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HIGHER INTEGRABILITY FOR THE GRADIENT OF SOLUTIONS TO SOME NONLINEAR ELLIPTIC SYSTEMS

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We consider the nonlinear elliptic system $-div(\mathscr{A}(x, Du(x)) = 0$ where $\mathscr{A}(x, \xi)$ is only hölder continuous with p-growth. We no longer assume differentiability on $\mathscr{A}$ and we prove higher integrability of the gradient Du using fractional Sobolev spaces.

1. Introduction.

Difference quotient technique has been successfully used to prove regularity for weak solutions $u \in W^{1,p}(\Omega)$ of nonlinear elliptic systems

$$-div\,(\mathscr{A}(x, Du(x))) = 0, \qquad x \in \Omega, \tag{1.1}$$

when $\mathscr{A} = \mathscr{A}(x, \xi)$ is differentiable with respect to x and ξ, [17], [13], [18]. Recently, [16], [10], [9] have dealt with $\mathscr{A} = \mathscr{A}(\xi)$ when no differentiability is assumed. In the present paper we further go on developing the technique

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employed in [16] by allowing $\mathscr{A}$ to depend also on x. Here Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, $\mathscr{A} : \Omega \times \mathbb{R}^{n\times N} \to \mathbb{R}^{n\times N}$ satisfies the following assumptions

$$|\mathscr{A}(x,\xi)| \leq c(1+|\xi|^{p-1}), \tag{1.2}$$

$$|\mathscr{A}(x,\xi) - \mathscr{A}(y,\xi)| \leq c(\nu+|\xi|)^{p-1}|x-y|^{\theta_1}, \tag{1.3}$$

$$|\mathscr{A}(x,\xi) - \mathscr{A}(x,\eta)| \leq c(\nu+|\xi|+|\eta|)^{p-1-\theta_2}|\xi-\eta|^{\theta_2}, \tag{1.4}$$

$$m(\nu+|\xi|+|\eta|)^{p-2}|\xi-\eta|^2 \leq (\mathscr{A}(x,\xi)-\mathscr{A}(x,\eta))(\xi-\eta), \tag{1.5}$$

for every $\xi, \eta \in \mathbb{R}^{n\times N}$, for all $x, y \in \Omega$, where $p, c, \nu, m, \theta_1, \theta_2$ are constants with

$$2 \leq p, \quad 0 < c, \quad 0 \leq \nu \leq 1, \quad 0 < m, \quad 0 < \theta_1 \leq 1, \quad 0 < \theta_2 \leq 1. \tag{1.6}$$

In the sequel $u : \Omega \to \mathbb{R}^N$, $u \in W^{1,p}(\Omega)$ is a weak solution to (1.1), that is

$$\int_\Omega \mathscr{A}(x, Du(x))D\phi(x)\,dx = 0, \tag{1.7}$$

for every $\phi : \Omega \to \mathbb{R}^N$, with $\phi \in W_0^{1,p}(\Omega)$. In Section 3 we will prove the following

Theorem. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to* (1.1) *under* (1.2), ..., (1.6). *Then*

$$Du \in L^r_{\mathrm{loc}}(\Omega), \qquad \forall r < p\frac{n}{n-\gamma}, \quad \textit{where} \quad \gamma = \min\left\{2\theta_1, \frac{2}{2-\theta_2}\right\}. \tag{1.8}$$

Remark 1. When $\gamma = 2$, then (1.8) holds with $r = p\frac{n}{n-2}$ for $n \geq 3$ and any $r < \infty$ for $n = 2$.

Remark 2. This higher integrability result is achieved by difference quotient technique: hölder continuity of $\mathscr{A}$ allows us to gain only a fractional derivative of $|Du|^{(p-2)/2}Du$ but it is enough in order to improve on the integrability of Du. Fractional Sobolev spaces have been successfully used in [8], [6] for the so-called "natural growth conditions", in [5] for the nonlinear case $1 < p < 2$, in [7] for the linear case and in [3], [15], [16] when dealing with the so-called "p, q growth conditions".

Remark 3. If we consider the integral

$$\int_\Omega a(x)|Du(x)|^p\,dx, \tag{1.9}$$

where $2 \le p$, $a : \mathbb{R}^n \to \mathbb{R}$, and

$$a \in C^{0,\sigma}(\overline{\Omega}), \qquad 0 < \sigma \le 1, \qquad \inf_\Omega a > 0, \tag{1.10}$$

then the Euler equation of the functional (1.9) is (1.7) with $\mathscr{A}(x,\xi) = a(x)p|\xi|^{p-2}\xi$ and such an $\mathscr{A}$ verifies (1.2), ..., (1.6) with $\theta_1 = \sigma$, $\theta_2 = 1$ and $\nu = 0$.

Thus we have the following

Corollary. *If* $u \in W^{1,p}(\Omega)$ *minimizes the integral* (1.9) *under* (1.10)*, then*

$$Du \in L^r_{\rm loc}(\Omega), \qquad \forall r < p\frac{n}{n-2\sigma}. \tag{1.11}$$

Remark 4. When $\sigma = 1$, then (1.11) holds with $r = p\frac{n}{n-2}$ and any $r < \infty$ for $n = 2$.

Remark 5. In this paper we deal with the lack of differentiability of $\mathscr{A}(x,\xi)$, but we assume monotonicity (1.5). In [12], [19], [11], [14], they weaken the ellipticity condition (1.5). [19] deals with the existence of solutions while [12], [11], [14] regard their regularity. [12] considers the case $p = 2$ and its ellipticity is weaker than (1.5) but it assumes $\mathscr{A} \in C^1$ and $D\mathscr{A} \in L^\infty$. On the other hand, [11] deals also with $p > 2$, its ellipticity condition is weaker than (1.5) when $\nu > 0$: no degeneration ($\nu = 0$) seems to be allowed in [11]; it assumes $\mathscr{A} \in C^1$ but it removes the growth condition on $D\mathscr{A}$ which was contained in [12]. Eventually, looking at the results, let us point out the difference between our paper and [12], [11], [14]: they prove partial regularity, that is hölder continuity of Du in some open $\Omega_0 \subset \Omega$; thus Du might be discontinuous at some point $x_0 \in \Omega$; in our paper we improve the integrability of Du in the neighbourhood of every point $x_0 \in \Omega$.

2. Preliminaries.

For a vector-valued function $f(x)$, define the difference

$$\tau_{s,h} f(x) = f(x + he_s) - f(x),$$

where $h \in \mathbb{R}$, e_s is the unit vector in the x_s direction, and $s = 1, 2, \dots, n$. For $x_0 \in \mathbb{R}^n$, let $B_R(x_0)$ be the ball centered at x_0 with radius R. We will often suppress x_0 whenever there is no danger of confusion. We now state several lemmas that are crucial to our work. In the following $f : \Omega \to \mathbb{R}^k$, $k \geq 1$; B_ρ, B_R, $B_{2\rho}$ and B_{2R} are concentric balls.

Lemma 2.1. *If* $0 < \rho < R$, $|h| < R - \rho$, $1 \leq t < \infty$, $s \in \{1, \dots, n\}$, f, $D_s f \in L^t(B_R)$, *then*

$$\int_{B_\rho} |\tau_{s,h} f(x)|^t \, dx \leq |h|^t \int_{B_R} |D_s f(x)|^t \, dx.$$

(See [13], p. 45, [4], p. 28).

Lemma 2.2. *Let* $f \in L^t(B_{2\rho})$, $1 < t < \infty$, $s \in \{1, \dots, n\}$; *if there exists a positive constant* C *such that*

$$\int_{B_\rho} |\tau_{s,h} f(x)|^t \, dx \leq C|h|^t,$$

for every h *with* $|h| < \rho$, *then there exists* $D_s f \in L^t(B_\rho)$.

(See [13], p. 45, [4], p. 26).

Lemma 2.3. *If* $f \in L^2(B_{3\rho})$ *and for some* $d \in (0, 1)$ *and* $C > 0$

$$\sum_{s=1}^{n} \int_{B_\rho} |\tau_{s,h} f(x)|^2 \, dx \leq C|h|^{2d},$$

for every h *with* $|h| < \rho$, *then* $f \in L^r(B_{\rho/4})$ *for every* $r < 2n/(n - 2d)$.

Proof. The previous inequality tells us that $f \in W^{b,2}(B_{\rho/2})$ for every $b < d$, so we can apply the imbedding theorem for fractional Sobolev spaces ([2], Chapter VII).

Lemma 2.4. *For every* t *with* $1 \leq t < \infty$, *for every* $f \in L^t(B_{2R})$, *for every* h *with* $|h| < R$, *for every* $s = 1, 2, \dots, n$ *we have*

$$\int_{B_R} |f(x + he_s)|^t \, dx \leq \int_{B_{2R}} |f(x)|^t \, dx.$$

Lemma 2.5. *For every $p \geq 2$*

$$\left|\tau_{s,h}\left(|f(x)|^{(p-2)/2} f(x)\right)\right|^2 \leq k^3 \left(\frac{p}{2}\right)^2 \int_0^1 |f(x) + t\,\tau_{s,h} f(x)|^{p-2} |\tau_{s,h} f(x)|^2 \, dt$$

for every $f \in L^p(B_{2R})$, for every h with $|h| < R$, for every $s = 1, 2, \ldots, n$, for every $x \in B_R$.

Lemma 2.6. *For every $\gamma > -1$, for every $k \in \mathbb{N}$ there exist positive constants c_2, c_3 such that*

$$c_2(|v|^2 + |w|^2)^{\gamma/2} \leq \int_0^1 |v + tw|^\gamma \, dt \leq c_3(|v|^2 + |w|^2)^{\gamma/2} \tag{2.1}$$

for every $v, w \in \mathbb{R}^k$.

(See [1]).

The previous Lemma 2.6 allows us to easily get the following corollary.

Corollary 2.7. *For every $p \geq 2$, for every $k \in \mathbb{N}$ there exists a positive constant c_4 such that*

$$c_4 \int_0^1 |\lambda + t(\xi - \lambda)|^{p-2} \, dt \leq (|\lambda| + |\xi|)^{p-2} \tag{2.2}$$

for every $\lambda, \xi \in \mathbb{R}^k$.

3. Proof of the Theorem.

Let $R > 0$ be such that $\overline{B_{4R}} \subset \Omega$ and let B_ρ and B_R be concentric balls, $0 < \rho < R \leq 1$. Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a "cut off" function in $C_0^\infty(B_R)$ with $\eta \equiv 1$ on B_ρ, $0 \leq \eta \leq 1$. Fix $s \in \{1, \ldots, n\}$, take $0 < |h| < R$. Using $\phi = \tau_{s,-h}(\eta^2 \tau_{s,h} u)$ in (1.7) we get

$$\begin{aligned} (I) = \int_{B_R} \tau_{s,h}\,(\mathscr{A}(x, Du))\,\eta^2 \tau_{s,h} Du \, dx = \\ = -\int_{B_R} \tau_{s,h}\,(\mathscr{A}(x, Du))\, 2\eta D\eta\, \tau_{s,h} u \, dx = (II). \end{aligned} \tag{3.1}$$

We have

$$\begin{aligned}(3.2)\quad \tau_{s,h}(\mathscr{A}(x, Du(x))) &= \mathscr{A}(x + he_s, Du(x + he_s)) - \mathscr{A}(x, Du(x))\\ &= \mathscr{A}(x + he_s, Du(x + he_s)) - \mathscr{A}(x, Du(x + he_s)) +\\ &\quad + \mathscr{A}(x, Du(x + he_s)) - \mathscr{A}(x, Du(x)).\end{aligned}$$

In order to estimate (II), we use (3.2), (1.3) and (1.4), thus

$$\begin{aligned}(3.3)\quad |\tau_{s,h}(\mathscr{A}(x, Du(x)))| &\le c(1 + |Du(x + he_s)|)^{p-1}|h|^{\theta_1} +\\ &+ c(\nu + |Du(x)| + |Du(x + he_s)|)^{p-1-\theta_2}|\tau_{s,h}Du(x)|^{\theta_2},\end{aligned}$$

then

$$\begin{aligned}(3.4)\quad |(II)| &\le \int_{B_R} 2\eta(x)|D\eta(x)||\tau_{s,h}u(x)|c(1 + |Du(x + he_s)|)^{p-1}|h|^{\theta_1}\,dx\\ &+ \int_{B_R} 2\eta(x)|D\eta(x)||\tau_{s,h}u(x)|c(\nu + |Du(x)| +\\ &+ |Du(x + he_s)|)^{p-1-\theta_2}|\tau_{s,h}Du(x)|^{\theta_2}\,dx = (III) + (IV).\end{aligned}$$

When dealing with (III), we use the properties of the "cut off" function η, then Hölder inequality, eventually Lemma 2.1 and 2.4:

$$\begin{aligned}(3.5)\quad (III) &\le c_6|h|^{\theta_1}\int_{B_R}(1 + |Du(x + he_s)|)^{p-1}|\tau_{s,h}u(x)|\,dx \le\\ &\le c_7|h|^{\theta_1}\left(\int_{B_R}(1 + |Du(x + he_s)|^p)\,dx\right)^{\frac{p-1}{p}}\left(\int_{B_R}|\tau_{s,h}u(x)|^p dx\right)^{\frac{1}{p}} \le\\ &\le c_8|h|^{\theta_1}\left(\int_{B_{2R}}(1 + |Du(x)|^p)\,dx\right)^{\frac{p-1}{p}}\left(\int_{B_{2R}}|D_s u(x)|^p dx\right)^{\frac{1}{p}}|h| \le c_9|h|^{\theta_1+1},\end{aligned}$$

for some positive constants c_6, c_7, c_8, c_9 independent on h. On the other hand, using Young inequality, we deal with (IV) as follows:

$$\begin{aligned}(3.6)\quad (IV) &= \int_{B_R}(\eta(x))^{\theta_2}\big(\nu + |Du(x)| +\\ &+ |Du(x + he_s)|\big)^{\frac{(p-2)\theta_2}{2}}|\tau_{s,h}Du(x)|^{\theta_2}2c|D\eta(x)|\big(\nu + |Du(x)| +\\ &+ |Du(x + he_s)|\big)^{p-1-\theta_2-\frac{(p-2)\theta_2}{2}}(\eta(x))^{1-\theta_2}|\tau_{s,h}u(x)|\,dx \le\\ &\le \frac{m}{2}\int_{B_R}(\eta(x))^2(\nu + |Du(x)| + |Du(x + he_s)|)^{p-2}|\tau_{s,h}Du(x)|^2\,dx +\\ &+ c_{10}\int_{B_R}(\nu + |Du(x)| + |Du(x + he_s)|)^{[p-1-\theta_2-\frac{(p-2)\theta_2}{2}]\frac{2}{2-\theta_2}}|\tau_{s,h}u(x)|^{\frac{2}{2-\theta_2}}\,dx\\ &= (V) + (VI),\end{aligned}$$

for some positive constant c_{10} independent of h. If $2 < p$ or $\theta_2 < 1$, by Hölder inequality with exponents $\frac{p(2-\theta_2)}{2}$ and $\frac{p(2-\theta_2)}{p(2-\theta_2)-2}$, using Lemma 2.1 and 2.4 we have

$$\begin{aligned}(VI) \leq c_{10}\bigg(\int_{B_R}\Big(\nu+|Du(x)|+\\ +|Du(x+he_s)|\Big)^p\,dx\bigg)^{\frac{p(2-\theta_2)-2}{p(2-\theta_2)}}\bigg(\int_{B_R}|\tau_{s,h}u(x)|^p\,dx\bigg)^{\frac{2}{p(2-\theta_2)}} \leq\\ \leq c_{11}\bigg(\int_{B_{2R}}(1+|Du(x)|^p)\,dx\bigg)^{\frac{p(2-\theta_2)-2}{p(2-\theta_2)}}|h|^{\frac{2}{2-\theta_2}}\bigg(\int_{B_{2R}}|D_su(x)|^p dx\bigg)^{\frac{2}{p(2-\theta_2)}},\end{aligned}$$

thus

$$(VI) \leq c_{12}|h|^{\frac{2}{2-\theta_2}}, \tag{3.7}$$

for some positive constant c_{12} independent on h. When $2 = p$ and $\theta_2 = 1$ we check (3.7) directly. Let us estimate (I) from below. We use (3.2)

$$\begin{aligned}(I) = \int_{B_R}(\mathscr{A}(x+he_s, Du(x+he_s)) -\\ -\mathscr{A}(x, Du(x+he_s)))\eta^2(x)\tau_{s,h}Du(x)\,dx +\\ +\int_{B_R}(\mathscr{A}(x, Du(x+he_s)) - \mathscr{A}(x, Du(x)))\eta^2(x)\tau_{s,h}Du(x)\,dx\\ = (VII) + (VIII).\end{aligned} \tag{3.8}$$

We apply (1.5) so that

$$\begin{aligned}m\int_{B_R}(\nu+|Du(x)|+\\ +|Du(x+he_s)|)^{p-2}|\tau_{s,h}Du(x)|^2\eta^2(x)\,dx \leq (VIII).\end{aligned} \tag{3.9}$$

In order to deal with (VII), we use (1.3), (1.6), Hölder inequality and

Lemma 2.4:

$$|(VII)| \leq \int_{B_R} c(\nu + |Du(x)| + |Du(x+he_s)|)^{p-1}|h|^{\theta_1}|\tau_{s,h}Du(x)|\eta^2(x)\,dx =$$
$$= \int_{B_R}(\nu + |Du(x)| + |Du(x+he_s)|)^{\frac{p-2}{2}}\eta(x)|\tau_{s,h}Du(x)|\cdot$$
$$\cdot\, c|h|^{\theta_1}(\nu + |Du(x)| + |Du(x+he_s)|)^{\frac{p}{2}}\eta(x)\,dx \leq$$
$$\leq \left\{\int_{B_R}(\nu + |Du(x)| + |Du(x+he_s)|)^{p-2}\eta^2(x)|\tau_{s,h}Du(x)|^2\,dx\right\}^{\frac{1}{2}}\cdot$$
$$\cdot\, c|h|^{\theta_1}\left\{\int_{B_R}(\nu + |Du(x)| + |Du(x+he_s)|)^{p}\eta^2(x)\,dx\right\}^{\frac{1}{2}} \leq$$
$$\leq \frac{m}{4}\int_{B_R}(\nu + |Du(x)| + |Du(x+he_s)|)^{p-2}\eta^2(x)|\tau_{s,h}Du(x)|^2\,dx +$$
$$+ c_{13}|h|^{2\theta_1}\int_{B_R}(1 + |Du(x)|^p + |Du(x+he_s)|^p)\,dx \leq$$
$$\leq \frac{m}{4}\int_{B_R}(\nu + |Du(x)| + |Du(x+he_s)|)^{p-2}\eta^2(x)|\tau_{s,h}Du(x)|^2\,dx +$$
$$+ c_{13}|h|^{2\theta_1}\,2\int_{B_{2R}}(1 + |Du(x)|^p)\,dx$$

thus

$$|(VII)| \leq \frac{m}{4}\int_{B_R}(\nu + |Du(x)| + |Du(x+he_s)|)^{p-2}\eta^2(x)|\tau_{s,h}Du(x)|^2\,dx + c_{14}|h|^{2\theta_1}, \tag{3.10}$$

for some positive constant c_{14} independent on h. Now (3.8),..., (3.10) merge into

$$\frac{3}{4}m\int_{B_R}(\nu + |Du(x)| + |Du(x+he_s)|)^{p-2}|\tau_{s,h}Du(x)|^2\eta^2(x)\,dx - c_{14}|h|^{2\theta_1} \leq (I). \tag{3.11}$$

Using (3.1), (3.4), ..., (3.7), (3.11) recalling $0 \leq \nu$, $2\theta_1 \leq \theta_1 + 1$ and $|h| < 1$, we get

$$\frac{m}{4}\int_{B_R}(\eta(x))^2\big(|Du(x)| + |Du(x+he_s)|\big)^{p-2}|\tau_{s,h}Du(x)|^2\,dx \leq c_{15}|h|^{\min\{2\theta_1,\frac{2}{2-\theta_2}\}}, \tag{3.12}$$

for some positive constant c_{15} independent on h. Now we use Lemma 2.5 and Corollary 2.7 in order to get

$$\int_{B_R} \left|\tau_{s,h}\left(|Du(x)|^{(p-2)/2}\, Du(x)\right)\right|^2 \eta^2(x)\, dx \leq c_{16}|h|^{\gamma}, \tag{3.13}$$

for every $s = 1, \ldots, n$, for every h with $|h| < R$, where c_{16} is a positive constant independent on h and γ is defined in (1.8). Since $\eta = 1$ on B_ρ, if $\gamma < 2$ inequality (3.13) allows us to apply Lemma 2.3 in order to get

$$|Du|^{(p-2)/2}\, Du \in L^r(B_{\rho/4}), \qquad \forall r < 2n/(n-\gamma).$$

We remark that $|\,|Du|^{(p-2)/2}\, Du| = |Du|^{p/2}$, thus (1.8) is completely proven. If $\gamma = 2$ we use (3.13), Lemma 2.2 and Sobolev imbedding theorem for $W^{1,2}$, thus Remark 1 is completely proven. □

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