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SYMMETRIC *Q*-BESSEL FUNCTIONS

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q analog of bessel functions, symmetric under the interchange of q and q^{-1} are introduced. The definition is based on the generating function realized as product of symmetric q-exponential functions with appropriate arguments. Symmetric q-Bessel function are shown to satisfy various identities as well as second-order q-differential equations, which in the limit $q \to 1$ reproduce those obeyed by the usual cylindrical Bessel functions. A brief discussion on the possible algebraic setting for symmetric q-Bessel functions is also provided.

1. Introduction.

Many special function of mathematical physics have been shown to admit generalizations to a base q, which are usually reported as q-special functions.

Interest in such q functions is motivated by the recent and increasing relevance of q analysis, originally suggested almost a century ago [22], in exactly solvable models in statistical mechanics ([1],[23]). Like ordinary special functions, q-analogs satisfy second order q-differential equations and various identities or recurrence relations.

Basic hypergeometric series are the prototype of q-special functions, their properties and applications have been deeply investigated in ([9],[12]).

Basic analogs of Bessel functions have been introduced by Jackson [13] and Swarthow [21] as q-generalizations of the power series expansions , which

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defines the ordinary cylindrical Bessel functions. Three different types of such q-extension can be recognized, each of them satisfy recurrence relations, second-order q-differential equations and addition theorems, which reduce to those holding for the usual Bessel functions in the limit $q \to 1$.

In analogy with the usual special functions, q-functions have recently been shown to admit an algebraic interpretation as matrix elements of q-exponentials of quantum-algebra generators on appropriate representation species. In [10], [11], for instance, a quantum-algebraic framework for q-Bessel functions is provided as well as for the basic hypergeometric function. Finally, in [6] the generating function method is proposed as alternative unifying formalism, where the various q-Bessel functions can be framed.

The above quoted investigations use the method of the *standard* q-analysis.

The discovery of quantum groups and algebras ([14], [8], [24]), characterized by deformed communication relations, which generalize the canonical commutation relations, has led to interest in q-analysis, which is symmetric under the interchange $q \to q^{-1}$.

Within that context, a lot of attention is devoted to the so-called q-oscillators ([2], [18]) and to their possible physical applications in such fields as atomic and nuclear physics [18], quantum optics [4] and superintegrable systems [5]. In this connection, it is natural to investigate the possibility of introducing q-functions, which are symmetric under the interchange of q and $1/q$. They are called symmetric to distinguish them from the standard ones. Symmetric q-exponential and gamma functions have been extensively studied in [16], [17].

In this paper, we address the problem of defining symmetric q-Bessel functions; in particular we follow the approach developed in [6] using indeed the symmetric q-exponential function to realize the generating functions.

Accordingly, in Section 2 we briefly review the definition and the relevant properties of the symmetric q-exponential function. In Section 3 symmetric q-Bessel functions are defined and are shown to satisfy various identities, which, in the limit $q \to 1$, reproduce the well-known recurrence relation obeyed by the usual cylindrical Bessel functions.

In Section 4 we recognize the possibility of introducing shifting operators, which are then used to obtain the second-order q-differential equations obeyed by the symmetric q-Bessel functions.

Finally, concluding comments on the possible algebraic setting for these functions as well as on the possible modified versions of them are given in Section 5.

2. Symmetric q-exponential functions.

Before entering the specific topic of the paper, let us briefly review the properties of the q-exponential functions, which will be basic to the forthcoming discussion on symmetric q-Bessel functions. Definitions of functions in q-analysis are borrowed from ordinary analysis through an appropriate generalization or "*q-deformation*". Accordingly, the q-exponential function is introduced as eigenfunction of the q-differentiation.

Hence, since standard and symmetric differentiations can be introduced in q-analysis, standard and symmetric q-exponential functions are defined.
For completeness'sake, we report in the following the definition of standard q-derivative, although for a detailed account of the relevant properties the reader is addressed to [16], [17].
Standard q-derivatives is indeed defined as

(1) $$\frac{d_q f(z)}{d_q z} = \frac{f(qz) - f(z)}{(q-1)z}$$

which suggests to introducing the differentiation operator

(2) $$\hat{D}_q = [(q-1)z]^{-1}\{q^{zd/dz} - 1\}.$$

On the other hand, as noticed, in quantum groups significant role is played by operators or functions, which are symmetric under the interchange of q with $1/q$. Symmetric q-derivative of an entire analytical function $f(z)$, $z \in C$, is indeed introduced through the definition ([16], [17], [20])

(3) $$\frac{d_q}{d(z:q)} f(z) = \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z}$$

which in analogy with the standard q-differentiation suggests to introducing the operator

(4) $$\hat{D}_{(z:q)} \equiv \frac{q}{(q^2-1)z}[q^{zd/dz} - q^{-zd/dz}].$$

The following link with the standard q-differentiation operator $\hat{D}_q$ is immediately recognized:

(5) $$\hat{D}_{(z:q)} = \frac{q}{q+1}[\hat{D}_q + \frac{1}{q}\hat{D}_{1/q}]$$

and the more direct relation with $\hat{D}_{q^2}$ can be stated

$$\hat{D}_{(z:q)} f(z) = q\hat{D}_{q^2} q^{-zd/dz} f(z) = q\hat{D}_{q^2} f(q^{-1}z). \tag{6}$$

It is also evident that the operator (5) is invariant with respect to the interchange $q \to 1/q$; explicitly

$$\hat{D}_{(z:q)} = \hat{D}_{(z:1/q)}. \tag{7}$$

Let us briefly review the main properties of the operator (5). Acting on powers of z, the operator $\hat{D}_{(z:q)}$ gives

$$\hat{D}_{(z:q)} z^n = [n]_q z^{n-1} \tag{8}$$

where the symbol $[n]_q$ denotes the number

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{9}$$

frequently occurring in the study of q-deformed quantum mechanical simple harmonic oscillator where $0 < q < 1$ ([2], [18]).

We list some relations satisfies by $[n]_q$, since they will be used in the following. It is easy to prove that

$$\begin{aligned} &[n]_q = [n]_{1/q} = -[-n]_q \\ &[m+n]_q = q^n[m]_q + q^{-m}[n]_q = q^m[n]_q + q^{-n}[m]_q \\ &[0]_q = 0 \quad [1]_q = 1. \end{aligned} \tag{10}$$

It is also useful to report the properties of the symmetric q-differentiation, which satisfies a sum rule, a product rule and, in special cases, a chain rule as follows

$$\begin{aligned} \hat{D}_{(z:q)}[f(z) + g(z)] &= D_{(z:q)} f(z) + \hat{D}_{(z:q)} g(z) \\ \hat{D}_{(z:q)}[f(z)g(z)] &= g(q^{-1}z)\hat{D}_{(z:q)} f(z) + f(qz)\hat{D}_{(z:q)} g(z) = \\ &= g(qz)\hat{D}_{(z:q)} f(z) + f(q^{-1}z)\hat{D}_{(z:q)} g(z) \\ \hat{D}_{(z:q)}[f(\alpha z)] &= \alpha\hat{D}_{(\alpha z:q)} f(\alpha z) \\ \hat{D}_{(z:q)} f(z^n) &= [n]_q z^{n-1}\hat{D}_{(z^n:q^n)} f(z^n)\,. \end{aligned} \tag{11}$$

As already noticed, q-analogs of exponential functions are defined as eigenfunctions of differentiation operators. Consequently, standard and symmetric q-exponential functions can be introduced according to whether the operators $\hat{D}_q$ or $\hat{D}_{(z:q)}$ are considered.

Limiting ourselves to the operator $\hat{D}_{(z:q)}$, symmetric q-exponential function $E_q(z)$ can be introduced according to

$$\hat{D}_{(z:q)} E_q(z) = E_q(z) \tag{12}$$

with furthermore $E_q(z)$ being requested to be regular at $z = 0$ and $Eq(0) = 1$.

If $q \in C$, $q \neq 0$ and $|q| \neq 1$, there is a unique function satisfying the required conditions. Hence, since $E_{1/q}(z)$ satisfies (12) as well, we have that $E_q(z) = E_{1/q}(z)$, thus confirming that E_q is symmetric in the parameter q with respect to the interchange of q and q^{-1}.

The power series expansion of E_q is given as

$$E_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \tag{13}$$

with the q-factorial $[n]_q!$ being defined as

$$[n]_q! = \begin{cases} 1 & n = 0 \\ \prod_{r=1}^{n} [r]_q & n \geq 1. \end{cases} \tag{14}$$

The condition (12) is easily verified to hold according to the rule (8). The power series (13) has infinite radius of convergence, and hence $E_q(z)$ is an entire function for all $q \neq 0$, $|q| \neq 1$.

It is worth stressing that the exponential function $E_q(z)$ does not satisfy the semigroup property, i.e. $E_q(x)E_q(y) \neq E_q(x + y)$.

Using indeed the power series expansion (13), we can write

$$E_q(x)E_q(y) = \sum_{r=0}^{\infty} \frac{x^r}{[r]_q!} \sum_{k=0}^{\infty} \frac{y^k}{[k]_q!} . \tag{15}$$

An appropriate rehandling of the sums entering the above expression allows to recast the product $E_q(x)E_q(y)$ in the form

$$E_q(x)E_q(y) = \sum_{n=0}^{\infty} \frac{\mathscr{L}_n(x, y)}{[n]_q!} \tag{16}$$

where $\mathscr{Z}_n(x, y)$ denotes the function

$$\mathscr{Z}_n(x, y) = \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix}_q x^{n-h} y^h \tag{17}$$

and $\begin{bmatrix} n \\ h \end{bmatrix}_q$ can be understood as the q-analog of the binomial coefficient:

$$\begin{bmatrix} n \\ h \end{bmatrix}_q \equiv \frac{[n]_q!}{[h]_q![n-h]_q!} \,. \tag{18}$$

However, the function $\mathscr{Z}_n(x, y)$ is not the q-analog of binomial formula, giving the n-th power of the sum $x + y$, which is indeed understood in the form

$$[x+y]_q^n \equiv \sum_{h=0}^{n} \begin{bmatrix} n \\ h \end{bmatrix} x^{n-h} y^h q^{-h(n+1)}. \tag{19}$$

Accordingly, the product (15) does not turn into q-exponential of $x + y$: $E_q(z)E_q(y) \neq E_q(x + y)$.

It might be interesting to derive the link between $\mathscr{Z}_n$ and $[x + y]_q^n$, which indeed reads

$$[x+y]_q^n \equiv \mathscr{Z}_n(x, yq^{-n+1}). \tag{20}$$

In addition, let us notice that the q-binomial formula (19) satisfies the product rule:

$$[x+y]_q^n [x + q^{-2n} y]_q^m = [x+y]_q^{n+m} \tag{21}$$

and the following formulae for the q-derivatives with respect to x and y can be derived

$$\begin{aligned} \hat{D}_{(x:q)}[x+y]_q^n &= [n]_q [x + y/q]_q^{n-1} \\ \hat{D}_{(y:q)}[x+y]_q^n &= [n]_q \, q^{-2n} [qx + y]_q^{n-1} \end{aligned} \tag{22}$$

which give the usual formulae in the limit $q \to 1$. Finally, it is worth stressing that the q-analogue of the binomial formula is not symmetric in the parameter q under the interchange $q \leftrightarrow q^{-1}$.

In the next section we use the symmetric q-exponential function $E_q(z)$ to *generate* symmetric q-Bessel functions.

3. Symmetric q-Bessel functions.

Let us consider the product of symmetric q-exponential functions as

$$\mathcal{G}(x;t:q) = E_q\Big(\frac{xt}{2}\Big)E_q\Big(-\frac{xt}{2}\Big) \tag{23}$$

whose expression as series of t-powers can be easily obtained using the power series expansion of E_q given in (13). Explicitly, we have

$$\mathcal{G}(x;t:q) = \sum_{n=-\infty}^{+\infty} t^n \sum_{k=0}^{\infty} \frac{(-1)^s (x/2)^{n+2k}}{[n+k]_q![k]_q!}. \tag{24}$$

Introducing symmetric q-Bessel functions $J_n(x:q)$ in full analogy with the standard case [10], that is as the coefficients of the expansion

$$\mathcal{G}(x;t:q) \equiv \sum_{n=-\infty}^{+\infty} t^n\, J_n(x:q) \tag{25}$$

it is immediate to get the explicit expression of $J_n(x:q)$ as x-power series

$$J_n(x:q) = \sum_{k=0}^{+\infty} \frac{(-1)^k (x/2)^{n+2k}}{[n+k]_q![k]_q!}. \tag{26}$$

It is evident that $J_n(x:q)$ is symmetric under the interchange $q \leftrightarrow q^{-1}$

$$J_n(x:q) = J_n(x:1/q). \tag{27}$$

Using the relation

$$[n]_q! = (n)_{q^2}!q^{-2/n(n-1)} \tag{28}$$

we can write (26) as

$$\begin{aligned} J_n(x:q) = q^{n^2/2} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2\sqrt{q}}\right)^{n+2k} q^{k(n+k)}}{(k)_{q^2}!(n+k)_{q^2}!} = \\ q^{-n^2/2} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x\sqrt{q}}{2}\right)^{n+2k} q^{k(n+k)}}{(k)_{1/q^2}!(n+k)_{1/q^2}!} \end{aligned} \tag{29}$$

where the symbol $(n)_q$ means

$$(30a)\qquad (n)_q = \frac{1-q^n}{1-q}$$

and correspondingly

$$(30b)\qquad (n)_q! = \prod_{k=1}^{n} (k)_q .$$

The functions (26) are characterized by recurrence relations, which can be easily obtained by taking the derivative of $\mathscr{G}$ with respect to x. In fact, an account of the two possible expression for the symmetric q-derivative of a product as reported in (11), we obtain the two relations involving the q-derivative $J_n(x : q) \equiv \hat{D}_{(x:q)} J_n(x : q)$, and the contiguous functions $J_{n-1}(x : q)$, $J_{n+1}(x : q)$, namely

$$(31)\qquad \begin{aligned} 2J'_n(x:q) &= q^{\frac{n-1}{2}} J_{n-1}\Big(\frac{x}{\sqrt{q}} : q\Big) - q^{\frac{n+1}{2}} J_{n+1}(x\sqrt{q} : q) \\ 2J'_n(x:q) &= q^{-\frac{n-1}{2}} J_{n-1}(x\sqrt{q} : q) - q^{-\frac{n+1}{2}} J_{n+1}\Big(\frac{x}{\sqrt{q}} : q\Big) \end{aligned}$$

which can be obtained from each other by changing $q \to 1/q$. On the other hand, by multiplying the expression (26) by $[n]_q$ and exploiting the property of $[n]_q$, according to which (see Eq. (10))

$$[n]_q = q^s[n+s]_q - q^{n+s}[s]_q$$

the further relation can be inferred

$$(32a)\qquad \frac{2[n]_q}{x} J_n(x:q) = q^{-\frac{(n-1)}{2}} J_{n-1}(x\sqrt{q} : q) + q^{\frac{n+1}{2}} J_{n+1}(x\sqrt{q} : q)$$

which by changing $q \to 1/q$ turns into

$$(32b)\qquad \frac{2}{x}[n]_q J_n(x:q) = q^{\frac{n-1}{2}} J_{n-1}\Big(\frac{x}{\sqrt{q}} : q\Big) + q^{-\frac{(n+1)}{2}} J_{n+1}\Big(\frac{x}{\sqrt{q}} : q\Big).$$

Useful relations can be obtained by combining the above equations or directly exploiting the series expansion (26).

For instance, from (31) and (32) one gets

$$q^{\frac{n-1}{2}} J_n\Big(\frac{x}{\sqrt{q}} : q\Big) - q^{-\frac{(n-1)}{2}} J_{n-1}(x\sqrt{q} : q) = \\ = q^{\frac{n+1}{2}} J_{n+1}(x\sqrt{q} : q) - q^{-\frac{(n+1)}{2}} J_{n+1}\Big(\frac{x}{\sqrt{q}} : q\Big) \tag{33}$$

by putting $x \to x/\sqrt{q}$ and $x \to x\sqrt{q}$ in the above equation, the further relations follows

$$\frac{q^2-1}{q}[n]_q J_{n+1}(x : q) = q^{-1}\left[J_{n+1}\Big(\frac{x}{q} : q\Big) + q^n J_{n-1}\Big(\frac{x}{q} : q\Big)\right] - \\ - q\left[J_{n+1}(xq : q) + q^{-n} J_{n-1}(xq : q)\right] \tag{34}$$

and

$$\frac{q^2-1}{q}[n]_q J_{n-1}(x : q) = q\left[J_{n-1}(xq : q) + q^n J_{n+1}(xq : q)\right] - \\ - q^{-1}\left[J_{n-1}\Big(\frac{x}{q} : q\Big) + q^{-n} J_{n+1}\Big(\frac{x}{q} : q\Big)\right]. \tag{35}$$

Correspondingly, by using the expression (26) the following relations can be proved

$$\begin{aligned} q^{-\frac{(n-1)}{2}} J_{n-1}\left(x\sqrt{q} : q\right) &= \frac{x}{2q}(1-q^2)J_n(x : q) + q^{\frac{n-1}{2}} J_{n-1}\Big(\frac{x}{\sqrt{q}} : q\Big) \\ q^{-\frac{(n+1)}{2}} J_{n+1}\left(x\sqrt{q} : q\right) &= \frac{x}{2q}(1-q^2)J_{n+2}(x : q) + q^{\frac{n+1}{2}} J_{n+1}\Big(\frac{x}{\sqrt{q}} : q\Big) \end{aligned} \tag{36}$$

which state a further link between the contiguous functions $J_n(\bullet : q)$, $J_{n-1}(\bullet : q)$ and $J_{n+1}(\bullet : q)$. It is worth stressing that there does not exist the analog for the ordinary functions. Let us not that in the relations (31) the functions J_{n-1} and J_{n+1} appear with different arguments $x/\sqrt{q}$ and $x\sqrt{q}$ respectively or viceversa. Conversely, in the relations (32) the functions J_{n-1} and J_{n+1} have the same argument, namely $x\sqrt{q}$ and $x/\sqrt{q}$. Exploiting the relations (36), one can rewrite the basic recurrences (31) in a form symmetric with respect to the arguments of J_{n-1} and J_{n+1}; and correspondingly one can turn (32) in a fashion where J_{n-1} and J_{n+1} have different arguments.

Explicitly, we have

$$\left[2\hat{D}_{(x:q)} - (1-q^2)\frac{x}{2q}\right] J_n(x : q) = \\ = q^{\frac{n-1}{2}} J_{n-1}\Big(\frac{x}{\sqrt{q}} : q\Big) - q^{-\frac{(n+1)}{2}} J_{n+1}\Big(\frac{x}{\sqrt{q}} : q\Big) \tag{37a}$$

which changing q into $1/q$ gives

(37b)
$$\left[2\hat{D}_{(x:q)} + (1-q^2)\frac{x}{2q}\right] J_n(x:q) = \\ = q^{-\frac{(n-1)}{2}} J_{n-1}\Big(x\sqrt{q} : q\Big) - q^{\frac{n+1}{2}} J_{n+1}\Big(x\sqrt{q} : q\Big).$$

Similarly, combining (32) and (36) one gets

(38a)
$$\left[\frac{2[n]_q}{x} - \frac{x}{2q}(1-q^2)\right] J_n(x:q) = \\ = q^{\frac{(n-1)}{2}} J_{n-1}\Big(\frac{x}{\sqrt{q}} : q\Big) + q^{\frac{n+1}{2}} J_{n+1}\Big(x\sqrt{q} : q\Big)$$

(38b)
$$\left[\frac{2[n]_q}{x} + \frac{x}{2q}(1-q^2)\right] J_n(x:q) = \\ = q^{-\frac{(n-1)}{2}} J_{n-1}\Big(x\sqrt{q} : q\Big) + q^{-\frac{n+1}{2}} J_{n+1}\Big(\frac{x}{\sqrt{q}} : q\Big).$$

The brief discussion clearly display the wealth of possible relations, which can be drawn, involving $J_n(\bullet : q)$, the contiguous functions $J_{n-1}(\bullet : q)$, $J_{n+1}(\bullet : q)$ and the derivative $J_n'(\bullet : q)$. This is a direct consequence of that the base q differs from unity: $q \neq 1$; in fact, as noticed in connection with (36), some of these relations do not have the analog in the limit $q \to 1$, i.e. for the ordinary Bessel functions.

4. Shifting operators and differential equations.

In the theory of ordinary Bessel functions the relevant recurrence relations allow to recognize shifting operators $\hat{E}_-$ and $\hat{E}_+$, which turn the functions of order n into the functions of the order $n-1$ and $n+1$, respectively. They play a significant role within the framework of group-theoretical interpretation of ordinary Bessel functions. In a purely mathematical context they allow to derive the differential equation obeyed by the Bessel functions in a straightforward way. Similarly, shifting operators can be introduced for q-Bessel functions as well, although in this case on turning the function of order n into that of order $n-1$ or $n+1$ they also rescale the argument by the factor $\sqrt{q}$ or $1/\sqrt{q}$. Accordingly, four types of shifting operators can be defined. Indeed, combining appropriately the recurrence relations (32) and (37), it is easy to obtain

$$\begin{aligned}\hat{E}_-^{(q,n)}(x) &= q^{\frac{n-1}{2}}\left\{\frac{[n]_q}{x} + \hat{D}_{(x:q)} + (1-q^2)\frac{x}{4q}\right\}\\ \hat{E}_+^{(q,n)}(x) &= q^{-\frac{(n+1)}{2}}\left\{\frac{[n]_q}{x} - \hat{D}_{(x:q)} - (1-q^2)\frac{x}{4q}\right\}\end{aligned} \tag{39}$$

which act on $J_n(x:q)$ according to

$$\begin{aligned}\hat{E}_-^{(q,n)}(x)J_n(x:q) &= J_{n-1}(x\sqrt{q}:q)\\ \hat{E}_+^{(q,n)}(x)J_n(x:q) &= J_{n+1}(x\sqrt{q}:q)\end{aligned} \tag{40}$$

the argument of the functions J_{n-1} and J_{n+1} being rescaled by the factor $\sqrt{q}$.

Correspondingly, the further operators $\hat{E}_-^{(1/q,n)}(x)$, $\hat{E}_+^{(1/q,n)}(x)$ explicitly given by

$$\begin{aligned}\hat{E}_-^{(1/q,n)}(x) &= q^{-\frac{(n-1)}{2}}\left\{\frac{[n]_q}{x} + \hat{D}_{(x:q)} - (1-q^2)\frac{x}{4q}\right\}\\ \hat{E}_+^{(1/q,n)}(x) &= q^{\frac{n+1}{2}}\left\{\frac{[n]_q}{x} - \hat{D}_{(x:q)} + (1-q^2)\frac{x}{4q}\right\}\end{aligned} \tag{41}$$

can be recognized, which differ from the above (39) since the argument of the functions J_{n-1} and J_{n+1} is rescaled by $1/\sqrt{q}$; namely

$$\begin{aligned}\hat{E}_-^{(1/q,n)}(x)J_n(x:q) &= J_{n-1}\left(\frac{x}{\sqrt{q}}:q\right)\\ \hat{E}_+^{(1/q,n)}(x)J_n(x:q) &= J_{n+1}\left(\frac{x}{\sqrt{q}}:q\right).\end{aligned} \tag{42}$$

It is needless to say that the operators (41) can be obtained from (39) by simply changing q into $1/q$. However they are crucial in deriving the differential equation obeyed by $J_n(x:q)$.

Indeed, it is evident the following identity

$$\hat{E}_-^{(1/q,n+1)}(x\sqrt{q})\hat{E}_+^{(q,n)}(x)J_n(x:q) = J_n(x:q) \tag{43}$$

where it has been explicitly indicated that the operator $\hat{E}_+^{(q,n)}$ acts on a function of order n and argument x, whilst $\hat{E}_-^{(1/q,n+1)}$ acts on the function of order $n+1$ and with rescaled argument $x\sqrt{q}$.

Since

$$\hat{E}_{-}^{(1,/q,n+1)}(x\sqrt{q}) = q^{-\frac{n+1}{2}}\left\{\frac{[n+1]_q}{x} + \hat{D}_{(x:q)} - \frac{x}{4}(1-q^2)\right\} \tag{44}$$

after some algebra we end up with

$$\hat{\mathscr{L}}_q^{(n)}(x)J_n(x:q) = 0 \tag{45}$$

where the q-Bessel operator $\hat{\mathscr{L}}_q^{(n)}$ has the rather complicated expression

$$\begin{aligned}\hat{\mathscr{L}}_q^{(n)}(x) = {} & \frac{[n]_q^2}{2qx^2}(1+q^2) - \frac{1-q^2}{4q}(q^n+q^{-n}) - \frac{1-q^2}{2}[n]_q + \\ & + \frac{x^2}{32q^3}(1+q^2)(1-q^2)^2 - (q^n+q^{-n})\frac{1}{2x}\hat{D}_{(x:q)} - \\ & - \hat{D}^2_{(x:q)} - q^{n+1}\end{aligned} \tag{46}$$

which reduces to the ordinary Bessel operator in the limit $q \to 1$.

The above operator is not symmetric under the interchange of q and q^{-1}, thus providing a further q-differential equation obeyed by $J_n(x:q)$

$$\begin{aligned}\hat{\mathscr{L}}_{1/q}^{(n)}(x) = {} & \frac{[n]_q^2}{2qx^2}(1+q^2) + \frac{1}{4q}(1-q^2)(q^n+q^{-n}) + \\ & + \frac{1-q^2}{2q^2}[n]_q + \frac{x^2}{32q^3}(1+q^2)(1-q^2)^2 - \\ & - (q^n+q^{-n})\frac{1}{2x}\hat{D}_{(x:q)} - \hat{D}^2_{(x:q)} - q^{-(n+1)}\,.\end{aligned} \tag{47}$$

Before closing the section, let us discuss the symmetric properties of $J_n(x:q)$ with respect to the index n and the argument x, multiplication and addition formulas being discussed in [23].

It is easy to prove that

$$\begin{aligned} J_n(-x:q) &= (-1)^n J_n(x:q) \\ J_n(x:q) &= J_n(-x:q) \end{aligned} \tag{48}$$

strongly reminiscent of the corresponding relations obeyed by the usual Bessel functions.

5. Concluding remarks.

The analysis performed in the previous section confirms the possibility of introducing q-analogs of Bessel function, which are symmetric in the parameter q with respect to the interchange of q and $1/q$. In full analogy with the ordinary case, recurrence relations, shifting operators and q-differential equations can be obtained.

The approach we follow is based on the generating function; which is defined as the product of symmetric q-exponential functions with appropriate arguments.

As already noticed, q-Bessel functions linked to the standard q-exponential function $e_q(z)$ have been introduced ([13], [21], [6]); three different types of functions can be recognized in this case as a consequence of the non-invariance of the exponential e_q with respect to the interchange $q \leftrightarrow q^{-1}$. The properties of these functions have been deeply investigated, also within the context of the theory of group representation. In [10],[11] it has been shown that q-generalizations of Bessel functions appear in the realization of the two-dimensional Euclidean quantum algebra $\mathscr{E}(2)$. Similarly, symmetric q-Bessel functions can be understood as matrix elements of the representation of $\mathscr{E}(2)$ on the Hilbert space of all the linear combinations of the functions z^n, $z \in C$ and $n \in N$. The group elements are realized as product of exponentials of the three generators, the symmetric q-exponential function E_q being involved. In this connection, it is needless to say that the generating function method can be immediately recovered within the group representation framework, as shown in [7]. As a conclusion, let us stress that symmetric q-analog of modified Bessel functions can be defined quite straightforwardly.

In particular we note that they are specified by the generating function

$$E_q\Big(\frac{xt}{2}\Big)E_q(-\frac{xt}{2}) = \sum_{-\infty}^{+\infty} t^n I_n(x:q). \tag{49}$$

The analytical continuation formula

$$i^n I_n(x:q) = J_n(ix:q) \tag{50}$$

is a consequence of the identity

$$E_q\Big(\frac{ixt}{2}\Big)E_q\Big(-\frac{ix}{2t}\Big) = E_q\Big(\frac{x(it)}{2}\Big)E_q\Big(\frac{x}{2(it)}\Big). \tag{51}$$

The q-differential equation satisfied by $I_n(x:q)$ can be obtained from Eq. (47), by replacing x with ix.

Other special functions like q-Hermite and q-Laguerre polynomials or functions can be introduced and furthermore an algebraic setting for them can be recognized [7].

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