ *be a nondegenerate hypersurface of decomposable type, with vanishing unimodular affine Pick invariant* $J = 0$. *Then, each of its components* $f^1, f^2, \ldots, f^n$, *must be of either parabolic or logarithmic type, or the three more kinds of types that are obtained from those two original types by suitable reflections in the* $x-$ *and* $y$*-axis. All of the solutions belonging to the original types share the common feature that their second derivatives,* $(f^k)'' = y > 0$, *satisfy, in each case, the classifying nonlinear, ordinary differential equation* $y' = \tilde{K} y^{3/2}$, *with* $\tilde{K} := C_k = 0$ *for the parabolic type and* $\tilde{K} > 0$ *for the remaining, logarithmic type.*

**Theorem 2.2.** *Let* $X : M^n \to E^{n+1}$ *be a nondegenerate hypersurface of decomposable type, with constant, nonvanishing unimodular affine Pick invariant,* $J = constant \neq 0$. *Then, one of the components must be of the type described by the equation* $(f^{k_0})(t^{k_0}) = (t^{k_0})^{-2/(n+1)}$, *or the corresponding ones which are obtained from the latter by suitable reflections on the* $x-$ *and* $y$*-axis. All of the remaining components* $f_j$, *with* $j \neq k_0$, *must be of parabolic type.*

## 3. Reduction of Mean Curvature and Scalar Curvature Equations.

As in [7], we can reduce in a unified way equations (1.11) through (1.14) by the substitution $x = t^k$, $(f^k)'' = y = g(x)$. This gives the following second order, nonlinear ordinary differential equation, common to all mentioned cases,

$$\frac{y''}{y^r} - s\frac{(y')^2}{y^{r+1}} = K, \tag{3.1}$$

where we have renamed, for convenience, the constant $C_k$ by $K$.

Let us recall that there is a standard method for calculating a first integral of the last equation by means of the change of variables $(y')^2 = z$. This furnishes for (3.1)

$$\frac{1}{2}\, y\, dz - [sz + Ky^{r+1}]dy = 0. \tag{3.2}$$

We shall consider four cases of current geometrical interest regarding the constant values of $r$ and $s$ :

1) $r = 2$, $s = \frac{2n+3}{n+2}$; (Lemma 1.1: vanishing unimodular affine mean curvature);

2) $r = 2 - \frac{1}{n+2}$, $s = \frac{2n+3}{n+2}$, (Lemma 1.2: constant nonvanishing equiaffine mean curvature);

3) $r = 2$, $s = \frac{7n+10}{4(n+2)}$; (Lemma 1.3: vanishing scalar curvature);

4) $r = 2 - \frac{1}{n+2}$, $s = \frac{7n+10}{4(n+2)}$; (Lemma 1.4: constant nonvanishing scalar curvature).

We can assume $y > 0$ and take $\mu = y^{-(1+2s)}$ as integrating factor. Thus (3.2) becomes

$$\frac{1}{2}y^{-2s}dz - [szy^{-(1+2s)} + K\ y^{r-2s}]dy = 0, \tag{3.3}$$

which is an exact equation.

Now, in order to integrate the latter equation we have to separate two cases:

A) This comprises the above *geometrical cases* 1), 2), 4), and 3) with $n > 2$. (Observe that here $r + 1 - 2s > 0$).

B) Corresponds to the remaining, limiting case 3) with $n = 2$. (Now, we have $r + 1 - 2s = 0$).

In order not to make our arguments too involved, during the course of the present exposition we shall only consider case A), leaving case B) for future development.

We obtain that, in the first case, the solution is given by

$$y' = y^{\frac{r+1}{2}} \left( C y^{2s-r-1} - \frac{2K}{2s-r-1} \right)^{1/2}. \tag{3.4}$$

Again, by analyzing all of possible cases of current geometrical interest, it is easy to show that the complete set of integral curves of the last two numbered equations is obtained by assuming first $y > 0$, and symmetrizing afterwards with respect to the $x-$ and $y$-axis.

We record the above considerations in a lemma.

**Lemma 3.1.** *The analysis of solutions of the nonlinear, second order ordinary differential equation* (3.1)*, including further integration, can be reduced to the analysis of positive solutions of the nonlinear, first order ordinary differential equations* (3.4)*. There exist three remaining classes of solutions, whose corresponding properties, including further integration, can be obtained by reflections on the $x-$ and $y$-axis.*

## 4. Qualitative analysis for the inverse function.

**Type I: C** $= 0$, $K = 0$. This is the most simple of solutions: we obtain immediately $y = \tilde{C}$. Hence, after two more integrations we get for the corresponding component the solution of the form $f^k(t^k) = C_k(t^k)^2 + D_k t^k + E_k$, ($k$ not summed), and, by an affine change of coordinates, we further write

$$f^k(t^k) = (t^k)^2, \text{ (parabolic type).} \tag{4.1}$$

**Type II: C** $= 0$, $K < 0$. We have to separate two subcases: $r = 2$; $r = 2 - \frac{1}{n+2}$.

If $r = 2$, it follows that $y = 4\left\{\left(-\frac{2K}{2s-3}\right)^{1/2} x + \tilde{C}\right\}^{-2}$. After two more integrations, and normalization of the corresponding constants by means of suitable affine changes of coordinates, one gets for the component of the hypersurface the expression

$$f^k(t^k) = -\log(t^k), \text{ (logarithmic type).} \tag{4.2}$$

Similarly, if $r = 2 - \frac{1}{n+2}$, we obtain

$$f^k(t^k) = (t^k)^{-\frac{2}{n+1}}. \tag{4.3}$$

**Type III: C** $> 0$, $K = 0$. Here it is again possible to integrate explicitly. We observe that $s > 1$ and obtain, after a suitable affine change of coordinates the expression

$$(f^k)''(t^k) = (t^k)^{\frac{1}{1-s}}. \tag{4.4}$$

Here, let us consider first geometrical cases 1) and 2), so that $s \neq 3/2$. Hence, if we perform two more integrations and normalize the constants, we obtain the solution

$$f^k(t^k) = \frac{(1-s)^2}{(2-s)(3-2s)}(t^k)^{\frac{3-2s}{1-s}}. \tag{4.5}$$

In geometrical cases 3) and 4), we have that $s = 3/2$ only for dimension $n = 2$. Thus, if $n > 2$, we obtain again the function represented by equation (4.5) as solution. On the other hand, for the subcase $n = 2$ the solution becomes

$$f^k(t^k) = -\log t^k. \tag{4.6}$$

Now, for the remaining of cases, where no longer explicit solutions can be computed in an immediate way, we are in pursue of analyzing the local, as well as global, behaviour of $y$ as a function of $x : y = g(x)$, when this represents a solution to equation (3.4). However, it is more convenient, first, to look at $x$ as a function of $y$. For this purpose we consider this equation in its equivalent form

$$\frac{dx}{dy} = \frac{1}{y^{\frac{r+1}{2}}\left(Cy^{2s-r-1} - \frac{2K}{2s-r-1}\right)^{1/2}}. \tag{4.7}$$

By fixing a point $(x_0, y_0)$, with $y_0 > 0$, as initial condition, we can describe a solution to the latter equation by

$$x = x_0 + \int_{y_0}^{y} \frac{dt}{t^{\frac{r+1}{2}}\left(Ct^{2s-r-1} - \frac{2K}{2s-r-1}\right)^{1/2}}. \tag{4.8}$$

We shall use this last expression in order to accomplish our goal of analyzing the behaviour of solutions for the remaining of cases. Prior to that, we make a couple of observations that shall be of help.

Firstly, we observe that the local behaviour shall depend mainly on the first two derivatives of $x$, with respect to $y$. So we now calculate the second one, and put together with the first, in the following equations

$$\begin{aligned} \frac{dx}{dy} &= y^{-\frac{r+1}{2}}\left(Cy^{2s-r-1} - \frac{2K}{2s-r-1}\right)^{-1/2}, \\ \frac{d^2x}{dy^2} &= \frac{-sCy^{2s-r-1} + \frac{(r+1)K}{2s-r-1}}{y^{\frac{r+3}{2}}\left(Cy^{2s-r-1} - \frac{2K}{2s-r-1}\right)^{3/2}}. \end{aligned} \tag{4.9}$$

Secondly, we observe, too, that the global behaviour of the solutions shall also depend on the analysis in the neighbourhood of singular points, as well as on the questions of convergence, or divergence, of the integral in (4.8).

**Type IV: C** $> 0$, $K > 0$. The graph of $x$ versus $y$, defined by equation (4.8), is fully characterized by the following conditions: there exist real numbers $\bar{x}_1, x_1$ (with $\bar{x}_1 < x_0 < x_1$) such that:

$$\lim_{y \to \infty} x(y) = x_1, \tag{4.10}$$

$$\lim_{y \to \bar{y}_0} x(y) = \bar{x}_1; \tag{4.11}$$

$$\frac{dx}{dy} > 0, \frac{d^2x}{dy^2} < 0, \forall\, y \in (\bar{y}_0, \infty). \tag{4.12}$$

In fact, we observe first that, in order for the square root to make sense in (4.8), we must have that $w := Cy^{2s-r-1} - (2K)/(2s - r - 1) > 0$. We choose $y_0 > \bar{y}_0 := ((2K)/[(2s - r - 1)C])^{1/(2s-r-1)}$, and make the substitution $u = ([(2s - r - 1)C]/(2K))^{1/(2s-r-1)}\, t = (\bar{y}_0)^{-1}\, t$, in order to write (4.8) in the form

$$x = x_0 + \frac{\bar{y}_0^{1-s}}{C^{1/2}} \int_{z_0}^{z} \frac{du}{u^{(r+1)/2} \left(u^{2s-r-1} - 1\right)^{1/2}}, \tag{4.13}$$

with $z = (\bar{y}_0)^{-1}\, y$, and corresponding value for $z_0$ in terms of $y_0$.

It then follows that the integral in (4.13) can be estimated. In fact, there exists $u_0 (> z_0)$ such that $u^{2s-r-1} - 1 > 1$, for all $u > u_0$, and hence, for $n \geq 2$, we have that $\frac{r+1}{2} \geq 11/8$. Therefore, since the integral $\int_{z_0}^{\infty} u^{-11/8} du$ is convergent, it follows that there exists a real number $x_1 (> x_0)$ such that (4.10) holds.

On the other hand, if we write $p = 2s - r - 1$ and $v := u^p - 1$, for the integral in (4.13) we have

$$\int_{z_0}^{z} \frac{du}{u^{\frac{r+1}{2}} (u^p - 1)^{1/2}} = \frac{1}{p} \int_{s_0}^{s} \frac{dv}{(v + 1)^{\frac{1}{p}(\frac{r+1}{2} + p - 1)}\, v^{1/2}}$$

with $s = z^p - 1$, $s_0 = z_0^p - 1$. Then, for $1 < z < z_0$, and since $\frac{r+1}{2} + p - 1 > 0$, it holds the inequality

$$0 < \int_{s_0}^{s} \frac{dv}{(v + 1)^{\frac{1}{p}(\frac{r+1}{2} + p - 1)}\, v^{1/2}} < \int_{s}^{s_0} \frac{dv}{v^{1/2}}.$$

This last proves (4.11) holds. Finally, (4.12) follows by direct calculation from (4.9).

**Type V: C** $> 0$, $K < 0$. Equation (4.8) defines $x$ as a function of $y$ with the following properties: there exists a real number $x_2 > 0$ such that

$$\lim_{y \to \infty} x(y) = x_2\,, \tag{4.14}$$

$$\lim_{y \to 0} x(y) = -\infty, \tag{4.15}$$

$$\frac{dx}{dy} > 0,\ \frac{d^2x}{dy^2} < 0. \tag{4.16}$$

We take as initial point $(x_0, y_0)$, such that $y_0 > 0$. Then we can write equation (4.8) in the form

$$x = x_0 + \frac{1}{C^{1/2}} \int_{y_0}^{y} \frac{dt}{t^{(r+1)/2}(t^p - a)^{1/2}}, \tag{4.17}$$

where $a = \frac{2K}{(2s-r-1)C} (< 0)$, and $p = 2s - r - 1$. Hence, since $t^p - a > t^p$ for every $t > 0$, it follows easily that there exists $x_2 > 0$ such that (4.14) holds.

On the other hand, $0 < t < y_0$ implies that $0 < t^p - a < y_0^p - a$, and hence the integral above can again be estimated, in this case by taking $y < y_0$, as follows

$$\int_{y}^{y_0} \frac{dt}{t^{(r+1)/2}(t^p - a)^{1/2}} > \frac{1}{(y_0^p - a)^{1/2}} \int_{y}^{y_0} \frac{dt}{t^{(r+1)/2}} \geq \frac{1}{(y_0^p - a)^{1/2}} \int_{y}^{y_0} \frac{dt}{t^{3/2}},$$

and since the right-hand member diverges to $+\infty$ as $y$ converges to 0, equation (4.15) holds. Finally, (4.16) follows directly from (4.9).

**Type VI: C** $< 0$, $K < 0$. We have in this case the following properties:

$$\lim_{y \to 0} x(y) = -\infty, \tag{4.18}$$

$$\lim_{y \to \bar{y}_0} x(y) = x_3 > x_0, \tag{4.19}$$

$$\frac{dx}{dy} > 0, \tag{4.20}$$

$$\frac{d^2x}{dy^2} < 0, \text{ in } (0, y_0^{\#}); \; \frac{d^2x}{dy^2}(y_0^{\#}) = 0; \; \frac{d^2x}{dy^2} > 0, \text{ in } (y_0^{\#}, \bar{y}_0), \tag{4.21}$$

where $y_0^{\#} := \left(\frac{(r+1)K}{(2s-r-1)sC}\right)^{1/(2s-r-1)}$.

We take, in this last case, $(x_0, y_0)$ in such a way that $0 < y_0 < \bar{y}_0$, with the right limit $\bar{y}_0 := ((2K)/[2s - r - 1)C])^{1/(2s-r-1)}$ being one of the singularities of (4.8). Hence , it follows that

$$x = x_0 + \frac{1}{(-C)^{1/2}} \int_{y_0}^{y} \frac{dt}{t^{(r+1)/2}(b - t^p)^{1/2}}, \tag{4.22}$$

where, similarly to the previous case, $b = (2K)/[(2s - r - 1)C] > 0$, $p = 2s - r - 1$.

Now, since the inequality $(b - t^p)^{1/2} < b^{1/2}$ implies the estimate

$$\int_{y}^{y_0} \frac{dt}{t^{(r+1)/2}(b - t^p)^{1/2}} > \frac{1}{b^{1/2}} \int_{y}^{y_0} \frac{dt}{t^{(r+1)/2}} \geq \frac{1}{b^{1/2}} \int_{y}^{y_0} \frac{dt}{t^{3/2}},$$

for $y < y_0$, then, equation (4.18) follows at once.

Next, we make the substitution $0 < u = b - t^p$ in order to obtain, for $y > y_0$ the estimate

$$\int_{y_0}^{y} \frac{dt}{t^{(r+1)/2}(b - t^p)^{1/2}} = -\frac{1}{p} \int_{b-y_0^p}^{b-y^p} \frac{du}{(b-u)^{1+\frac{r-1}{2p}} u^{1/2}} <$$

$$< -\frac{1}{p y_0^{p+\frac{r-1}{2}}} \int_{b-y_0^p}^{b-y^p} \frac{du}{u^{1/2}},$$

from which it follows (4.19).

Finally, equation (4.9) implies directly (4.20) and (4.21).

## 5. Further integration of the direct function.

We have computed in the previous section explicit solutions for three of the types under consideration. Moreover, from the analysis practiced on the remaining cases, which furnish the inverse functions of the solutions, $x = x(y)$, one gets the unified conclusion that the latter are always invertible, allowing to define $y$ as a function of $x$ : say $y = g(x)$, for all possible cases. Thus, in order to obtain the components of the functions defining the hypersurfaces we need, in accordance with (3.1), to further integrate $y = g(x)$ two more times.

This can be done for all of types, as we shall see in the current section. A fundamental tool for accomplishing that goal will be the non-linear, ordinary differential equations (3.4), which we now rewrite as

$$\frac{dy}{dx} = g'(x) = [g(x)]^{(r+1)/2}\left(C[g(x)]^{2s-r-1} - \frac{2K}{2s-r-1}\right)^{1/2}. \tag{5.1}$$

We shall use this expression in order to analyze separately the diverse remaining cases, and shall agree in denoting by $G(x)$, $F(x)$ the first and, respectively, the second integral of $g(x)$ in all of possible cases.

**Type IV: C** $> 0$, $K > 0$. We easily obtain that, as a consequence of (4.10), (4.11) and (4.12), the function $y = g(x)$, where $g : (\bar{x}_1, x_1) \to R$, is characterized by the three following conditions

$$\lim_{x\to x_1} g(x) = +\infty\,, \tag{5.2}$$

$$\lim_{x\to \bar{x}_1} g(x) = \bar{y}_0 := \left(\frac{2K}{2s-r-1)C}\right)^{1/(2s-r-1)} > 0\,, \tag{5.3}$$

$$\frac{dg}{dx} > 0,\ \frac{d^2g}{dx^2} > 0,\ \forall x \in (\bar{x}_1, x_1). \tag{5.4}$$

**Lemma 5.1.** *The first integral of* $g(x)$, $G : (\bar{x}_1, x_1) \to R$, *satisfies the following conditions:*

$$\lim_{x\to x_1} G(x) = +\infty\,, \tag{5.5}$$

$$\lim_{x\to \bar{x}_1} G(x) = \overline{G}_1 \in R\,, \tag{5.6}$$

$$G'(x) = g(x) > 0,\ G''(x) = g'(x) > 0,\ \forall x \in (\bar{x}_1, x_1). \tag{5.7}$$

*Proof.* By taking $(\tilde{x}_0, \tilde{y}_0)$, with $\tilde{x}_0 \in (\bar{x}_1, x_1)$, a typical integral of the function $g$ can be written

$$G(x) = \tilde{y}_0 + \int_{\tilde{x}_0}^{x} g(t)\,dt. \tag{5.8}$$

Hence, if we define $a := [(2s - r - 1)C]/(2K)$, the use of (5.1) allows to write the latter as

$$G(x) = \tilde{y}_0 + \left(\frac{2s - r - 1}{2K}\right)^{1/2} \int_{\tilde{x}_0}^{x} \frac{g'(t)\,dt}{[g(t)]^{(r-1)/2}\left(a[g(t)]^{2s-r-1} - 1\right)^{1/2}}. \tag{5.9}$$

Next, by integrating by parts one obtains

$$G(x) = \tilde{y}_0 + \left(\frac{2s - r - 1}{2K}\right)^{1/2} \frac{2}{3 - r} \left(\frac{[g(t)]^{(3-r)/2}}{\left(a[g(t)]^{2s-r-1} - 1\right)^{1/2}}\right)_{\tilde{x}_0}^{x} + \tag{5.10}$$
$$+ \frac{a(2s - r - 1)^{3/2}}{(2K)^{1/2}(3 - r)} \int_{\tilde{x}_0}^{x} \frac{[g(t)]^{2s-\frac{3}{2}r-\frac{1}{2}} g'(t)\,dt}{\left(a[g(t)]^{2s-r-1} - 1\right)^{3/2}}.$$

Now, since for $x > \tilde{x}_0$ the third term in the right-hand side of this identity is greater that 0, while the second one diverges to $+\infty$ as $x$ converges to $x_1$, it follows that equation (5.5) holds.

(5.6) follows directly from (5.3) and (5.8); (5.7) is a consequence of (5.2) through (5.4).

**Lemma 5.2.** *The second integral of $g(x)$, $F : (\bar{x}_1, x_1) \to R$, satisfies the following conditions:*

$$\begin{aligned} &F'(x) = G(x) \text{ ranges from } \overline{G}_1 \text{ to } +\infty, \\ &F''(x) = G'(x) > 0,\ \forall x \in (\bar{x}_1, x_1)\,, \end{aligned} \tag{5.11}$$

$$\lim_{x \to \bar{x}_1} F(x) = \overline{F}_1 \in R. \tag{5.12}$$

*The other limit behaves as follows: in geometrical cases 1) $r = 2$, $s = \frac{2n+3}{n+2}$; 2) $r = 2 - \frac{1}{n+2}$, $s = \frac{2n+3}{n+2}$; 3) $r = 2$, $s = \frac{7n+10}{4(n+2)}$, (here it holds $n > 2$), 4) $r = 2 - \frac{1}{n+2}$, $s = \frac{7n+10}{4(n+2)}$, with $n > 2$, we have that*

$$\lim_{x \to x_1} F(x) = F_1 \in R; \tag{5.13}$$

*while in case 4) with $n = 2$,*

$$\lim_{x \to x_1} F(x) = +\infty. \tag{5.14}$$

*Proof.* We write for $F : (\bar{x}_1, x_1) \to R$, and $\hat{x}_0 \in (\bar{x}_1, x_1)$,

$$F(x) = \hat{y}_0 + \int_{\hat{x}_0}^{x} G(t)\, dt. \tag{5.15}$$

It is immediate to obtain, from Lemma 5.1 above, that the first two derivatives of $F$ behave as indicated in (5.11), while it is also easy to see, on the other hand, that (5.12) holds.

To complete the study on the global behaviour of $F$ we need to determine $\lim\limits_{x \to x_1} F(x)$, and to achieve this we have to analyze in more detail $G(x)$ near that limit point. Let us consider the integral part in equation (5.9). We shall treat separately two cases, namely:

$A_1$) $\frac{1}{n+2} < 2s - r - 1$; $A_2$) $\frac{1}{n+2} \geq 2s - r - 1$.

In case $A_1$), if we fix a real number $q$ such that $\frac{1}{n+2} < q < 2s - r - 1$, it follows from (5.2) that there exists $t_1$ in the open interval $(\tilde{x}_0, x_1)$ such that for every $t \in (t_1, x_1)$ it holds

$$[g(t)]^q < a[g(t)]^{2s-r-1} - 1. \tag{5.16}$$

From this we obtain the estimate

$$\int_{t_1}^{x} \frac{g'(t)\, dt}{[g(t)]^{(r-1)/2} \left(a[g(t)]^{2s-r-1} - 1\right)^{1/2}} < \int_{t_1}^{x} \frac{g'(t)\, dt}{[g(t)]^{(r+q-1)/2}} = \tag{5.17}$$
$$= \frac{2}{3-r-q} \left([g(x)]^{(3-r-q)/2} - [g(t_1)]^{(3-r-q)/2}\right),$$

for every $x$ in the open interval $(t_1, x_1)$. Next, we estimate the integral of $[g(t)]^p$, with $p = (3 - r - q)/2$. We can write, by using (5.1) and (5.16), that

$$\int_{t_1}^{x} [g(t)]^p\, dt = \left(\frac{2s - r - 1}{2K}\right)^{1/2} \int_{t_1}^{x} \frac{[g(t)]^{p - \frac{r+1}{2}} g'(t)\, dt}{\left(a[g(t)]^{2s-r-1} - 1\right)^{1/2}} <$$
$$< \left(\frac{2s - r - 1}{2K}\right)^{1/2} \int_{t_1}^{x} [g(t)]^{p - \frac{q}{2} - \frac{r+1}{2}} g'(t)\, dt =$$
$$= \left(\frac{2s - r - 1}{2K}\right)^{1/2} \frac{1}{p - \frac{r-1}{2} - \frac{q}{2}} \left([g(x)]^{p-(r+q-1)/2} - [g(t_1)]^{p-(r+q-1)/2}\right).$$

The last member has a finite limit as $x \to x_1$, because for both cases of current geometrical interest, namely $r = 2$ and $r = 2 - \frac{1}{n+2}$, we have that

$$p - (r + q - 1)/2 = 2 - (r + q) = \left\{ \begin{array}{ll} -q, & \text{if } \quad r = 2 \\ \frac{1}{n+2} - q, & \text{if } \quad r = 2 - \frac{1}{n+2} \end{array} \right\} < 0.$$

Hence, it follows in case $A_1$) that

$$\lim_{x\to x_1} F(x) = F_1 \in R. \tag{5.18}$$

To treat now case $A_2$), we fix a real number $q$ in the closed interval $[2s - r - 1, \frac{1}{n+2}]$. Then it follows from (5.2) that there exists $t_1$ in the open interval $(\tilde{x}_0, x_1)$ such that for every $t \in (t_1, x_1)$ it holds

$$a[g(t)]^q > a[g(t)]^q - 1 \geq a[g(t)]^{2s-r-1} - 1. \tag{5.19}$$

From this we obtain the estimate

$$\begin{aligned} \int_{t_1}^{x} \frac{g'(t)\,dt}{[g(t)]^{(r-1)/2}\left(a[g(t)]^{2s-r-1} - 1\right)^{1/2}} &> \frac{1}{a^{1/2}} \int_{t_1}^{x} \frac{g'(t)\,dt}{[g(t)]^{(r+q-1)/2}} = \\ &= \frac{1}{a^{1/2}} \frac{2}{3-r-q} \left([g(x)]^{(3-r-q)/2)} - [g(t_1)]^{(3-r-q)/2}\right), \end{aligned} \tag{5.20}$$

for every $x$ in the open interval $(t_1, x_1)$. Next, we estimate the integral of $[g(t)]^p$, with $p = (3 - r - q)/2$. We can write, by using (5.1) and (5.19), that

$$\begin{aligned} \int_{t_1}^{x} [g(t)]^p\,dt &= \left(\frac{2s-r-1}{2K}\right)^{1/2} \int_{t_1}^{x} \frac{[g(t)]^{p-\frac{r+1}{2}} g'(t)\,dt}{\left(a[g(t)]^{2s-r-1} - 1\right)^{1/2}} > \\ &> \left(\frac{2s-r-1}{2K}\right)^{1/2} \frac{1}{a^{1/2}} \int_{t_1}^{x} [g(t)]^{p-\frac{q}{2}-\frac{r+1}{2}} g'(t)\,dt = \\ &= \left(\frac{2s-r-1}{2K}\right)^{1/2} \frac{a^{-1/2}}{p - \frac{r-1}{2} - \frac{q}{2}} \left([g(x)]^{p-(r+q-1)/2)} - [g(t_1)]^{p-(r+q-1)/2}\right). \end{aligned}$$

Now, we have, as before, that

$$p - (r+q-1)/2 = 2 - (r+q) = \begin{cases} -q, & \text{if } \ r = 2, \\ \frac{1}{n+2} - q, & \text{if } \ r = 2 - \frac{1}{n+2}. \end{cases}$$

Hence, if $r = 2$ there is no conclusion. But, if $r = 2 - \frac{1}{n+2}$ and if, besides, one can choose $q < \frac{1}{n+2}$, then it follows that $\lim_{x\to x_1} F(x) = +\infty$.

Thus, let us analyze next, the diverse cases 1) through 4) of current geometrical interest:

1) If $r = 2$, $s = \frac{2n+3}{n+2}$. It follows that $2s - r - 1 = \frac{n}{n+2} > \frac{1}{n+2}$, since $n \geq 2$, (case $A_1$) ), and hence (5.13) holds.
2) If $r = 2 - \frac{1}{n+2}$, $s = \frac{2n+3}{n+2}$. It follows that $2s - r - 1 = \frac{n+1}{n+2} > \frac{1}{n+2}$, and hence, again, (5.13) holds.
3) If $r = 2$, $s = \frac{7n+10}{4(n+2)}$. Recall that in the present case we also have $n > 2$. Thus

$$2s - r - 1 = \frac{n-2}{2(n+2)} = \begin{cases} > \frac{1}{n+2} & \text{if} \quad n > 4, \\ \leq \frac{1}{n+2} & \text{if} \quad n \leq 4 \text{ (i.e. } n = 3, 4). \end{cases}$$

We are in case $A_1$), if $n > 4$, and obtain that (5.13) holds.

On the other hand, if $n = 3, 4$ we can take $q$ such that $0 < q < 2s - r - 1$, repeat the argument as in case $A_1$) and obtain, that (5.13) holds.

4) If $r = 2 - \frac{1}{n+2}$, $s = \frac{7n+10}{4(n+2)}$. It follows that

$$2s - r - 1 = \frac{n}{2(n+2)} = \begin{cases} > \frac{1}{n+2} & \text{if} \quad n > 2, \quad (A_1)) \\ \leq \frac{1}{n+2} & \text{if} \quad n \leq 2 \text{ (i.e. } n = 2) \quad (A_2)). \end{cases}$$

Thus, if $n > 2$, we have proven once again (5.13).

If $n = 2$, we can take $q \in (2s - r - 1, \frac{1}{n+2}) = \left(\frac{1}{8}, \frac{1}{4}\right)$ (case $A_2$)) and obtain (5.14).

The lemma is proved.

**Type V: C** $> 0$, $K < 0$. In this case, and according to (4.14) through (4.16) we have for the direct function $y = g(x)$, with $g : (-\infty, x_2) \to R$, the three characterizing conditions described next

$$\lim_{x \to x_2} g(x) = +\infty\,, \tag{5.21}$$

$$\lim_{x \to -\infty} g(x) = 0\,, \tag{5.22}$$

$$\frac{dg}{dx} > 0, \ \frac{d^2g}{dx^2} > 0, \ \forall x \in (-\infty, x_2). \tag{5.23}$$

**Lemma 5.3.** *The first integral of $g(x)$, $G : (-\infty, x_2) \to R$, satisfies the following conditions:*

$$\lim_{x \to x_2} G(x) = +\infty\,, \tag{5.24}$$

$$G'(x) = g(x) > 0, \ G''(x) = g'(x) > 0, \forall x \in (-\infty, x_2)\,, \tag{5.25}$$

$$\lim_{x \to -\infty} G(x) = G_{-\infty} \in R. \tag{5.26}$$

*Proof.* Let us choose a point $(\tilde{x}_0, \tilde{y}_0)$, with $\tilde{x}_0 < x_2$. Then, by (5.1), a first integral of $g$ is written

$$G(x) = \tilde{y}_0 + \frac{1}{C^{1/2}} \int_{\tilde{x}_0}^{x} \frac{g'(t)\,dt}{[g(t)]^{\frac{r-1}{2}} \left([g(t)]^{2s-r-1} + b\right)^{1/2}}, \tag{5.27}$$

with $b = -(2K)/[(2s - r - 1)C] > 0$.

Next, we integrate again by parts, in order to obtain that

$$\begin{aligned} G(x) = \tilde{y}_0 + \frac{1}{C^{1/2}} \left( \frac{2}{3-r} \frac{[g(t)]^{\frac{3-r}{2}}}{\left([g(t)]^{2s-r-1} + b\right)^{1/2}} \right)_{\tilde{x}_0}^{x} + \\ + \frac{1}{C^{1/2}} \frac{2s-r-1}{3-r} \int_{\tilde{x}_0}^{x} \frac{[g(t)]^{2s-\frac{3r+1}{2}} g'((t)\,dt}{\left([g(t)]^{2s-r-1} + b\right)^{3/2}}. \end{aligned} \tag{5.28}$$

Thus, by observing that the term containing the integral is greater than zero while the other diverges to $+\infty$ as $x$ converges to $x_2$, it follows (5.24). It is also immediate to check (5.25).

Now, if we fix a real number $q$, with $0 < q < 2s - r - 1$, we can find a real number $t_2 \in (-\infty, x_2)$ such that $[g(t)]^q < [g(t)]^{2s-r-1} + b$, for every $t < t_2$. Hence, the integral in the last member of (5.27) can be estimated. In fact, for $x < t_2$ we have

$$\begin{aligned} 0 < \int_{x}^{t_2} \frac{g'(t)\,dt}{[g(t)]^{\frac{r-1}{2}} \left([g(t)]^{2s-r-1} + b\right)^{1/2}} < \int_{x}^{t_2} \frac{g'(t)\,dt}{[g(t)]^{\frac{r-1+q}{2}}} = \\ = \frac{2}{3-r-q} \left([g(t)]^{(3-r-q)/2}\right)_{x}^{t_2}, \end{aligned} \tag{5.29}$$

and since this last has a positive limit as $x \to -\infty$, it follows that (5.26) holds.

**Lemma 5.4.** *The second integral of $g(x)$, $F : (-\infty, x_2) \to R$, satisfies the following conditions:*

$$\begin{aligned} \frac{dF}{dx} &= G(x) \ \textit{ranges from} \ G_{-\infty} \ \textit{to} \ +\infty, \\ \frac{d^2F}{dx^2} &= \frac{dG}{dx} = g(x) > 0. \end{aligned} \tag{5.30}$$

*For $r = 2$, we have that*

$$\lim_{x \to -\infty} F(x) = \begin{cases} +\infty, & \textit{if} \quad G_{-\infty} < 0, \\ -\infty, & \textit{if} \quad G_{-\infty} \geq 0; \end{cases} \tag{5.31}$$

*and for* $r = 2 - \frac{1}{n+2}$*, on the other hand, it holds*

$$\lim_{x \to -\infty} F(x) = F_{-\infty} \in R. \tag{5.32}$$

*The other limit behaves as follows: in geometrical cases* 1), 2), 3) *(recall that here it holds* $n > 2$*), and* 4) *with* $n > 2$*, we have that*

$$\lim_{x \to x_2} F(x) = F_2 \in R; \tag{5.33}$$

*while in case* 4) *with* $n = 2$*,*

$$\lim_{x \to x_2} F(x) = +\infty. \tag{5.34}$$

*Proof.* We proceed to integrate the function $G$, by choosing a point $(\hat{x}_0, \hat{y}_0)$ with $\hat{x}_0 < x_2$, and define $F : (-\infty, x_2) \to R$, by

$$F(x) = \hat{y}_0 + \int_{\hat{x}_0}^{x} G(t)\, dt. \tag{5.35}$$

It is immediate to verify (5.30).

On the other hand, it is fairly obvious to get conclusions as to behaviour of $F$ near $-\infty$ in the cases $G_{-\infty} > 0$, $G_{-\infty} < 0$. In order to see what happens in the remaining, limiting case of $G_{-\infty} = 0$, we observe, by using (5.22), that there exists a real number $t_0 \in (-\infty, x_2)$ such that $[g(t)]^{2s-r-1} + b < 2b$, for every $t < t_0$. Hence, for $x < x_0 < t_0$, and by also using (5.1), we can write the estimate

$$\int_x^{x_0} \frac{[g(t)]^{\frac{3-r}{2}}\, dt}{\left([g(t)]^{2s-r-1} + b\right)^{1/2}} > (2b)^{-1/2} \int_x^{x_0} [g(t)]^{\frac{3-r}{2}}\, dt =$$

$$= \frac{(2b)^{-1/2}}{C^{1/2}} \int_x^{x_0} \frac{[g(t)]^{1-r} g'(t)\, dt}{\left([g(t)]^{2s-r-1} + b\right)^{1/2}} > \frac{(2b)^{-1}}{C^{1/2}} \int_x^{x_0} [g(t)]^{1-r} g'(t)\, dt =$$

$$= \begin{cases} [(2b)^{-1}]/(C^{1/2})\,(\log g(x_0) - \log g(x))\,, & \text{if} \quad r = 2, \\ [(2b)^{-1}(n+2)]/(C^{1/2}) \left([g(x_0)^{\frac{1}{n+2}} - [g(x)]^{\frac{1}{n+2}}\right), & \text{if} \quad r = 2 - \frac{1}{n+2}. \end{cases}$$

Thus, for $r = 2$, the right-hand side diverges to $+\infty$ when $x \to -\infty$ and, then, (5.31) holds. On the other hand, for $r = 2 - \frac{1}{n+2}$, there is no conclusion from the above equation. However, in this situation we can write

$$G(x) = \int_{-\infty}^{x} g(t)\, dt, \tag{5.36}$$

and it follows, from (5.1) and (5.29), that $0 < G(x) < \frac{2}{3-r-q}\frac{1}{C^{1/2}}[g(x)]^{\frac{3-r-q}{2}}$, which implies the estimate

$$\begin{aligned} 0 &< \int_x^{t_2} G(t)\,dt < \frac{2}{3-r-q}\frac{1}{C^{1/2}}\int_x^{t_2}[g(t)]^{\frac{3-r-q}{2}}\,dt = \\ &= \frac{2}{3-r-q}\frac{1}{C}\int_x^{t_2}\frac{g'(t)\,dt}{[g(t)]^{r-1+q}\left([g(t)]^{2s-r-1}+b\right)^{1/2}} < \\ &< \frac{2}{3-r-q}\frac{1}{C}\int_x^{t_2}\frac{g'(t)\,dt}{[g(t)]^{r-1+q}} = \frac{2}{3-r-q}\frac{1}{C}\frac{1}{2-r-q}\left([g(t)]^{2-r-q}\right)_x^{t_2}. \end{aligned}$$

But, $2-r-q = \frac{1}{n+2}$, and since we can take $q$ such that $0 < q < \frac{1}{n+2}$, (5.32) follows.

To finish the analysis of the function $F$ for this case it remains only to study its behaviour near $x_2$. For this purpose we consider first the integral in the last member of (5.27). From there, since $b > 0$, we get for $x > x_0$ the inequality

$$\begin{aligned} 0 &< \int_{\tilde{x}_0}^{x}\frac{g'(t)\,dt}{[g(t)]^{(r-1)/2}\left([g(t)]^{2s-r-1}+b\right)^{1/2}} < \\ &< \int_{\tilde{x}_0}^{x}\frac{g'(t)\,dt}{[g(t)]^{s-1}} = \frac{1}{2-s}\left([g(x)]^{2-s}-[g(\tilde{x}_0)]^{2-s}\right). \end{aligned}$$

We estimate, next, the integral of the variable part in this last member, by also using again (5.1), to obtain

$$\begin{aligned} &\int_{\tilde{x}}^{x}[g(t)]^{2-s}\,dt = \int_{\tilde{x}_0}^{x}\frac{[g(t)]^{2-s-\frac{r+1}{2}}g'(t)\,dt}{\left(C[g(t)]^{2s-r-1}-\frac{2K}{2s-r-1}\right)^{1/2}} < \\ &< \frac{1}{C^{1/2}}\int_{x_0}^{x}[g(t)]^{2-2s}g'(t)\,dt = \frac{1}{C^{1/2}}\frac{1}{3-2s}\left([g(x)]^{3-2s}-[g(x_0)]^{3-2s}\right). \end{aligned}$$

Now, for $s = \frac{2n+3}{n+2}$ we have $3-2s = -\frac{n}{n+2} < 0$, and for $s = \frac{7n+10}{4(n+2)}$, $3-2s = \frac{-n+2}{2(n+2)}$, which is $< 0$ if, and only if $n \geq 3$.

Thus, in geometrical cases 1), 2), 3) with $n > 2$, and 4) with $n > 2$, (5.33) holds.

We consider next subcase 4) with $n = 2$ and write

$$G(x) = \tilde{y}_0 + \frac{1}{C^{1/2}}\int_{\tilde{x}_0}^{x}\frac{g'(t)\,dt}{[g(t)]^{3/8}\left([g(t)]^{1/8}+b\right)^{1/2}}.$$

There exists $t_3 < x_2$ such that, for every $t \in (t_3, x_2)$, $[g(t)]^{1/4} > [g(t)]^{1/8} + b$, hence

$$\int_{t_3}^{x} \frac{g'(t)\,dt}{[g(t)]^{3/8}\left([g(t)]^{1/8}+b\right)^{1/2}} > \int_{t_3}^{x} \frac{g'(t)\,dt}{[g(t)]^{1/2}} = 2\left([g(x)]^{1/2} - [g(t_3)]^{1/2}\right).$$

Now, by integrating again we obtain

$$\int_{t_3}^{x} [g(t)]^{1/2}\,dt = \frac{1}{C^{1/2}} \int_{t_3}^{x} \frac{g'(t)\,dt}{[g(t)]^{7/8}\left([g(t)]^{1/8}+b\right)^{1/2}} >$$

$$> \frac{1}{C^{1/2}} \int_{t_3}^{x} \frac{g'(t)\,dt}{[g(t)]} = \frac{1}{C^{1/2}} \left(\log g(x) - \log g(t_3)\right),$$

and from this we get (5.34) checked.

**Type VI: C** $< 0$, $K < 0$. To trat this last type we make use of (4.18) through (4.21) and obtain that the direct function $y = g(x)$, $g : (-\infty, x_3) \to R$, has the following properties

(5.37) $$\lim_{x \to -\infty} g(x) = 0,$$

(5.38) $$\lim_{x \to x_3} g(x) = \overline{y}_0 := \left(\frac{2K}{(2s-r-1)C}\right)^{1/(2s-r-1)},$$

(5.39) $$\frac{dg}{dx} > 0, \text{ for every } x \in (-\infty, x_3),$$

(5.40) $$\frac{d^2g}{dx^2} > 0, \text{ in } (-\infty, x_0^{\#});\ \frac{d^2g}{dx^2}(x_0^{\#}) = 0;\ \frac{d^2g}{dx^2} < 0, \text{ in } (x_0^{\#}, x_3),$$

where $x_0^{\#} = g^{-1}(y_0^{\#})$ and $y_0^{\#} = \left(\frac{(r+1)K}{(2s-r-1)sC}\right)^{1/(2s-r-1)}$.

**Lemma 5.5.** *The first integral of* $g(x)$, $G : (-\infty, x_3) \to R$, *satisfies the following conditions:*

(5.41) $$\lim_{x \to -\infty} G(x) = G_{-\infty} \in R,$$

(5.42) $$\lim_{x \to x_3} G(x) = G_3 \in R,$$

(5.43) $$\frac{dG}{dx} = g(x) > 0,\ \frac{d^2G}{dx^2} = \frac{dg}{dx} > 0.$$

*Proof.* In order to integrate $g$, in the present case, we choose a point $(\tilde{x}_0, \tilde{y}_0)$, with $\tilde{x}_0 < x_3$, and use again (5.1) in order to define $a =: \frac{2K}{(2s-r-1)C} > 0$, and $G : (-\infty, x_3) \to R$, by

$$
\begin{aligned}
G(x) &:= \tilde{y}_0 + \int_{\tilde{x}_0}^{x} g(t)\,dt \\
&= \tilde{y}_0 + \frac{1}{(-C)^{1/2}} \int_{\tilde{x}_0}^{x} \frac{g'(t)\,dt}{[g(t)]^{(r-1)/2}\left(a - [g(t)]^{2s-r-1}\right)^{1/2}}\,.
\end{aligned} \tag{5.44}
$$

Now, there exists some $t_3 < x_3$ such that $a/2 < a - [g(t)]^{2s-r-1}$, for every $t < t_3$. Hence, we can write, for $x < t_3$, the estimate

$$
\begin{aligned}
0 &< \int_{x}^{t_3} \frac{g'(t)\,dt}{[g(t)]^{\frac{r-1}{2}}\left(a - [g(t)]^{2s-r-1}\right)^{1/2}} < \left(\frac{2}{a}\right)^{1/2} \int_{x}^{t_3} \frac{g'(t)\,dt}{[g(t)]^{\frac{r-1}{2}}} = \\
&= \left(\frac{2}{a}\right)^{1/2} \frac{2}{3-r}\left([g(t_3)]^{\frac{3-r}{2}} - [g(x)]^{\frac{3-r}{2}}\right),
\end{aligned} \tag{5.45}
$$

and since the right-hand side has a finite limit as $x \to -\infty$, (5.41) is proved.

On the other hand, it is obvious that (5.42) follows from (5.38), and also that (5.43) is a consequence of equations (5.37) through (5.39).

**Lemma 5.6.** *The second integral of $g(x)$, $F : (-\infty, x_3) \to R$, satisfies the following conditions:*

$$\lim_{x \to x_3} F(x) = F_3 \in R, \tag{5.46}$$

$$
\begin{aligned}
&\frac{df}{dx} = G(x) \textit{ ranges from } G_{-\infty} \quad \textit{to} \quad G_3, \ \textit{while} \\
&\frac{d^2F}{dx^2} = \frac{dG}{dx} > 0.
\end{aligned} \tag{5.47}
$$

*For $r = 2$, the other limit is given by*

$$\lim_{x \to -\infty} F(x) = \begin{cases} +\infty, & \textit{if} \quad G_{-\infty} < 0, \\ -\infty, & \textit{if} \quad G_{-\infty} \geq 0; \end{cases} \tag{5.48}$$

*while for $r = 2 - \frac{1}{n+2}$ it holds*

$$\lim_{x \to -\infty} F(x) = \begin{cases} +\infty, & \textit{if} \quad G_{-\infty} < 0, \\ F_{-\infty} \in R, & \textit{if} \quad G_{-\infty} = 0, \\ -\infty, & \textit{if} \quad G_{-\infty} > 0. \end{cases} \tag{5.49}$$

*Proof.* We integrate once again by choosing $(\hat{x}_0, \hat{y}_0)$, with $\hat{x} < x_3$, and define

$$F(x) = \hat{y}_0 + \int_{\hat{x}_0}^{x} G(t)\,dt. \tag{5.50}$$

It is obvious that (5.46) follows from (5.42), while (5.43) furnishes (5.47).

We analyze the limit case where $G_{-\infty} = 0$. Here the function $G$ can be expressed as

$$G(x) = \int_{-\infty}^{x} g(t)\,dt. \tag{5.51}$$

Hence, by using again (5.1), we can further write

$$G(x) = \frac{1}{(-C)^{1/2}} \int_{-\infty}^{x} \frac{g'(t)\,dt}{[g(t)]^{(r-1)/2}\left(a - [g(t)]^{2s-r-1}\right)^{1/2}}, \tag{5.52}$$

and, from the inequality $a - [g(t)]^{2s-r-1} < a$, derive the estimate

$$\begin{aligned} G(x) > \left(-\frac{1}{aC}\right)^{1/2} \int_{-\infty}^{x} [g(t)]^{\frac{1-r}{2}} g'(t)\,dt = \\ = \left(-\frac{1}{aC}\right)^{1/2} \frac{2}{3-r} [g(x)]^{\frac{3-r}{2}}. \end{aligned} \tag{5.53}$$

Hence we get, for $x < \hat{x}_0$,

$$\begin{aligned} \int_{x}^{\hat{x}_0} G(t)\,dt &> \left(-\frac{1}{aC}\right)^{1/2} \frac{2}{3-r} \int_{x}^{\hat{x}_0} [g(t)]^{\frac{3-r}{2}}\,dt = \\ &= \left(-\frac{1}{aC}\right)^{1/2} \frac{2}{3-r} \left(-\frac{1}{C}\right)^{1/2} \int_{x}^{\hat{x}_0} \frac{[g(t)]^{1-r} g'(t)\,dt}{\left(a - [g(t)]^{2s-r-1}\right)^{1/2}} > \\ &> -\frac{1}{aC}\frac{2}{3-r} \int_{x}^{\hat{x}_0} \frac{g'(t)\,dt}{[g(t)]^{r-1}} = \\ &= \begin{cases} -\dfrac{1}{aC}\dfrac{2}{3-r}\left(\log g(\hat{x}_0) - \log g(x)\right), & \text{if } \ r = 2, \\ -\dfrac{(n+2)}{aC}\dfrac{2}{3-r}\left([g(\hat{x}_0)]^{\frac{1}{n+2}} - [g(x)]^{\frac{1}{n+2}}\right), & \text{if } \ r = 2 - \dfrac{1}{n+2}. \end{cases} \end{aligned} \tag{5.54}$$

Then, for $r = 2$, since the last right-hand member diverges to $+\infty$ as $x$ diverges to $-\infty$, it follows that $F(x)$ itself diverges to $-\infty$ in that case, and this proves (5.48).

We also observe that there is no conclusion from the above in the case where $r = 2 - \frac{1}{n+2}$. Thus, for the latter case, we use inequality (5.45) in order to write, for $x < t_3$, the estimate

$$G(x) = \int_{-\infty}^{x} g(t)\,dt < \frac{1}{(-C)^{1/2}} \left(\frac{2}{a}\right)^{1/2} \frac{2}{3-r} \big[g(x)\big]^{\frac{3-r}{2}},$$

and from this further obtain that

$$\begin{aligned}
\int_x^{t_3} G(t)\,dt &< \left(-\frac{2}{aC}\right)^{1/2} \frac{2}{3-r} \int_x^{t_3} [g(t)]^{\frac{3-r}{2}}\,dt = \\
&= -\frac{1}{C}\left(\frac{2}{a}\right)^{1/2} \frac{2}{3-r} \int_x^{t_3} \frac{[g(t)]^{1-r} g'(t)\,dt}{\left(a - [g(t)]^{2s-r-1}\right)^{1/2}} < \\
&< -\frac{1}{C}\frac{2}{a}\frac{2}{3-r} \int_x^{t_3} [g(t)]^{1-r} g'(t)\,dt = \\
&= -\frac{n+2}{C}\frac{2}{a}\frac{2}{3-r} \left([g(t_3)]^{1/(n+2)} - [g(x)]^{1/(n+2)}\right).
\end{aligned}$$

This proves (5.49), and finishes the analysis of the function $F$ in this case and, therefore, in all of possible cases.

## 6. Unimodular Affine Mean Curvature.

We proceed to use the qualitative analysis developed in the two previous sections in order to accomplish one of our goals: the local and global classification of those hypersurfaces of decomposable type whose unimodular affine mean curvature is constant:

**Theorem 6.1.** *Let $X : M^n \to E^{n+1}$ be a nondegenerate hypersurface of decomposable type, with vanishing unimodular affine mean curvature (maximal-minimal hypersurface if $X(M)$ is, besides, locally strongly convex). Then, each of its components $f^1, f^2, \ldots, f^n$, must be of one of the* original types I through VI*, whose properties are enumerated below, or the corresponding three more kinds of types than are obtained from those original types by suitable reflections in the $x$- and $y$-axis. All of the solutions belonging to the original types share the common feature that their second derivatives, $(f^k)'' = y > 0$, satisfy, in each case, the classifying, non-linear, ordinary differential equation*

$$y' = y^{3/2}\left(C y^{n/(n+2)} - \frac{2K}{n}\right)^{1/2}.$$

**Type I:** $\mathbf{C}=0, K=0$*:* $f^k : R \to R$*, given by* $f^k(t^k) = (t^k)^2$*, (parabolic type).*

**Type II:** $\mathbf{C}=0, K<0$*:* $f^k : (0,+\infty) \to R$*, given by* $f^k(t^k) = -\log(t^k)$*, (logarithmic type).*

**Type III:** $\mathbf{C}>0, K=0$*:* $f^k : (-\infty,0) \to R$*, defined by* $f^k(t^k) = -(-t^k)^{n/(n+1)}$.

**Type IV:** $\mathbf{C}>0, K>0$*:* $f^k$ *defined on a finite open interval, i.e.* $f_k : \left(\bar{t}_1^k, t_1^k\right) \to R$*, such that*

$$\lim_{t^k\to\bar{t}_1^k} f^k(t^k) = \bar{f}_1^k \in R,\ \lim_{t^k\to t_1^k} f^k(t^k) = f_1^k \in R,$$

$$\lim_{t^k\to\bar{t}_1^k} (f^k)'(t^k) = (\bar{f}_1^k)' \in R,\ \lim_{t^k\to t_1^k} (f^k)'(t^k) = +\infty,$$

$$\lim_{t^k\to\bar{t}_1^k} (f^k)''(t^k) = \left(\frac{2K}{nC}\right)^{(n+2)/n},\ \lim_{t^k\to t_1^k} (f^k)''(t^k) = +\infty,$$

$$(f^k)'''(t^k) > 0, (f^k)^{iv}(t^k) > 0,\ \textit{for every}\ t^k \in (\bar{t}_1^k, t_1^k).$$

**Type V:** $\mathbf{C}>0, K<0$*:* $f^k$ *defined on a semi-infinite interval,* $f^k : \left(-\infty, t_2^k\right) \to R$*, such that*

$$\lim_{t^k\to-\infty} f^k(t^k) = \begin{cases} +\infty, & \textit{if}\ \ (f^k)'_{-\infty} < 0,\\ -\infty, & \textit{if}\ \ (f^k)'_{-\infty} \geq 0,\end{cases} \qquad \lim_{t^k\to t_2^k} f^k(t^k) = f_2^k \in R,$$

$$\lim_{t^k\to-\infty} (f^k)'(t^k) = (f^k)'_{-\infty} \in R, \quad \lim_{t^k\to t_2^k} (f_k)'(t^k) = +\infty,$$

$$\lim_{t^k\to-\infty} (f^k)''(t^k) = 0, \qquad \lim_{t^k\to t_2^k} (f^k)''(t^k) = +\infty,$$

$$(f^k)'''(t^k) > 0, (f^k)^{iv}(t^k) > 0,\ \textit{for every}\ t^k \in (-\infty, t_2^k).$$

**Type VI:** $\mathbf{C}<0, K<0$*:* $f^k : (-\infty, t_3^k) \to R$*, with the following properties*

$$\lim_{t^k\to-\infty} f^k(t^k) = \begin{cases} +\infty, & \textit{if}\ \ (f^k)'_{-\infty} < 0,\\ -\infty & \textit{if}\ \ (f^k)'_{-\infty} \geq 0,\end{cases} \qquad \lim_{t^k\to t_3^k} f^k(t^k) = f_3^k \in R,$$

$$\lim_{t^k\to-\infty} (f^k)'(t^k) = (f^k)'_{-\infty} \in R, \qquad \lim_{t^k\to t_3^k} (f^k)'(t^k) = (f_3^k)' \in R,$$

$$\lim_{t^k\to-\infty}(f^k)''(t^k)=0,\qquad \lim_{t^k\to t_3^k}(f^k)''(t^k)=(f_3^k)'':=\left(\frac{2(n+2)K}{nC}\right)^{(n+2)/n},$$

$$(f^k)'''(t^k)>0,\ \textit{for every}\ \ t^k\in(-\infty,t_3^k),$$

$$(f^k)^{iv}(t^k)>0\quad \textit{for}\quad t^k\in(-\infty,t_0^k),$$

$$(f^k)^{iv}(t_0^k)=0,$$

$$(f^k)^{iv}(t^k)<0\quad \textit{for}\quad t^k\in(t_0^k,t_3^k),$$

*where* $t_0^k=\left((f^k)''\right)^{-1}\left([3(n+2)^2K]/[n(2n+3)C]\right)^{(n+2)/n}$.

*Proof.* This follows directly from Lemma 1.1 and the analysis practiced in the two previous sections, as applied to equation (1.11), *geometrical case* 1), i.e. $r=2$, $s=\frac{2n+3}{n+2}$; and in particular by equations numbered (4.1) for Type I; (4.2) for Type II; (4.5) for Type III; (5.2), (5.3), (5.4), Lemmas 5.1 and 5.2, for Type IV; (5.21), (5.22), (5.23), Lemmas 5.3 and 5.4 for Type V; and (5.37), (5.38), (5.39), (5.40), Lemmas 5.5 and 5.6 for Type VI.

**Theorem 6.2.** *Let* $X:M^n\to E^{n+1}$ *be a nondegenerate hypersurface of decomposable type, with constant, nonvanishing unimodular affine mean curvature. Then,* $n-1$ *of its components must be of parabolic type; the remaining one, labeled* $f^{k_0}$, *with* $1\le k_0\le n$, *must be of one of the original Types II, IV, V, or VI, whose properties are enumerated below, or the corresponding three more kinds of types that are obtained from those original types by suitable reflections in the* $x-$ *and* $y$*-axis. All of the solutions belonging to the original types share the common feature that their second derivatives,* $(f^k)''=y>0$, *satisfy, in each case, the classifying, non-linear, ordinary differential equation*

$$y'=y^{\frac{3n+5}{2(n+2)}}\left(Cy^{\frac{n+1}{n+2}}-\frac{2(n+2)K}{n+1}\right)^{1/2}.$$

**Type II:** $\mathbf{C}=0$, $K<0$*:* $f^k:(0,+\infty)\to R$, *given by* $f^k(t^k)=(t^k)^{-\frac{2}{n+1}}$.

**Type IV:** $\mathbf{C}>0$, $K>0$*:* $f^k$ *defined on a finite open interval, i.e.* $f_k:(\bar t_1^k,t_1^k)\to R$, *such that*

$$\lim_{t^k\to\bar t_1^k}f^k(t^k)=\bar f_1^k\in R,\ \lim_{t^k\to t_1^k}f^k(t^k)=f_1^k\in R,$$

$$\lim_{t^k\to\bar t_1^k}(f^k)'(t^k)=(\bar f_1^k)'\in R,\ \lim_{t^k\to t_1^k}(f^k)'(t^k)=+\infty,$$

$$\lim_{t^k\to\bar t_1^k}(f^k)''(t^k)=\left(\frac{2(n+2)K}{(n+1)C}\right)^{\frac{n+2}{n+1}},\ \lim_{t^k\to t_1^k}(f^k)''(t^k)=+\infty,$$

$$(f^k)'''(t^k) > 0, (f^k)^{iv}(t^k) > 0, \text{ for every } t^k \in \left(\bar{t}_1^k, t_1^k\right).$$

**Type V : C** $> 0, K < 0$: *$f^k$ defined on a semi-infinite interval, $f^k$ : $(-\infty, t_2^k) \to R$, such that*

$$\lim_{t^k \to -\infty} f^k(t^k) = f_{-\infty}^k \qquad \lim_{t^k \to t_2^k} f^k(t^k) = f_2^k \in R,$$

$$\lim_{t^k \to -\infty} (f^k)'(t^k) = (f^k)'_{-\infty} \in R, \qquad \lim_{t^k \to t_2^k} (f_k)'(t^k) = +\infty,$$

$$\lim_{t^k \to -\infty} (f^k)''(t^k) = 0, \qquad \lim_{t^k \to t_2^k} (f^k)''(t^k) = +\infty,$$

$$(f^k)'''(t^k) > 0, (f^k)^{iv}(t^k) > 0, \text{ for every } t^k \in (-\infty, t_2^k).$$

**Type VI: C** $< 0$, $K < 0$: *$f^k : (-\infty, t_3^k) \to R$, with the following properties*

$$\lim_{t^k \to -\infty} f^k(t^k) = \begin{cases} +\infty, & \text{if } (f^k)'_{-\infty} < 0, \\ \bar{f}_{-\infty}^k \in R, & \text{if } (f^k)'_{-\infty} = 0, \\ -\infty, & \text{if } (f^k)'_{-\infty} > 0, \end{cases} \qquad \lim_{t^k \to t_3^k} f^k(t^k) = f_3^k \in R,$$

$$\lim_{t^k \to -\infty} (f^k)'(t^k) = (f^k)'_{\infty} \in R, \qquad \lim_{t^k \to t_3^k} (f^k)'(t^k) = (f_3^k)' \in R,$$

$$\lim_{t^k \to -\infty} (f^k)''(t^k) = 0, \qquad \lim_{t^k \to t_3^k} (f^k)''(t^k) = (f_3^k)'' := \left(\frac{2(n+2)K}{(n+1)C}\right)^{\frac{n+2}{n+1}},$$

$$(f^k)'''(t^k) > 0, \text{ for every } t^k \in (-\infty, t_3^k),$$
$$(f^k)^{iv}(t^k) > 0 \quad \text{for} \quad t^k \in (-\infty, t_0^k),$$
$$(f^k)^{iv}(t_0^k) = 0,$$
$$(f^k)^{iv}(t^k) < 0 \quad \text{for} \quad t^k \in (t_0^k, t_3^k),$$

*where* $t_0^k = \left((f^k)''\right)^{-1} ([(n+2)(3n+5)K]/[(n+1)(2n+3)C])^{(n+2)/(n+1)}$.

*Proof.* The properties described follow directly from Lemma 1.2 and the analysis practiced in the previous section, as applied to equation (1.12), *geometrical case* 2), i.e. $r = 2 - \frac{1}{n+2}$, $s = \frac{2n+3}{n+2}$. Observe, too, that the distinguished component can not be of either Type I or III, since the constant in equation (1.12) must be nonvanishing. Thus, excluded those two cases from the analysis, the rest of the argument is a consequence of equations numbered (4.3), for Type II; (5.2), (5.3), (5.4), Lemmas 6.1 and 6.2 for Type IV; (6.21), (6.22), (6.23), Lemmas 6.3 and 6.4 for Type V; and (6.37), (6.38), (6.39), (6.40), Lemmas 6.5 and 6.6 for Type VI.

## 7. Unimodular Affine Scalar Curvature.

**Theorem 7.1.** *Let $X : M^n \to E^{n+1}$ be a nondegenerate hypersurface of decomposable type, with dimension strictly greater than two, $n > 2$, and with vanishing unimodular affine scalar curvature. Then, each of its components $f^1, f^2, \ldots, f^n$, must be of one of the original Types I through VI, whose properties are enumerated below, or the corresponding three more kinds of types that are obtained from those original types by suitable reflections in the $x$- and $y$-axis. All of the solutions belonging to the original types share the common feature that their second derivatives, $(f^k)'' = y > 0$, satisfy, in each case, the classifying, non-linear, ordinary differential equation*

$$y' = y^{3/2}\left(Cy^{\frac{n-2}{2(n+2)}} - \frac{4(n+2)K}{n-2}\right)^{1/2}.$$

**Type I: C** $= 0$, $K = 0$: *$f^k : R \to R$, given by $f^k(t^k) = (t^k)^2$, (parabolic type).*

**Type II: C** $= 0$, $K < 0$: *$f^k : (0, +\infty) \to R$, given by $f^k(t^k) = -\log(t^k)$, (logarithmic type).*

**Type III: C** $> 0$, $K = 0$: *$f^k : (-\infty, 0) \to R$, defined by $f^k(t^k) = -(-t^k)^{2(n-2)/(3n+2)}$.*

**Type IV: C** $> 0$, $K > 0$: *$f^k$ defined on a finite open interval, i.e. $f_k : (\bar{t}^k_1, t^k_1) \to R$, such that*

$$\lim_{t^k \to \bar{t}^k_1} f^k(t^k) = \bar{f}^k_1 \in R, \qquad \lim_{t^k \to t^k_1} f^k(t^k) = f^k_1 \in R,$$

$$\lim_{t^k \to \bar{t}^k_1} (f^k)'(t^k) = (\bar{f}^k_1)' \in R, \qquad \lim_{t^k \to t^k_1} (f^k)'(t^k) = +\infty,$$

$$\lim_{t^k \to \bar{t}^k_1} (f^k)''(t^k) = \left(\frac{4(n+2)K}{(n-2)C}\right)^{[2(n+2)]/(n-2)}, \quad \lim_{t^k \to t^k_1} (f^k)''(t^k) = +\infty,$$

$$(f^k)'''(t^k) > 0, (f^k)^{iv}(t^k) > 0, \ \ \textit{for every} \ \ t^k \in (\bar{t}^k_1, t^k_1).$$

**Type V: C** $> 0$, $K < 0$: *$f^k$ defined on a semi-infinite interval, $f^k : (-\infty, t^k_2) \to R$, such that*

$$\lim_{t^k \to -\infty} f^k(t^k) = \begin{cases} +\infty, & \textit{if} \quad (f^k)'_{-\infty} < 0, \\ -\infty, & \textit{if} \quad (f^k)'_{-\infty} \geq 0, \end{cases} \qquad \lim_{t^k \to t^k_2} f^k(t^k) = f^k_2 \in R,$$

$$\lim_{t^k \to -\infty} (f^k)'(t^k) = (f^k)'_{-\infty} \in R, \quad \lim_{t^k \to t_2^k} (f_k)'(t^k) = +\infty,$$

$$\lim_{t^k \to -\infty} (f^k)''(t^k) = 0, \qquad \lim_{t^k \to t_2^k} (f^k)''(t^k) = +\infty,$$

$$(f^k)'''(t^k) > 0, (f^k)^{iv}(t^k) > 0, \ \textit{for every} \ t^k \in (-\infty, t_2^k).$$

**Type VI: C** $< 0$, $K < 0$*:* $f^k : (-\infty, t_3^k) \to R$*, with the following properties*

$$\lim_{t^k \to -\infty} f^k(t^k) = \begin{cases} +\infty, & \textit{if} \quad (f^k)'_{-\infty} < 0, \\ -\infty, & \textit{if} \quad (f^k)'_{-\infty} \geq 0, \end{cases} \qquad \lim_{t^k \to t_3^k} f^k(t^k) = f_3^k \in R,$$

$$\lim_{t^k \to -\infty} (f^k)'(t^k) = (f^k)'_{-\infty} \in R, \qquad \lim_{t^k \to t_3^k} (f^k)'(t^k) = (f_3^k)' \in R,$$

$$\lim_{t^k \to -\infty} (f^k)''(t^k) = 0, \qquad \lim_{t^k \to t_3^k} (f^k)''(t^k) = (f_3^k)'' := \left( \frac{4(n+2)K}{(n-2)C} \right)^{\frac{2(n+2)}{n-2}},$$

$$(f^k)'''(t^k) > 0, \ \textit{for every} \ t^k \in (-\infty, t_3^k),$$

$$(f^k)^{iv}(t^k) > 0 \quad \textit{for} \quad t^k \in (-\infty, t_0^k),$$

$$(f^k)^{iv}(t_0^k) = 0,$$

$$(f^k)^{iv}(t^k) < 0 \quad \textit{for} \quad t^k \in (t_0^k, t_3^k),$$

*where* $t_0^k = \left((f^k)''\right)^{-1} \left([24(n+2)^2 K]/[(n-2)(7n+10)C]\right)^{[2(n+2)]/(n-2)}$.

*Proof.* The present theorem follows from Lemma 1.3 and the analysis performed in 4 and 5, as applied to equation (1.13), *geometrical case* 3), i.e. $r = 2$, $s = \frac{7n+10}{4(n+2)}$, with $n > 2$. In particular, the equations that apply here are the ones numbered: (5.1) for Type I; (5.2) for Type II; (5.5) for Type III; (5.2), (5.3), (5.4), Lemmas 5.1 and 5.2 for Type IV; (5.21), (5.22), (5.23), Lemmas 5.3 and 5.4 for Type V; and (5.37), (5.38), (5.39), (5.40), Lemmas 5.5 and 5.6 for Type VI.

**Theorem 7.2.** *Let* $X : M^n \to E^{n+1}$ *be a nondegenerate hypersurface of decomposable type, with constant, nonvanishing unimodular affine scalar curvature. Then,* $n - 1$ *of its components must be of parabolic type; the remaining one, labeled* $f^{k_0}$*, with* $1 \leq k_0 \leq n$*, must be of one the original Types II, IV, V, or VI, whose properties are enumerated below, or the corresponding three more kinds of types that are obtained from those original types by suitable reflections in the* $x-$ *and* $y$*-axis. All of the solutions belonging to the original types share* *the common feature that their second derivatives,* $(f^k)'' = y > 0$, *satisfy, in each case, the classifying, non-linear, ordinary differential equation*

$$y' = y^{\frac{3n+5}{2(n+2)}} \left( C y^{\frac{n}{2(n+2)}} - \frac{4(n+2)K}{n} \right)^{1/2}.$$

**Type II:** $\mathbf{C} = 0,\ K < 0$: $f^k : (0, +\infty) \to R$, *given by* $f^k(t^k) = (t^k)^{-\frac{2}{n+1}}$.

**Type IV:** $\mathbf{C} > 0,\ K > 0$: $f^k$ *defined on a finite open interval, i.e.* $f_k : (\bar{t}_1^k, t_1^k) \to R$, *such that*

$$\lim_{t^k \to \bar{t}_1^k} f^k(t^k) = \bar{f}_1^k \in R, \qquad \lim_{t^k \to t_1^k} f^k(t^k) = \begin{cases} +\infty, & \textit{if} \quad n = 2, \\ f_1^k \in R, & \textit{if} \quad n > 2, \end{cases}$$

$$\lim_{t^k \to \bar{t}_1^k} (f^k)'(t^k) = (\bar{f}_1^k)' \in R, \qquad \lim_{t^k \to t_1^k} (f^k)'(t^k) = +\infty,$$

$$\lim_{t^k \to \bar{t}_1^k} (f^k)''(t^k) = \left( \frac{4(n+2)K}{nC} \right)^{(n+2)/(n+1)}, \quad \lim_{t^k \to t_1^k} (f^k)''(t^k) = +\infty,$$

$$(f^k)'''(t^k) > 0, (f^k)^{iv}(t^k) > 0, \quad \textit{for every} \quad t^k \in (\bar{t}_1^k, t_1^k).$$

**Type V:** $\mathbf{C} > 0,\ K < 0$: $f^k$ *defined on a semi-infinite interval,* $f^k : (-\infty, t_2^k) \to R$, *such that*

$$\lim_{t^k \to -\infty} f^k(t^k) = f_{-\infty}^k, \quad \lim_{t^k \to t_2^k} f^k(t^k) = \begin{cases} +\infty, & \textit{if} \quad n = 2, \\ f_2^k \in R, & \textit{if} \quad n > 2, \end{cases}$$

$$\lim_{t^k \to -\infty} (f^k)'(t^k) = (f^k)'_{-\infty} \in R, \qquad \lim_{t^k \to tk_2} (f_k)'(t^k) = +\infty,$$

$$\lim_{t^k \to -\infty} (f^k)''(t^k) = 0, \qquad \lim_{t^k \to t_2^k} (f^k)''(t^k) = +\infty,$$

$$(f^k)'''(t^k) > 0, (f^k)^{iv}(t^k) > 0, \quad \textit{for every} \quad t^k \in (-\infty, t_2^k).$$

**Type VI:** $\mathbf{C} < 0,\ K < 0$: $f^k : (-\infty, t_3^k) \to R$, *with the following properties*

$$\lim_{t^k \to -\infty} f^k(t^k) = \begin{cases} +\infty, & \textit{if} \quad (f^k)'_{-\infty} < 0, \\ \bar{f}_{-\infty}^k \in R, & \textit{if} \quad (f^k)'_{-\infty} = 0, \\ -\infty, & \textit{if} \quad (f^k)'_{-\infty} > 0, \end{cases} \quad \lim_{t^k \to t_3^k} f^k(t^k) = f_3^k \in R,$$

$$\lim_{t^k \to -\infty} (f^k)'(t^k) = (f^k)'_{-\infty} \in R, \qquad \lim_{t^k \to t_3^k} (f^k)'(t^k) = (f_3^k)' \in R,$$

$$\lim_{t^k \to -\infty} (f^k)''(t^k) = 0, \quad \lim_{t^k \to t_3^k} (f^k)''(t^k) = (f_3^k)'' := \left(\frac{4(n+2)K}{nC}\right)^{\frac{2(n+2)}{n}},$$

$$(f^k)'''(t^k) > 0, \ \textit{for every} \ t^k \in (-\infty, t_3^k),$$

$$(f^k)^{iv}(t^k) > 0 \quad \textit{for} \quad t^k \in (-\infty, t_0^k),$$

$$(f^k)^{iv}(t_0^k) = 0,$$

$$(f^k)^{iv}(t^k) < 0 \quad \textit{for} \quad t^k \in (t_0^k, t_3^k),$$

*where* $t_0^k = \left((f^k)''\right)^{-1} ([8(n+2)(3n+5)K/[n(7n+10)C])^{[2(n+2)]/n}$.

*Proof.* In this occasion the properties described follow from Lemma 1.4 and the analysis practiced in former sections, as applied to equation (1.14), *geometrical case* 4), i.e. $r = 2 - \frac{1}{n+2}$, $s = \frac{7n+10}{4(n+2)}$. Again, the distinguished component can not be of either Type I or III, since the constant in equation (1.14) must be nonvanishing. Thus, excluded those two cases from the analysis, the rest of the argument is a consequence of equations numbered (4.3), for Type II; (5.2), (5.3), (5.4), Lemmas 5.1 and 5.2 for Type IV, (5.21), (5.22), (5.23), Lemmas 5.3 and 5.4 for Type V; and (5.37), (5.38), (5.39), (5.40), Lemmas 5.5 and 5.6 for Type VI.

## 8. Final Remarks.

We have studied so far, for hypersurfaces of decomposable type, the cases when the scalar invariants represented by the Pick Invariant J ([8]), the Unimodular Affine Mean Curvature L (7), and the Scalar Curvature R (8), are separately constant. One question that arises naturally is the consideration of the case when those three invariants are simultaneously constant. Let us recall that these are related among them by the so-called Higher Dimensional Affine Theorema Egregium. For dimension $n = 2$, i.e. for surfaces in three-dimensional affine space, this problem has been considered in full generality by Dilllen, Martínez, Milán, García Santos and Vrancken (see [5]). In our case, the results to be presented are direct consequences of the theory developed in previous sections and are valid for every dimension $n \geq 2$.

**Corollary 8.1.** *Let* $X : M^n \to E^{n+1}$ *be a nondegenerate hypersurface of decomposable type, with vanishing Pick Invariant* $J \equiv 0$. *Then, the Unimodular Affine Mean Curvature and the Riemannian Scalar Curvature are also, simultaneously, vanishing:* $L \equiv 0$ *and* $R \equiv 0$.

*Proof.* By Theorem 2.1 all of the components must be of parabolic or logarithmic types. Then, by using the characterizing condition represented by equation (2.31) in [8], with the additional condition that the sum of the constants be zero: $\sum C_k = 0$ (Lemma 2.1), the result follows by direct computations and the use of the further characterizing conditions described by Lemma 1.1, for vanishing unimodular affine mean curvature, and Lemma 1.3, for vanishing scalar curvature.

**Corollary 8.2.** *Let $X : M^n \to E^{n+1}$ be a nondegenerate hypersurface of decomposable type, with constant, nonvanishing unimodular affine Pick Invariant $J \neq 0$. Then, the Unimodular Affine Mean Curvature and the Riemannian Scalar Curvature are also identically equal to (different) nonvanishing constants: $L =$ constant $\neq 0$ and $R =$ constant $\neq 0$.*

*Proof.* In this occasion, by Theorem 3.2, $n-1$ components are of parabolic type, while the remaining one is of the form $(f_0^k)(t_0^k) = (t^{k_0})^{-2/(n+1)}$, and this is exactly the so-called Type II, in both Theorem 6.2, which classifies hypersurfaces of decomposable type with constant, nonvanishing unimodular affine mean curvature $L \neq 0$; and Theorem 7.2, which renders the corresponding result for constant, nonvanishing scalar curvature $R \neq 0$. The corollary is proved.

In particular, for dimension $n = 2$, and for $J =$ constant $\neq 0$, we have essentially the graphs of $f(x, y) = x^2 \pm y^{-2/3}$, while for $J \equiv 0$, we have, besides both kinds of paraboloids, the graph of $g(x, y) = \log x - \log y$, and this last can be easily seen to represent a ruled surface, by means of suitable reparametrization: $X(u, v) = (u \cdot v, u/v, 2 \log v)$. Compare to [5].

## REFERENCES

[1] W. Blaschke, *Vorlesungen über Differentialgeometrie,* II, Springer-Verlag, Berlin, 1923.

[2] E. Calabi, *Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens,* Mich. Math. J., 5 (1958), pp. 105-126.

[3] E. Calabi, *Complete affine hyperspheres I,* Ist. Naz. Alta Mat. Sym. Mat., 10 (1972), pp. 19-38.

[4] S.-Y. Cheng - S.-T. Yau, *Complete affine hypersurfaces. Part I. The completeness of affine metrics,* Comm. Pure Appl. Math., 39 (1986), pp. 839-886.

[5] F. Dillen - A. Martínez - F. Milán - F. García Santos - L. Vrancken, *On the Pick invariant, the affine mean curvature and the Gauss curvature of affine surfaces,* Results in Math., 20 (1991), pp. 622-642.

[6] S. Gigena, *On a conjecture by E. Calabi,* Geom. Dedicata, 11 (1981), pp. 387-396.

[7] S. Gigena, *Constant affine mean curvature hypersurfaces of decomposable type,* Proceedings of Symposia in Pure Mathematics, American Mathematical Society, 54 (1993), Part 3, pp. 289-316.

[8] S. Gigena, *El invariante de Pick para hipersuperficies descomponibles,* Math. Notae, 37 (1993/94), pp. 87-104.

[9] K. Jörgens, *Uber die Lösungen der Differentialgleichung* $r.t - s^2 = 1$, Math. Ann., 127 (1954), pp. 130-134.

[10] A.V. Pogorelov, *On the improper affine hyperspheres,* Geom. Dedicata, 1 (1972), pp. 33-46.

[11] T. Sasaki, *Hyperbolic affine hyperspheres,* Nagoya Math. J., 77 (1980), pp. 107-123.

[12] R. Schneider, *Zur affinen Differentialgeometrie im Grossen I,* Math. Z., 101 (1967), pp. 375-406.

[13] R. Schneider, *Zur affinen Differentialgeometrie im Grossen II,* Math. Z., 102 (1967), pp. 1-8.

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