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# ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS IN UNBOUNDED DOMAINS OF $\mathbb{R}^2$

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*Dedicated to Professor Francesco Guglielmino on his seventieth birthday*

In this paper we are concerned with second order elliptic equations in unbounded domains $\Omega$ of $\mathbb{R}^2$. We establish existence and uniqueness theorems under the assumptions that the leading coefficients are bounded and measurable in $\Omega$ and satisfy a suitable condition at infinity.

## Introduction.

Let $\Omega$ a sufficiently regular open subset of $\mathbb{R}^2$.

In $\Omega$ we consider the second order linear differential operator

$$Lu := -\sum_{i,j=1}^{2} a_{ij} u_{x_i x_i} + \sum_{i=1}^{2} a_i u_{x_i} + au, \tag{1}$$

which is uniformly elliptic with symmetric, bounded and measurable leading coefficients, i.e.

$$a_{ji} = a_{ij} \in L^\infty(\Omega), \quad \sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \geq \nu|\xi|^2 \quad \text{a.e. in } \Omega \quad \forall \xi \in \mathbb{R}^2, \tag{2}$$

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where $\nu$ is a positive constant.

As well known, the Dirichlet problem

$$u \in W^2(\Omega) \cap W_0^1(\Omega), \quad Lu = f, \quad f \in L^2(\Omega), \tag{3}$$

has been exhaustively studied (see [9]) under the only assumption (2) in the case of a bounded domain $\Omega$.

Indeed, assuming that $a_{ij}$ satisfy (2), whilst $a_i$ and $a$ are bounded and measurable, G. Talenti [9] has established for a solution $u$ of (3) the estimate

$$|u_{xx}|_{2,\Omega} \leq c(|f|_{2,\Omega} + |u|_{2,\Omega}) \tag{4}$$

with $c$ independent of $u$; by using (4) and an uniqueness result of C. Pucci (see [8]), he has also shown that problem (3) is uniquely solvable when

$$\operatorname*{essinf}_{\Omega} a \geq 0.$$

In this paper we study the same problem (3) when $\Omega$ is an unbounded domain.

After recalling (see Sec. 1) definitions and properties of the spaces of Morrey type $M^p(\Omega)$, $VM^p(\Omega)$, $\widetilde{M}^p(\Omega)$, $M_0^p(\Omega)$, introduced and studied in [11], [14], we prove (see Sec. 2) that the a-priori bound (4) still holds true, assuming (2) and

$$a_i \in \widetilde{M}^s(\Omega) \quad \text{for some } s > 2, \quad a \in \widetilde{M}^2(\Omega). \tag{5}$$

Plainly, in the case of unbounded domains the above estimate (4) is not sufficient to get an existence and uniqueness result.

In order to do this our method proceeds through a $L^\infty$-bound of Pucci type

$$\sup_{\Omega} |u| \leq c|f|_{2,\Omega} \tag{6}$$

and a $W^2$-estimate of type

$$\|u_{xx}\|_{W^2(\Omega)} \leq c(|f|_{2,\Omega} + |u|_{2,\Omega_0}), \tag{7}$$

where $\Omega_0$ is a bounded open subset of $\Omega$, to be satisfied by a solution $u$ of problem (3) in an unbounded domain $\Omega$, with $c$ and $\Omega_0$ independent of $u$ and $f$.

By virtue of the already given assumptions on the coefficients of the operator $L$, the a-priori bound (6) is contained in a recent paper (see [15]).

For the estimate (7) we need further conditions (at infinity) about the boundary of $\Omega$ and the behavior of the coefficients of $L$. Precisely, in order to get (7), we suppose $\partial\Omega$ has non-negative curvature outside some closed ball $\overline{B}_{r_0}$ of sufficiently large radius $r_0$ and centered at the origin, a.e. with respect to the one-dimensional Hausdorff measure on $\partial\Omega$, and the coefficients of $L$ satisfy (2) together with the following conditions:

$$a_i \in M_0^s(\Omega) \text{ for some } s > 2, \quad a = a' + b \in \widetilde{M}^s(\Omega), \quad a' \in M_0^2(\Omega), \tag{8}$$

and

$$\mu^{-2} \operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_1}} \sum_{i=1}^{2} (e_{ij} - g a_{ij})^2 + \mu_1^{-2} \operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_1}} (e - gb)^2 < 1 \tag{9}$$

for a sufficiently large $r_1$ with $\mu, \mu_1 \in \mathbb{R}_+$ and $e_{ij}, e \in L^\infty(\Omega)$ such that

$$e_{ji} = e_{ij}, \quad \sum_{i,j=1}^{2} e_{ij}\xi_i\xi_j \geq \mu|\xi|^2 \quad \text{a.e. in } \Omega \quad \forall \xi \in \mathbb{R}^2,$$

$$(e_{ij})_{x_k}, e_{x_k} \in M_0^s(\Omega) \text{ for some } s > 2, \quad \operatorname*{essinf}_{\Omega} e \geq \mu_1,$$

$$g \in L^\infty(\Omega), \quad \operatorname*{essinf}_{\Omega} g > 0.$$

We notice that (9) implies

$$\operatorname*{essinf}_{\Omega\setminus\overline{B}_{r_1}} b > 0. \tag{10}$$

On the other side, we remark that (9) holds true for any $b$ satisfying (10) if the coefficients $a_{ij}$ converge at infinity (see Remark 3.5) and that for any matrix-function with coefficients $a_{ij}$ satisfying (2) there exists a $b$ verifying (9) (see (2.5)).

Alternatively, we prove (7), for a sufficiently regular domain $\Omega$, when conditions (2), (8), (10) are verified and the operator

$$L_0 u := -\sum_{i,j=1}^{2} a_{ij} u_{x_i x_i} \tag{11}$$

can be approximated (at infinity) by means of an uniformly elliptic operator $-\sum_{i,j=1}^{2} \alpha_{ij} u_{x_i x_i}$ having coefficients $\alpha_{ij}$ such that

$$\alpha_{ji} = \alpha_{ij} \in L^\infty(\Omega), \quad (\alpha_{ij})_{x_k} \in M_0^s(\Omega) \text{ for some } s > 2.$$

Finally, by using (7) together with the results of [15], we show that (3) is a zero index problem, uniquely solvable when $a' = 0$.

We also remark that such conclusions can fail when (10) is not satisfied. For instance (see [2]) we have uniqueness, but not always existence, when we consider the Dirichlet problem

$$u \in W^2(\mathbb{R}^2), \quad -\Delta u = f, \quad f \in L^2(\mathbb{R}^2).$$

## 1. The spaces of Morrey type $M^p(\Omega)$, $VM^p(\Omega)$, $\widetilde{M}^p(\Omega)$, $M_0^p(\Omega)$.

In this section we introduce the notations which will be used throughout the paper.

For $x \in \mathbb{R}^2$ and $r \in \mathbb{R}_+$ we set

$$B(x,r) := \{y \in \mathbb{R}^2 : |y - x| < r\},$$

in particular $B_r := B(0,r)$.

We denote by $\zeta_1$ a function of class $C_0^\infty(\mathbb{R}^2)$ such that

$$0 \le \zeta_1 \le 1, \quad \zeta_1 = 1 \text{ on } \overline{B}_1, \quad \zeta_1 = 0 \text{ on } \mathbb{R}^2 \setminus B_2,$$

and put

$$\zeta_r(x) := \zeta_1(x/r), \quad x \in \mathbb{R}^2.$$

For an open subset $\Omega$ of $\mathbb{R}^2$ we let

$$\Omega(x,r) := \Omega \cap B(x,r), \quad \Omega(x) := \Omega(x,1), \quad \Omega_r := \Omega(0,r)$$

and denote by $\Sigma(\Omega)$ the $\sigma$-algebra of the Lebesgue-measurable subsets of $\Omega$.

For $p \in [1, +\infty]$, if $A \in \Sigma(\Omega)$ and $g \in L^p(A)$, we put

$$|A| := \text{ Lebesgue-measure of } A,$$

$$\chi_A := \text{ characteristic function of } A,$$

$$|g|_{p,A} := \|g\|_{L^p(A)}.$$

Introducing $\mathcal{D}(\overline{\Omega})$, the class of the restrictions to $\Omega$ of the functions in $C_0^\infty(\mathbb{R}^2)$, and $L^p_{\mathrm{loc}}(\overline{\Omega})$, the class of the functions $g : \Omega \to \mathbb{R}$ such that $\zeta g \in L^p(\Omega)$ for every $\zeta \in \mathcal{D}(\overline{\Omega})$, we define $M^p(\Omega)$ as the space of the functions $g \in L^p_{\mathrm{loc}}(\overline{\Omega})$ such that

$$\|g\|_{M^p(\Omega)} := \sup_{x\in\Omega} |g|_{p,\Omega(x)} < +\infty, \tag{1.1}$$

endowed with the norm given in (1.1).

We also need the following subspaces of $M^p(\Omega)$:

$VM^p(\Omega)$, the subspace of the functions $g \in M^p(\Omega)$ such that

$$\eta_p[g, \Omega](\tau) := \sup_{x\in\Omega} |g|_{p,\Omega(x,\tau)} \to 0 \quad \text{as } \tau \to 0;$$

$\widetilde{M}^p(\Omega)$, the subspace of the functions $g \in M^p(\Omega)$ such that

$$\sigma_p[g, \Omega](\tau) := \sup_{\substack{A\in\Sigma(\Omega)\\ |A(x)|\le\tau\ \ \forall x\in\Omega}} \|\chi_A g\|_{M^p(\Omega)} \to 0 \text{ as } \tau \to 0;$$

$M_0^p(\Omega)$, the subspaces of the functions $g \in M^p(\Omega)$ such that

$$\theta_p[g, \Omega](r) := \|(1-\zeta_r)u\|_{M^p(\Omega)} \to 0 \quad \text{as } r \to \infty.$$

Clearly, it turns out that $\widetilde{M}^p(\Omega) \subset VM^p(\Omega)$ and for every $g \in \widetilde{M}^p(\Omega)$

$$\eta_p[g, \Omega](\tau) \le \sigma_p[g, \Omega](\tau);$$

moreover (see Lemma 2.1 of [11])

$$M_0^p(\Omega) \subset \widetilde{M}^p(\Omega).$$

Furthermore we call:

modulus of continuity of $g \in VM^p(\Omega)$ any function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\eta(\tau) \to 0 \quad \text{as } \tau \to 0,\ \ \eta_p[g, \Omega](\tau) \le \eta(\tau)\ \forall\tau \in \mathbb{R}_+;$$

modulus of continuity of $g \in \widetilde{M}^p(\Omega)$ any function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sigma(\tau) \to 0 \quad \text{as } \tau \to 0,\ \ \sigma_p[g, \Omega](\tau) \le \sigma(\tau) \quad \forall\tau \in \mathbb{R}_+;$$

modulus of continuity of $g \in M_0^p(\Omega)$ any function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\theta(r) \to 0 \quad \text{as } r \to +\infty,\ \ \sigma_p[g, \Omega](1/r) + \theta_p[g, \Omega](r) \le \theta(r) \quad \forall r \in \mathbb{R}_+ .$$

The above-mentioned spaces have been introduced in [10] and represent the particular case $\lambda = 0$ of the spaces $M^{p,\lambda}(\Omega)$, which have been defined in [14].

From [10] and [14] we also infer the following two lemmas.

**Lemma 1.1.** *$\widetilde{M}^p(\Omega)$ is the closure of $L^\infty(\Omega)$ in $M^p(\Omega)$; $M_0^p(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $M^p(\Omega)$.*

**Lemma 1.2.** *Let $k \in \mathbb{N}$, $p \in [2, +\infty[$, with $p > 2$ if $k = 1$, and suppose $\Omega$ endowed with the cone property. Then for every $g \in M^p(\Omega)$ and $u \in W^k(\Omega)$ we have $gu \in L^2(\Omega)$ and*

$$|gu|_{2,\Omega} \leq c\|g\|_{M^p(\Omega)}\|u\|_{W^2(\Omega)},$$

*where $c$ is a positive constant depending only on $p, k$ and the characteristic cone of $\Omega$.*

From the previous lemmas we easily deduce the following further results.

**Lemma 1.3.** *If the assumptions of Lemma 1.2 are verified and $g \in \widetilde{M}^p(\Omega)$, then for any $\varepsilon \in \mathbb{R}_+$ the bound*

$$|gu|_{2,\Omega} \leq \varepsilon\|u\|_{W^2(\Omega)} + c(\varepsilon)|u|_{2,\Omega}\,, \quad u \in W^k(\Omega),$$

*holds true with a positive constant $c(\varepsilon)$ depending only on $\varepsilon$, $p, k$, the modulus of continuity of $g \in \widetilde{M}^p(\Omega)$ and the characteristic cone of $\Omega$.*

**Lemma 1.4.** *If the assumptions of Lemma 1.2 are verified and $g \in M_0^p(\Omega)$, then there exist $c(\varepsilon) \in \mathbb{R}_+$ and an open subset $\Omega(\varepsilon) \subset\subset \Omega$ such that for any $\varepsilon \in \mathbb{R}_+$*

$$|gu|_{2,\Omega} \leq \varepsilon\|u\|_{W^2(\Omega)} + c(\varepsilon)|u|_{2,\Omega(\varepsilon)}\,, \quad \forall u \in W^k(\Omega),$$

*with $c(\varepsilon)$ and $\Omega(\varepsilon)$ depending only on $\varepsilon$, $p, k$, the modulus of continuity of $g \in M_0^p(\Omega)$ and the characteristic cone of $\Omega$.*

**Lemma 1.5.** *If the assumptions of Lemma 1.2 are verified, then for every $g \in M_0^p(\Omega)$ the operator*

$$u \in W^k(\Omega) \to gu \in L^2(\Omega)$$

*is compact.*

For a function $u$ defined on $\Omega$ having derivatives in the sense of the distributions, we will make use of the following notations:

$$u_x = \left(u_{x_1}^2 + u_{x_2}^2\right)^{\frac{1}{2}}, \quad u_{xx} = \left(u_{x_1x_1}^2 + 2u_{x_1x_2}^2 + u_{x_2x_2}^2\right)^{\frac{1}{2}}.$$

## 2. Preliminary lemmas.

In the sequel we suppose the open subset $\Omega$ of $\mathbb{R}^2$ has the uniform $C^2$-regularity property according to R.A. Adams [1] (see 4.6):

$i_1$) there exist $d \in \mathbb{R}_+$, $k \in \mathbb{N}$, an open covering $\{U_i\}_{i\in\mathbb{N}}$ of $\partial\Omega$ and diffeomorphisms $\Phi_i : U_i \to B_1$, $i \in \mathbb{N}$, of class $C^2$ such that

1) $\{x \in \Omega / \mathrm{dist}(x, \partial\Omega) < d\} \subset \bigcup_{i\in\mathbb{N}} \Phi_i^{-1}(B(0, 1/2))$;
2) every collection of $k + 1$ of the sets $U_i$ has empty intersection;
3) $\Phi_i(U_i \cap \Omega) = \{x \in B_1 / x_2 > 0\}$, $i \in \mathbb{N}$;
4) the components of $\Phi_i$ and $\Phi_i^{-1}$, together with first and second derivatives, are all bounded by a constant independent of $i \in \mathbb{N}$.

Let us consider the differential operator $L$ defined in (1) with principal term $L_0$ given by (11).

If (2) is verified, $a_i \in M^s(\Omega)$ for some $s > 2$, $a \in M^2(\Omega)$, then we put

$$\beta := \max\{\max_{i,j} |a_{ij}|_{\infty,\Omega}, \max_i \|a_i\|_{M^s(\Omega)}, \|a\|_{M^2(\Omega)}\}.$$

**Lemma 2.1.** *Assuming $i_1$), (2), $a_i \in VM^s(\Omega)$ for some $s > 2$, $a \in M^2(\Omega)$ and*

$$a_0 := \operatorname*{ess\,inf}_{\Omega} a > 0, \tag{2.1}$$

*we have the bound*

$$\sup_{\Omega} |u| \le c|Lu|_{2,\Omega}\ , \quad \forall u \in W^2(\Omega) \cap W_0^1(\Omega), \tag{2.2}$$

*where $c$ is a constant depending only on $\nu$, $\beta$, $a_0$ and the moduli of continuity of $a_i \in VM^s(\Omega)$.*

*Proof.* As a consequence of well known results about Sobolev spaces. A function $u \in W^2(\Omega) \cap W_0^1(\Omega)$ has the following properties:

$$u \in C^0(\overline{\Omega}), \quad u = 0 \quad \text{on } \partial\Omega, \quad \lim_{|x|\to+\infty} u(x) = 0.$$

So we deduce the assertion from the results of [15]. $\square$

Let us suppose

$i_2$) the coefficient of $L$ verify (2) and (5).

It is known (e.g., see [4], [9]) that the uniform ellipticity of $L$ in an open subset $\Omega$ of $\mathbb{R}^2$ is equivalent to Cordes' hypothesis:

$$\operatorname*{ess\,inf}_{\Omega} \frac{\left(\sum_{i=1}^{2} a_{ii}\right)^2}{\sum_{i,j=1}^{2} a_{ij}^2} > 1. \tag{2.3}$$

If we put

$$\varepsilon_0 := \operatorname*{ess\,inf}_{\Omega} \frac{\left(\sum_{i=1}^{2} a_{ii}\right)^2}{\sum_{i,j=1}^{2} a_{ij}^2} - 1, \quad \gamma := \operatorname*{ess\,inf}_{\Omega} \frac{\sum_{i=1}^{2} a_{ii}}{\sum_{i,j=1}^{2} a_{ij}^2}, \tag{2.4}$$

we have

$$\operatorname*{ess\,sup}_{\Omega} \sum_{i,j=1}^{2} (\delta_{ij} - \gamma a_{ij})^2 = 1 - \varepsilon_0$$

and so (2.3) is equivalent to the condition

$$\operatorname*{ess\,sup}_{\Omega} \sum_{i,j=1}^{2} (\delta_{ij} - \gamma a_{ij})^2 < 1. \tag{2.5}$$

**Lemma 2.2.** *Assuming $i_1$) and $i_2$), we have bound*

$$|u_{xx}|_{2,\Omega} \le c(|Lu + \lambda u|_{2,\Omega} + |u|_{2,\Omega}), \qquad \forall u \in W^2(\Omega) \cap W_0^1(\Omega) \text{ and } \forall \lambda \in [0, +\infty[, \tag{2.6}$$

*where $c$ is a constant depending only on $\Omega$, $\nu$, $\beta$ and the moduli of continuity of $a_i \in \widetilde{M}^s(\Omega)$, $i = 1, 2$, and of $a \in \widetilde{M}^2(\Omega)$.*

*Proof.* From Theorem 3 of [12] we have (2.6) with $L_0$ instead of $L$, and so we obtain the result by applying Lemma 1.3. $\square$

## 3. Conditions at infinity on the coefficients $a_{ij}$.

Let $\mu \in \mathbb{R}_+$ and $k \in \mathbb{N}$.

We denote by $E_k(\mu, \Omega)$ the class of the $k \times k$ matrix-functions $((e_{ij}))$ such that

$$e_{ji} = e_{ij} \in L^\infty(\Omega), \quad \sum_{i,j=1}^{k} e_{ij}\xi_i\xi_j \geq \mu|\xi|^2 \text{ a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}^k,$$

$$(e_{ij})_{x_k} \in M_0^s(\Omega) \quad \text{for some } s > 2.$$

Moreover we put

$$\mathcal{G}(\Omega) := \{g \in L^\infty(\Omega) : \operatorname*{ess\,inf}_{\Omega} g > 0\}.$$

We will use the pair $(a_{ij}, b)$ to indicate the operator

$$L_0 u + bu, \quad u \in W^2(\Omega),$$

with $L_0$ given by (11) and $b \in \widetilde{M}^2(\Omega)$ such that $\operatorname*{ess\,inf}_{\Omega \setminus \overline{B}_r} b > 0$ for some $r \in \mathbb{R}_+$.

**Hypothesis 3.1.** There exist $\mu, \mu_1, r_1 \in \mathbb{R}_+$, $e_{ij} \in E_2(\mu, \Omega)$, $e \in E_1(\mu_1, \Omega)$, $g \in \mathcal{G}(\Omega)$ such that

$$\mu^{-2} \operatorname*{ess\,sup}_{\Omega \setminus \overline{B}_{r_1}} \sum_{i,j=1}^{2} (e_{ij} - g a_{ij})^2 + \mu_1^{-2} \operatorname*{ess\,sup}_{\Omega \setminus \overline{B}_{r_1}} (e - gb)^2 < 1. \tag{3.1}$$

To be more explicit, we will also say that $(a_{ij}, b)$ verifies Hypothesis 3.1 (with respect to $(e_{ij}, e, g)$).

**Remark 3.2.** As a consequence of (2.5), in order that $(a_{ij}, b)$ verifies Hypothesis 3.1 (with respect to $(e_{ij}, e, \gamma)$, where $\gamma$ has been defined in (2.4)) it is sufficient that there exist $\mu_1, r_0 \in \mathbb{R}_+$, $e \in E_1(\mu_1, \Omega)$, such that

$$\operatorname*{ess\,sup}_{\Omega \setminus \overline{B}_{r_0}} |e - \gamma b| < \mu_1 \sqrt{\varepsilon_0}. \tag{3.2}$$

**Remark 3.3.** Let $\mu, r \in \mathbb{R}_+$, $e_{ij} \in E_2(\mu, \Omega)$, $g \in \mathcal{G}(\Omega)$, such that

$$\alpha = 1 - \mu^{-2} \operatorname*{ess\,sup}_{\Omega \setminus \overline{B}_r} \sum_{i,j=1}^{2} (e_{ij} - g a_{ij})^2 > 0. \tag{3.3}$$

As a consequence of Remark 4.1 of [3], Hypothesis 3.1 is satisfied (by $(a_{ij}, b)$) if there exist $r_0 \in \mathbb{R}_+$ such that

$$\operatorname*{essinf}_{\Omega\setminus\overline{B}_{r_0}} (gb) > (1 - \sqrt{\alpha}) \operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_0}} (gb). \tag{3.4}$$

**Remark 3.4.** From (2.5) and Remark 3.3 we deduce that Hypothesis 3.1 is satisfied (by $(a_{ij}, b)$) if there exists $r \in \mathbb{R}_+$ such that

$$\frac{\operatorname*{essinf}_{\Omega\setminus\overline{B}_r} (\gamma b)}{\operatorname*{esssup}_{\Omega\setminus\overline{B}_r} (\gamma b)} > 1 - \sqrt{\operatorname*{essinf}_{\Omega\setminus\overline{B}_r} \frac{(a_{11} + a_{22})^2}{a_{11}^2 + 2a_{12}^2 + a_{22}^2} - 1}\ .$$

**Remark 3.5.** As a consequence of Remark 3.3, Hypothesis 3.1 is satisfied (by $(a_{ij}, b)$), whatever $b$ is, in the case of

$$a_{ij} = a'_{ij} + a''_{ij}\,,\ (a'_{ij})_{x_k} \in M_0^s(\Omega) \text{ for some } s > 2,\ \lim_{|x|\to+\infty} a''_{ij} = a^0_{ij} \in \mathbb{R},$$

because (3.3) and (3.4) can be satisfied by taking $\mu = \nu/2, r_0 \in \mathbb{R}_+$, $e_{ij} = a'_{ij} + a^0_{ij}$, $g = 1$, such that

$$\operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_0}} |a''_{ij} - a^0_{ij}| < \frac{\nu}{2}\left[1 - \left(1 - \frac{\operatorname*{essinf}_{\Omega\setminus\overline{B}_{r_0}} b}{\operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_0}}}\right)^2\right]^{\frac{1}{2}}.$$

We also observe (see note (1) of M. Giaquinta [5] and Proposition 1 of M. Chicco [2]) that, if we set

$$g_0 := \frac{\mu^{-2} \sum\limits_{i,j=1}^{2} e_{ij} a_{ij} + \mu_1^{-2} eb}{\mu^{-2} \sum\limits_{i,j=1}^{2} a_{ij}^2 + \mu_1^{-2} b^2}, \tag{3.5}$$

then for any function $f : \Omega \to \mathbb{R}$ we have

$$\mu^{-2} \sum_{i,j=1}^{2} (e_{ij} - g_0 a_{ij})^2 + \mu_1^{-2}(e - g_0 b)^2 \le \mu^{-2} \sum_{i,j=1}^{2} (e_{ij} - f a_{ij})^2 + \mu_1^{-2}(e - f b)^2.$$

Therefore a pair $(a_{ij}, b)$ verifies Hypothesis 3.1 with respect to $(e_{ij}, e, g)$ if and only if $(a_{ij}, b)$ does it with respect to $(e_{ij}, e, g_0)$.

Moreover

$$\mu^{-2} \sum_{i,j=1}^{2} (e_{ij} - g_0 a_{ij})^2 + \mu_1^{-2}(e - g_0 b)^2 =$$

$$= \mu^{-2} \sum_{i,j=1}^{2} e_{ij}^2 + \mu_1^{-2} e^2 - \frac{\left(\mu^{-2} \sum\limits_{i,j=1}^{2} e_{ij} a_{ij} + \mu_1^{-2} eb\right)^2}{\mu^{-2} \sum\limits_{i,j=1}^{2} a_{ij}^2 + \mu_1^{-2} b^2},$$

and so $(a_{ij}, b)$ verifies Hypothesis 3.1 with respect to $(e_{ij}, e, g)$ if and only if

$$\operatorname*{ess\,sup}_{\Omega \setminus \overline{B}_{r_0}} \left[ \mu^{-2} \sum_{i,j=1}^{2} e_{ij}^2 + \mu_1^{-2} e^2 - \frac{\left(\mu^{-2} \sum\limits_{i,j=1}^{2} e_{ij} a_{ij} + \mu_1^{-2} eb\right)^2}{\mu^{-2} \sum\limits_{i,j=1}^{2} a_{ij}^2 + \mu_1^{-2} b^2} \right] < 1.$$

## 4. A-priori bounds.

We state in advance some lemmas.

**Lemma 4.1.** *If $\Omega$ has the uniform $C^2$-regularity property, then each $u \in W^2(\Omega) \cap W_0^1(\Omega)$ is the limit in $W^2(\Omega)$ of a sequence $\{u_n\}_{n \in \mathbb{N}}$ such that*

$$u_n \in W^2(\Omega) \cap C^2(\overline{\Omega}), \quad u_n = 0 \text{ on } \partial\Omega.$$

*Proof.* Let us take $v_n \in \mathcal{D}(\overline{\Omega})$, $n \in \mathbb{N}$, such that

$$v_n \to u \quad \text{in } W^2(\Omega). \tag{4.1}$$

By virtue of Theorem 5.4 of [11] for each $n \in \mathbb{N}$ there exists a solution $u_n \in W^2(\Omega) \cap W_0^1(\Omega)$ of the equation

$$-\Delta u_n + u_n = -\Delta v_n + v_n; \tag{4.2}$$

from Theorem 5.1 of [2] we deduce that $u_n \in W^{2,p}(\Omega)$ for every $p \in [2, +\infty[$; so in particular $u_n \in C^0(\overline{\Omega})$, whence, by known results (see [6]), $u_n \in C^2(\overline{\Omega})$.

On the other side, as a consequence of Theorem 4.2 of [11], the solution

$$u_n - u \in W^2(\Omega) \cap W_0^1(\Omega)$$

of the equation

$$-\Delta(u_n - u) + (u_n - u) = -\Delta(v_n - u) + (v_n - u)$$

satisfies a bound of the type

$$\|u_n - u\|_{W^2(\Omega)} \leq c| - \Delta(v_n - u) + (v_n - u)|,$$

with $c \in \mathbb{R}_+$ independent of $n$, whence the result. □

**Lemma 4.2.** *Let $\Omega$ have the uniform $C^2$-regularity property and $r_0 \in \mathbb{R}_+$ be such that the curvature is non-negative on $\partial\Omega \setminus \overline{B}_{r_0}$ a.e. with respect to the one-dimensional Hausdorff measure on $\partial\Omega$.*

*Let $u \in W^2(\Omega) \cap W_0^1(\Omega)$ and $r > r_0$.*

*If $e_{ij} \in E_2(\mu, \Omega \setminus \overline{B}_{r_0})$, then the function*

$$u_r := (1 - \zeta_r)u$$

*satisfies the inequality*

$$\mu^2 \int_\Omega (u_r)_{xx}^2 \, dx \leq \int_\Omega \Big| - \sum_{i,j=1}^{2} e_{ij}(u_r)_{x_i x_j} \Big|^2 dx + \tag{4.3}$$
$$+ \sum_{i,j,h,k=1}^{2} \int_\Omega [(e_{ij} e_{hk})_{x_j} (u_r)_{x_i} (u_r)_{x_h x_k} - (e_{ij} e_{hk})_{x_h} (u_r)_{x_i} (u_r)_{x_k x_j}] \, dx.$$

*Proof.* By virtue of Lemma 4.1 we can suppose

$$u \in W^2(\Omega) \cap C^2(\overline{\Omega}), \quad u = 0 \text{ on } \partial\Omega.$$

Setting

$$w_\rho := \zeta_\rho u_r, \quad \rho \in \mathbb{R}_+,$$

from classical results we deduce that

$$\mu^2 \int_\Omega (w_\rho)_{xx}^2\, dx + \tag{4.4}$$

$$+ \int_{\partial\Omega} \sum_{i,j,h,k=1}^{2} e_{ij} e_{hk} [(w_\rho)_{x_h x_k} (w_\rho)_{x_i} n_j - (w_\rho)_{x_j x_k} (w_\rho)_{x_i} n_h]\, d\ell \leq$$

$$\leq \int_\Omega | - \sum_{i,j=1}^{2} e_{ij} (w_\rho)_{x_i x_j} |^2\, dx +$$

$$+ \sum_{i,j,h,k=1}^{2} \int_\Omega [(e_{ij} e_{hk})_{x_j} (w_\rho)_{x_i} (w_\rho)_{x_h x_k} - (e_{ij} e_{hk})_{x_h} (w_\rho)_{x_i} (w_\rho)_{x_k x_j}]\, dx,$$

with $n = (n_1, n_2)$ the unit outward normal to $\partial\Omega$.

By proceeding as in [7] and using the assumption on the curvature, the line integral along $\partial\Omega$ turns out to be non-negative, and so (4.4) yields (4.3) for $w_\rho$ (in the place of $u_r$).

From this we get the result, letting $\rho \to +\infty$, by the dominated convergence theorem of Lebesgue. $\square$

We will consider the following two conditions alternatively:

$i_3$) Hypothesis 3.1 is satisfied and there exists $r_0 \in \mathbb{R}_+$ such that the curvature is non-negative on $\partial\Omega \setminus B_{r_0}$ a.e. with respect to the one-dimensional measure of Hausdorff on $\partial\Omega$;

$i_3'$) there exist $\mu, \mu_1 \in \mathbb{R}_+$, $((\alpha_{ij})) \in E_2(\mu, \Omega)$ and, for any $\varepsilon \in \mathbb{R}_+$, $r_\varepsilon \in \mathbb{R}_+$ such that

$$\operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_\varepsilon}} |\alpha_{ij} - a_{ij}| \leq \varepsilon\,, \quad \operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_\varepsilon}} b \geq \mu_1.$$

**Remark 4.3.** Condition $i_3'$) implies Hypothesis 3.1. In fact, if $i_3'$) holds true, then (3.1) is satisfied choosing $\mu, \mu_1 \in \mathbb{R}_+$, as given by $i_3'$), $e_{ij} = \alpha_{ij}$, $e = \mu$, $g = 1$, for a sufficiently large $r_1$.

We will set

$$\beta' \geq \max\{\beta, |e_{ij}|_{\infty,\Omega}, |e|_{\infty,\Omega}, |g|_{\infty,\Omega}\},$$

with $\beta$ defined in Section 2, and $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, $\gamma(\tau) \to 0$ as $\tau \to 0$, such that

$$\gamma(\tau) \geq \theta_s[(e_{ij})_x, \Omega] + \theta_s[e_x, \Omega],$$

if $i_3$) is verified, whilst

$$\beta' \geq \max\{\beta, |\alpha_{ij}|_{\infty,\Omega}, \operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_\varepsilon}} b\},$$

and

$$\gamma(\tau) \geq \theta_s[(\alpha_{ij})_x, \Omega],$$

if $i_3'$) is verified.

We will also make use of the following condition:

$i_4$) $a_i \in M_0^s(\Omega)$ for some $s > 2$, $a = a' + b$, with $a' \in M_0^2(\Omega)$.

**Lemma 4.4.** *If conditions $i_1$), $i_2$), $i_3$), $i_4$) are verified, then there exists $r^* \in \mathbb{R}_+$ such that*

$$\|(1-\zeta_r)u\|_{W^2(\Omega)} \leq c|L[(1-\zeta_r)u]|_{2,\Omega} \tag{4.5}$$

*for every $u \in W^2(\Omega) \cap W_0^1(\Omega)$ and $r > r^*$, where $c$ is a positive constant depending only on $\Omega$, $\mu$, $\mu_1$, $\beta'$, $\gamma(\tau)$, $\operatorname*{essinf}_{\Omega} g$, and the moduli of continuity of $a_i \in M_0^s(\Omega)$, $a' \in M_0^2(\Omega)$, $b \in \widetilde{M}^2(\Omega)$.*

*Proof.* Starting from inequality (4.3) and proceeding as in the proof of Lemma 6 of [13], we can find a bounded open subset $\Omega_0$ of $\Omega$ such that

$$\|(1-\zeta_r)u\|_{W^2(\Omega)} \leq c(|L[(1-\zeta_r)u]|_{2,\Omega} + |(1-\zeta_r)u|_{2,\Omega_0}) \tag{4.6}$$

for $r > \max\{r_0, r_1\}$, whence the result follows at once. □

**Theorem 4.5.** *If conditions $i_1$), $i_2$), $i_3$) or $i_3'$) (alternatively, $i_4$) are verified, then there exist $c \in \mathbb{R}_+$ and a bounded open subset $\Omega_0$ of $\Omega$ such that*

$$\|u\|_{W^2(\Omega)} \leq c(|Lu|_{2,\Omega} + |u|_{2,\Omega_0}), \quad \forall u \in W^2(\Omega) \cap W_0^1(\Omega), \tag{4.7}$$

*with $c$ and $\Omega_0$ depending only on $\Omega$, $\mu$, $\mu_1$, $\beta'$, $\gamma(\tau)$, $\operatorname*{essinf}_{\Omega} g$, and the moduli of continuity of $a_i \in M_0^s(\Omega)$, $a' \in M_0^2(\Omega)$, $b \in \widetilde{M}^2(\Omega)$.*

*Proof.* Firstly, we consider the case when $i_1$), $i_2$), $i_3$), $i_4$) are verified.

Let $r^* \in \mathbb{R}_+$ as in Lemma 4.4. By applying Lemma 2.2 to $\zeta_r u$ and using (4.5), for $r > r^*$ we have:

$$\|u\|_{W^2(\Omega)} \leq c_1(|Lu|_{2,\Omega} + |L(\zeta_r u)|_{2,\Omega} + |\zeta_r u|_{2,\Omega}). \tag{4.8}$$

From $i_2$), by virtue of Lemma 1.3, we deduce that

$$|L(\zeta_r u)|_{2,\Omega} \leq |Lu|_{2,\Omega} + \varepsilon \|u\|_{W^2(\Omega)} + c(\varepsilon)|u|_{2,\Omega_r} \tag{4.9}$$

with $\Omega_r$ a bounded open subset of $\Omega$, whence (4.7) in the present case.

Now, let us suppose that $i_1$), $i_2$), $i_3'$), $i_4$) are verified.

In this case (see, e.g., Theorem 4.4 of [11]) there exist $c_2$ and a bounded open subset $\Omega'$ of $\Omega$ such that

$$\|(1-\zeta_r)u\|_{W^2(\Omega)} \leq c_2\Big(|L[(1-\zeta_r)u] + \\ + \sum_{i,j=1}^{2}(a_{ij}-\alpha_{ij})[(1-\zeta_r)u]_{x_i x_i}|_{2,\Omega} + |(1-\zeta_r)u|_{2,\Omega'}\Big). \tag{4.10}$$

whence, by virtue of $i_3'$), choosing a sufficiently large $r_\varepsilon \in \mathbb{R}_+$ we get

$$\|(1-\zeta_r)u\|_{W^2(\Omega)} \leq c_2(|L[(1-\zeta_r)u]|_{2,\Omega} + |(1-\zeta_r)u|_{2,\Omega'}) + \varepsilon\|(1-\zeta_r)u\|_{W^2(\Omega)},$$

for $r \geq r_\varepsilon$, which yields an inequality of type (4.6) and so (4.5).

By arguing as in the first part of this proof, then we obtain (4.7). □

**Theorem 4.6.** *Let us suppose that the conditions of Theorem 4.5 are verified and assume*

$$a_0 := \operatorname*{ess\,inf}_{\Omega} a > 0. \tag{4.11}$$

*Then we have the estimate*

$$\|u\|_{W^2(\Omega)} \leq c|Lu|_{2,\Omega}\,, \quad \forall u \in W^2(\Omega) \cap W_0^1(\Omega), \tag{4.12}$$

*with $c$ depending only on $a_0$ the parameters occurring in the constant of the bound* (4.7).

*Proof.* The result is an obvious consequence of Theorem 4.5 and Lemma 4.2, since a modulus of continuity in $M_0^s(\Omega)$ is a modulus of continuity in $VM^s(\Omega)$, too. □

## 5. Existence theorems.

In this section we consider the problem

$$u \in W^2(\Omega) \cap W_0^1(\Omega), \quad Lu = f, \quad f \in L^2(\Omega). \tag{5.1}$$

**Theorem 5.1.** *If the conditions of Theorem 4.5 are verified, then* (5.1) *is a zero index problem.*

*If in addition* (4.11) *is verified, then problem* (5.1) *is uniquely solvable.*

*Proof.* Firstly, we consider the case when (4.11) is verified.

Let us set

$$L_\tau u := \tau Au + (1-\tau)Lu, \quad \tau \in [0, 1], \tag{5.2}$$

where

$$Au := -\sum_{i,j=1}^{2} e_{ij} u_{x_i x_i} + eu, \tag{5.3}$$

if we consider $i_3$),

$$Au := -\sum_{i,j=1}^{2} a_{ij} u_{x_i x_i} + bu, \tag{5.4}$$

if we consider $i_3'$).

In the case of assumption $i_3$), we observe that for every $\tau \in [0, 1]$

$$\nu^{-2} \sum_{i,j=1}^{2} [e_{ij} - g_\tau(\tau e_{ij} + (1-\tau)a_{ij})]^2 + \tag{5.5}$$

$$+ \mu^{-2}[e - g_\tau(\tau e + (1-\tau)b)]^2 \leq \nu^{-2} \sum_{i,j=1}^{2} (e_{ij} - g_0 a_{ij})^2 + \mu^{-2}(e - g_0 b)^2,$$

where

$$g_\tau := \frac{\mu^{-2} \sum_{i,j=1}^{2} e_{ij}[\tau e_{ij} + (1-\tau)a_{ij}] + \mu_1^{-2} e[\tau e + (1-\tau)b]}{\mu^{-2} \sum_{i,j=1}^{2} [\tau e_{ij} + (1-\tau)a_{ij}]^2 + \mu_1^{-2}[\tau e + (1-\tau)b]^2},$$

which is reduced to (3.5) for $\tau = 0$.

Since $(a_{ij}, b)$ verifies Hypothesis 3.1 with respect to $(e_{ij}, e, g_0)$, then for every $\tau \in [0, 1]$ the pair $([\tau e_{ij} + (1-\tau)a_{ij}], [\tau e + (1-\tau)b])$ verifies Hypothesis 3.1 with respect to $(e_{ij}, e, g_\tau)$.

Furthermore, since $\tau \to g_\tau$ is a continuous function, from Theorem 4.6 we deduce that there exists $c \in \mathbb{R}_+$ such that

$$\|u\|_{W^2(\Omega)} \le c|L_\tau u|_{2,\Omega} \quad \forall u \in W^2(\Omega) \cap W_0^1(\Omega) \text{ and } \forall \tau \in [0,1]. \tag{5.6}$$

In the case of assumption $i_3'$), the coefficients of $L_\tau$ satisfy condition $i_3'$) uniformly with respect to $\tau \in [0, 1]$ and so again Theorem 4.2 yields (5.6).

Now, we recall that, as a consequence of known results, the problem

$$u \in W^2(\Omega) \cap W_0^1(\Omega), \ Au = f, \ f \in L^2(\Omega), \tag{5.7}$$

is uniquely solvable. For instance, we can get this result observing that the proof of Theorem 5.4 of [11] remains unchanged if we suppose the coefficient of $u$ belongs to $\widetilde{M}^2(\Omega)$ rather than to $M^{t_0}(\Omega)$ for some $t_0 > 2$.

From the uniqueness and existence result for problem (5.7), together with (5.6), we can apply the classical method of continuity along a parameter in order to establish that problem (5.1) is uniquely solvable if (4.11) is verified.

If (4.11) is not verified, by applying the above conclusions to the operator $Lu - a'u$ and observing that, as a consequence of Lemma 1.5, the operator $u \in W^k(\Omega) \to a'u \in L^2(\Omega)$ is compact, we deduce that (5.1) is a zero index problem from well known results of functional analysis. $\square$

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