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ON THE RELAXATION OF SOME CLASSES OF UNBOUNDED INTEGRAL FUNCTIONALS

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Dedicated to Professor Francesco Guglielmino
on his seventieth birthday

Given a Borel function $g : \mathbb{R}^n \to [0, +\infty]$ having convex effective domain, but not necessarily bounded or with nonempty interior, locally bounded in the relative interior of its effective domain and verifying an upper semicontinuity type assumption in its effective domain, we prove that for every convex bounded open set Ω the relaxed functional in the $L^1(\Omega)$-topology of the integral $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n) \mapsto \int_\Omega g(\nabla u)dx$ is equal to

$$\int_\Omega g^{**}(\nabla u)dx + \int_\Omega (g^{**})^\infty(\frac{dD^s u}{d|D^s u|})d|D^s u|$$

for every $u \in BV(\Omega)$, g^{**} being the convex lower semicontinuous envelope of g and $(g^{**})^\infty$ its recession function.

Introduction.

Some studies in elastic-plastic torsion theory and electrostatics (see [1], [4], [18], [23], [25], [28], [31] and the book of G. Duvaut and J.L. Lions [19]) lead to various classes of minimum problems for integral functionals defined on

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spaces of admissible functions subject to pointwise constraints on the gradient that can be studied in the framework of a general theory on lower semicontinuity and relaxation for variational functionals of the type

$$G(\Omega, u) = \int_\Omega g(x, u, \nabla u)\, dx,$$

where g is a function taking its values in $\mathbb{R} \cup \{+\infty\}$.

When g is just real valued relaxation problems for such functionals are well studied in literature (see for example [5], [14], [20], [27] and the references quoted therein) whilst, when g is admitted to take the value $+\infty$ and consequently G may be not finite also on bounded sets of regular functions, i.e. G is what we call an unbounded integral functional, few relaxation results are available (see [20], [26]).

In the present paper we intend to start a study of the relaxation of unbounded integral functionals starting from the case in which g does not depend on x and u.

We prove a general integral representation result on BV-spaces for the relaxed functionals in L^1-topologies of the integrals $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n) \mapsto \int_\Omega g(\nabla u)\, dx$ (see Theorem 7.2 and Proposition 7.3) from which the following particular case can be deduced (see Corollary 7.4).

Let $g : \mathbb{R}^n \to [0, +\infty[$ be continuous, C be a convex subset of $\mathbb{R}^n$, I_C the indicator function of C defined by $I_C(z) = 0$ if $z \in C$ and $I_C(z) = +\infty$ if $z \in \mathbb{R}^n \setminus C$, $(g + I_C)^{**}$ the bipolar of $g + I_C$ and $((g + I_C)^{**})^\infty$ its recession function (see (1.3)), then for every convex bounded open set Ω

$$\begin{aligned}
&\inf\Big\{ \liminf_h \int_\Omega g(\nabla u_h) dx : \{u_h\} \subseteq W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n), \\
&\quad \text{for every } h \in \mathbb{N}\ \nabla u_h(x) \in C \text{ for a.e. } x \in \Omega,\ u_h \to u \text{ in } L^1(\Omega)\Big\} = \\
&\quad = \int_\Omega (g + I_C)^{**}(\nabla u)\, dx + \int_\Omega ((g + I_C)^{**})^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)\, d|D^s u| \\
&\qquad\qquad \text{for every } u \in BV(\Omega),
\end{aligned}$$

$BV(\Omega)$ being the set of the functions in $L^1(\Omega)$ having distributional partial derivatives that are Borel measures with finite total variations on Ω, ∇u the density of the absolutely continuous part of the vector measure Du with respect to Lebesgue measure, $D^s u$ its singular part and $\dfrac{dD^s u}{d|D^s u|}$ the Radon-Nikodym derivative of $D^s u$ with respect to the total variation $|D^s u|$ of $D^s u$.

Problems of this type are treated in Chapter X of [20] and in [26], where also some dependences on x and u in the integrand g are allowed, but limitedly to the case in which C is a ball of $\mathbb{R}^n$. On the contrary we are able to treat also the case in which C is just a convex set, possibly unbounded and with empty interior.

The main tools used to obtain our results are a recent integral representation theorem for unbounded functionals and an extension principle proved in [9] (see Theorem 1.5 and Proposition 1.6 in the next section) together with an inner regularity condition that we prove in an abstract setting (see Section 2).

The result of the present paper have been announced in [8].

1. Notations and preliminary results.

For every couple of open sets A and B of $\mathbb{R}^n$, $A \subset\subset B$ means that $\overline{A}$ is compact and $\overline{A} \subseteq B$.

Definition 1.1. *Let $\mathcal{E}$ be a set of open subsets of $\mathbb{R}^n$ and $\alpha : \mathcal{E} \to [0, +\infty]$. We say that α is increasing if $\alpha(A_1) \leq \alpha(A_2)$ whenever $A_1, A_2 \in \mathcal{E}$ and $A_1 \subseteq A_2$.*

If α is increasing, we define the inner regular envelope α_- of α as the function defined by

$$\alpha_- : A \in \mathcal{E} \mapsto \sup\left\{\alpha(B) : B \in \mathcal{E},\ B \subset\subset A\right\}$$

and say that α is inner regular if $\alpha(A) = \alpha_-(A)$ for every $A \in \mathcal{E}$.

Remark 1.2. It is clear that if α is increasing then

$$\alpha_-(A) \leq \alpha(A) \quad \text{for every} \quad A \in \mathcal{E}.$$

In the present paper we will consider functionals F depending on a open set Ω and a function u such that, for fixed u, $F(\cdot, u)$ is increasing. In this case, given an open set Ω and a function u, we will set $F_-(\Omega, u) = F(\cdot, u)_-(\Omega)$.

For every open set Ω we denote by $BV_{\mathrm{loc}}(\Omega)$ the set of the functions in $L^1_{\mathrm{loc}}(\Omega)$ that are in $BV(A)$ for every open set A with $A \subset\subset \Omega$.

Given an open set Ω and u in $BV_{\mathrm{loc}}(\Omega)$ we set, by Lebesgue decomposition theorem, $Du = D^a u + D^s u = \int_{\cdot} \nabla u dx + D^s u$, where $D^a u$ is the absolutely continuous part of Du with respect to Lebesgue measure and $D^s u$ its singular part; we also denote by $|Du|$ and $|D^s u|$ the total variations of the $\mathbb{R}^n$-valued measures Du and $D^s u$ and recall that $BV(\Omega)$ is a Banach space with norm

$$\|u\|_{BV(\Omega)} = \int_{\Omega} |u|\, dx + |Du|(\Omega).$$

Given a sequence $\{u_h\} \subseteq BV(\Omega)$ and $u \in BV(\Omega)$ we say that $\{u_h\}$ converges to u in w^*-$BV(\Omega)$, and write $u_h \to u$ in w^*-$BV(\Omega)$, if $u_h \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and the sequence $\{|Du_h|(\Omega)\}$ is bounded. Moreover, given a functional F on $BV(\Omega)$ we say that F is sequentially w^*-$BV(\Omega)$-lower semicontinuous if for every sequence $\{u_h\} \subseteq BV(\Omega)$, $u \in BV(\Omega)$ such that $u_h \to u$ in w^*-$BV(\Omega)$ it results $F(u) \leq \liminf_h F(u_h)$.

For a deeper study of BV-functions we refer to [21] and [32], here we just recall that (see for example Chapter 1 of [21]) for every bounded open set with Lipschitz boundary Ω the $BV(\Omega)$-bounded subsets of $BV(\Omega)$ are relatively compact in $BV(\Omega)$ endowed with the $L^1(\Omega)$-topology.

For every $f : \mathbb{R}^n \to]-\infty, +\infty]$ we denote by dom f the effective domain of f, i.e. $\mathrm{dom}\, f = \{z \in \mathbb{R}^n : f(z) < +\infty\}$, by co f the convex hull of f, i.e. the function

$$\mathrm{co}\, f : z \in \mathbb{R}^n \mapsto \sup\{\phi(z) : \phi : \mathbb{R}^n \to]-\infty, +\infty] \text{ convex, } \phi \leq f \text{ on } \mathbb{R}^n\}$$

and by f^{**} the bipolar of f, i.e. the function defined by (see for example [20], Proposition 4.1, page 18)

$$f^{**} : z \in \mathbb{R}^n \mapsto \sup\{\phi(z) : \phi : \mathbb{R}^n \to \mathbb{R} \text{ affine, } \phi \leq f \text{ on } \mathbb{R}^n\}.$$

Obviously co f turns out to be convex, f^{**} convex, lower semicontinuous and

$$f^{**}(z) \leq \mathrm{co}\, f(z) \leq f(z) \quad \text{for every} \quad z \in \mathbb{R}^n, \tag{1.1}$$

moreover we also have

$$f^{**}(z) = \sup\{\phi(z) : \phi : \mathbb{R}^n \to]-\infty, +\infty] \text{ convex, lower semicontinuous } \phi \leq f \text{ on } \mathbb{R}^n\} \quad \text{for every } z \in \mathbb{R}^n. \tag{1.2}$$

For every subset C of $\mathbb{R}^n$ we denote by aff(C) the affine hull of C, i.e. the intersection of all the affine subsets of $\mathbb{R}^n$ containing C. If C is also convex we denote by ri(C) the relative interior of C, i.e. the set of the interior points of C, in the topology of aff(C), once it is regarded as a subspace of aff(C) and by rb(C) the relative boundary of C, i.e. the set $\overline{C} \setminus \mathrm{ri}(C)$. When $\mathrm{aff}(C) = \mathbb{R}^n$ we write as usual $\mathrm{ri}(C) = C^0$ and $\mathrm{rb}(C) = \partial C$.

The following result holds (see for example Theorem 12.2, Corollary 7.4.1, Theorem 7.4 and Theorem 7.5 in [29]).

Proposition 1.3. *Let* $f : \mathbb{R}^n \to]-\infty, +\infty]$ *be convex, then* $\mathrm{ri}(\mathrm{dom}\, f^{**}) = \mathrm{ri}(\mathrm{dom}\, f)$, $\mathrm{rb}(\mathrm{dom}\, f^{**}) = \mathrm{rb}(\mathrm{dom}\, f)$ *and* $f^{**}(z) = f(z)$ *for every* $z \in \mathbb{R}^n \setminus \mathrm{rb}(\mathrm{dom}\, f)$.

Moreover for every $z_0 \in \mathrm{ri}(\mathrm{dom}\, f)$, $z \in \mathbb{R}^n$ *the limit* $\lim\limits_{t\to 1-} f(tz + (1-t)z_0)$ *exists and* $f^{**}(z) = \lim\limits_{t\to 1-} f(tz + (1-t)z_0)$.

Given $f : \mathbb{R}^n \to]-\infty, +\infty]$ convex, lower semicontinuous and $z_0 \in \mathrm{dom}\, f$ we define the recession function f^∞ of f by

$$f^\infty : z \in \mathbb{R}^n \mapsto \lim_{t\to+\infty} \frac{1}{t} f(z_0 + tz); \tag{1.3}$$

it is well known that the definition in (1.3) is independent on the choice of z_0 and that f^∞ is a nonnegative, convex, lower semicontinuous and positively 1-homogeneous function.

Let $f : \mathbb{R}^n \to [0, +\infty]$ be convex and lower semicontinuous for every open set Ω let $G(\Omega, \cdot)$ be the functional defined by

$$G(\Omega, \cdot) : u \in BV(\Omega) \mapsto \int_\Omega f(\nabla u)\, dx + \int_\Omega f^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)\, d|D^s u| \tag{1.4}$$

(in (1.4) and in the sequel we adopt the usual convention that $0\cdot(+\infty) = 0$), then the following lower semicontinuity result holds (see for example Corollary 3.4.2 in [5]).

Theorem 1.4. *Let* $f : \mathbb{R}^n \to [0, +\infty]$ *be convex, lower semicontinuous,* Ω *be an open set and* $G(\Omega, \cdot)$ *be given by* (1.4)*, then* $G(\Omega, \cdot)$ *is sequentially* w^*-$BV(\Omega)$*-lower semicontinuous.*

Let Ω be an open set. Given a sequence $\{u_h\} \subseteq W^{1,\infty}(\Omega)$ and $u \in W^{1,\infty}(\Omega)$ we say that $\{u_h\}$ converges to u in w^*-$W^{1,\infty}(\Omega)$, and write $u_h \to u$ in w^*-$W^{1,\infty}(\Omega)$, if $\{u_h\}$ converges to u weakly* in $L^\infty(\Omega)$ and $\{\nabla u_h\}$ converges to ∇u weakly* in $(L^\infty(\Omega))^n$. Moreover, given a functional F on $W^{1,\infty}(\Omega)$, we say that F is sequentially w^*-$W^{1,\infty}(\Omega)$-lower semicontinuous if for every sequence $\{u_h\} \subseteq W^{1,\infty}(\Omega)$, $u \in W^{1,\infty}(\Omega)$ such that $u_h \to u$ in w^*-$W^{1,\infty}(\Omega)$ it results $F(u) \leq \liminf\limits_h F(u_h)$.

For every measurable subset E of $\mathbb{R}^n$ we denote by $|E|$ the Lebesgue measure of E and by χ_E its characteristic function defined by $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \in \mathbb{R}^n \setminus E$.

For every $z \in \mathbb{R}^n$ we denote by u_z the function defined by $u_z : x \in \mathbb{R}^n \mapsto z \cdot x$.

We say that a function u on $\mathbb{R}^n$ is piecewise affine on $\mathbb{R}^n$ if it is continuous and if there exist $z_1, \dots, z_m \in \mathbb{R}^n$, $s_1, \dots, s_m \in \mathbb{R}$ and m pairwise disjoint polyedra $P_1, \dots, P_m$ having nonempty interiors with $\left|\mathbb{R}^n \setminus \bigcup_{j=1}^m P_j\right| = 0$ such that $u(x) = \sum_{j=1}^m (u_{z_j}(x) + s_j)\chi_{P_j}(x)$ for every $x \in \mathbb{R}^n$. We denote by $PA(\mathbb{R}^n)$ the set of the piecewise affine functions on $\mathbb{R}^n$ and, for every $u = \sum_{j=1}^m (u_{z_j} + s_j)\chi_{P_j}$ in $PA(\mathbb{R}^n)$, set $B_u = \bigcup_{j=1}^m (\overline{P}_j \setminus P_j^\circ)$.

Given an open set Ω, a function u defined on Ω, $x_0 \in \mathbb{R}^n$ and $t > 0$ we denote by $T[x_0]u$ and $O_t u$ the functions defined by $T[x_0]u : x \in \Omega - x_0 \mapsto u(x + x_0)$ and $O_t u : x \in \frac{1}{t}\Omega \mapsto \frac{1}{t}u(tx)$.

For every $r > 0$ and $x_0 \in \mathbb{R}^n$ let $Q_r(x_0)$ be the open cube of $\mathbb{R}^n$ with faces parallel to the coordinate planes centred in x_0 and with sidelength r and set $Q_r = Q_r(0)$.

Let α be a mollifier, i.e. a nonnegative function in $C_0^\infty(Q_1)$ such that $\int_{\mathbb{R}^n} \alpha(y)\, dy = 1$, then, for every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\varepsilon > 0$, we define the regularization u_ε of u as

$$
(1.5) \qquad u_\varepsilon : x \in \mathbb{R}^n \mapsto u_\varepsilon(x) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \alpha\Big(\frac{x-y}{\varepsilon}\Big) u(y)\, dy\,.
$$

Given an open set Ω and $x_0 \in \Omega$, we say that (see [11]) Ω is strongly star shaped with respect to x_0 if it is star shaped with respect to x_0 and if for every $x \in \overline{\Omega}$ the half open line segment joining x_0 to x, and not containing x, is contained in Ω. We say that an open set Ω is strongly star shaped if there exists $x_0 \in \Omega$ such that Ω is strongly star shaped with respect to x_0.

By the above definition it follows that if Ω is a bounded open set strongly star shaped with respect to x_0, then for every $t > 0$ the open set $x_0 + t(\Omega - x_0)$ is still strongly star shaped with respect to x_0 and $x_0 + s(\Omega - x_0) \subset\subset \Omega \subset\subset x_0 + t(\Omega - x_0)$ for every $s, t \in \mathbb{R}$ with $0 \leq s < 1 < t$, moreover it is clear that

$$
(1.6) \qquad \Omega \text{ convex} \Rightarrow \Omega \text{ strongly star shaped.}
$$

We now recall the following integral representation result (see Theorem 6.2 in [9]).

Theorem 1.5. *For every bounded open set* Ω *let* $F(\Omega, \cdot) : W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n) \to [0, +\infty]$ *verifying*

(1.7) $F(\Omega, u_z + c) = F(\Omega, u_z)$ *for every bounded open set* Ω, $z \in \mathbb{R}^n$, $c \in \mathbb{R}$, (1.8) *$F(\Omega - x_0, T[x_0]u_z) = F(\Omega, u_z)$ for every bounded open set Ω, $z \in \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$,*

(1.9) *for every $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ $F(\cdot, u)$ is increasing,*

(1.10) *$F(\Omega_1, u)+F(\Omega_2, u) \le F(\Omega, u)$ whenever $\Omega_1, \Omega_2, \Omega$ are bounded open sets with $\Omega_1 \cup \Omega_2 = \emptyset$, $\Omega_1 \cup \Omega_2 \subset\subset \Omega$, $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$,*

(1.11) *$F(\Omega, u) \le F(\Omega_1, u)+F(\Omega_2, u)$ whenever $\Omega, \Omega_1, \Omega_2$ are bounded open sets with $\Omega \subset\subset \Omega_1 \cup \Omega_2$, $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$,*

(1.12) *$\limsup_{r\to 0+} \dfrac{1}{r^n} F(Q_r(x_0), u) \ge F(Q_1(x_0), u(x_0)+\nabla u(x_0)\cdot(\cdot - x_0))$ for every $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, x_0 a.e. in $\mathbb{R}^n$,*

(1.13) *for every $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ $F(\cdot, u)$ is inner regular,*

(1.14) *for every bounded open set Ω $F(\Omega, \cdot)$ is sequentially w^*-$W^{1,\infty}(\Omega)$-lower semicontinuous,*

(1.15) *$F(\Omega, u) \le F(\Omega \setminus B_u, u)$ for every bounded open set Ω, $u \in PA(\mathbb{R}^n)$*

and let f_F be defined by $f_F : z \in \mathbb{R}^n \mapsto F(Q_1, u_z) \in [0, +\infty]$, then f_F is convex, lower semicontinuous and

(1.16) *$F(\Omega, u) = \displaystyle\int_\Omega f_F(\nabla u)\, dx$ for every bounded open set Ω, $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$.*

Conversely, given $f : \mathbb{R}^n \to [0, +\infty]$ convex, lower semicontinuous and defined, for every bounded open set Ω, the functional $F(\Omega, \cdot)$ by (1.16) *with $f_F = f$, it turns out that conditions* (1.7) $\div$ (1.15) *are verified by F.*

Let $\mathcal{A}_0$ be a family of bounded open sets verifying the following property

(1.17) for every $\Omega \in \mathcal{A}_0$ and every open set A with $A \subset\subset \Omega$ there exists $B \in \mathcal{A}_0$ such that $A \subset\subset B \subset\subset \Omega$.

Proposition 1.6. *Let $\mathcal{A}_0$ be a family of bounded open sets verifying* (1.17) *and $f : \mathbb{R}^n \to [0, +\infty]$ be convex and lower semicontinuous. For every bounded open set Ω let $F(\Omega, \cdot) : BV_{\mathrm{loc}}(\mathbb{R}^n) \to [0, +\infty]$ be such that for every $u \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ $F(\cdot, u)$ is increasing, for every $\Omega \in \mathcal{A}_0$ $F_-(\Omega, \cdot)$ is sequentially w^*-$BV(\Omega)$-lower semicontinuous and*

$$F_-(\Omega, u) \le \int_\Omega f(\nabla u)\, dx \quad \text{for every } \Omega \in \mathcal{A}_0,\ u \in C^\infty(\mathbb{R}^n),$$

then

$$F_-(\Omega, u) \le \int_\Omega f(\nabla u)\, dx + \int_\Omega f^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)\, d|D^s u| \qquad \text{for every } \Omega \in \mathcal{A}_0,\ u \in BV_{\mathrm{loc}}(\mathbb{R}^n).$$

Proof. Follows by Proposition 3.5 in [9]. □

In conclusion we prove the following lower semicontinuity result.

For every open set Ω we denote by $\mathcal{D}'(\Omega)$ the weak* topology of the space of the distributions on Ω.

Proposition 1.7. *Let $f : \mathbb{R}^n \to [0, +\infty]$ be convex and lower semicontinuous, then for every open set Ω the functional*

$$u \in BV_{\text{loc}}(\Omega) \mapsto \int_\Omega f(\nabla u)\, dx + \int_\Omega f^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)\, d|D^s u|$$

is sequentially $\mathcal{D}'(\Omega)$-lower semicontinuous.

Proof. Let us preliminarily recall that if for every $w \in BV_{\text{loc}}(\mathbb{R}^n)$, $\varepsilon > 0$ w_ε is the regularization of w given by (1.5), then by Lemma 3.3 in [9] we obtain that

$$\text{(1.18)} \quad \int_A f(\nabla w_\varepsilon)\, dx \le \int_B f(\nabla w)\, dx + \int_B f^\infty\Big(\frac{dD^s w}{d|D^s w|}\Big)\, d|D^s w| \ \text{ for every bounded open set } B, \text{ every open set } A \text{ with } A \subset\subset B,\ w \in BV_{\text{loc}}(\mathbb{R}^n),\ \varepsilon \in]0, \operatorname{dist}(A, \partial B)[.$$

Let Ω be a bounded open set, $u \in BV_{\text{loc}}(\Omega)$, $\{u_h\} \subseteq BV_{\text{loc}}(\Omega)$ with $u_h \to u$ in $\mathcal{D}'(\Omega)$, A be an open set with $A \subset\subset \Omega$, $\varepsilon \in]0, \operatorname{dist}(A, \partial\Omega)[$ and B be an open set with Lipschitz boundary such that $A \subset\subset B \subset\subset \Omega$ and $\operatorname{dist}(A, \partial B) > \varepsilon$. For every $h \in \mathbb{N}$ let v and v_h be the zero extensions of u and u_h out of B, then (see for example Chapter 1 of [21]) $v, v_h \in BV(\mathbb{R}^n)$, moreover, if v_ε and $v_{h,\varepsilon}$ are the regularizations of v and v_h given by (1.5), by (1.18) we get

$$\text{(1.19)} \quad \begin{aligned} \int_A f(\nabla v_{h,\varepsilon})\, dx &\le \int_B f(\nabla v_h)\, dx + \int_B f^\infty\Big(\frac{dD^s v_h}{d|D^s v_h|}\Big)\, d|D^s v_h| \le \\ &\le \int_\Omega f(\nabla u_h)\, dx + \int_\Omega f^\infty\Big(\frac{dD^s u_h}{d|D^s u_h|}\Big)\, d|D^s u_h| \quad \text{for every } h \in \mathbb{N}. \end{aligned}$$

Let us observe now that $v_{h,\varepsilon} \to v_\varepsilon$ in w^*-$BV(A)$ as h diverges, hence by Theorem 1.4 and (1.19) we deduce that

$$\text{(1.20)} \quad \begin{aligned} \int_A f(\nabla v_\varepsilon)\, dx &\le \liminf_h \int_A f(\nabla v_{h,\varepsilon})\, dx \le \\ &\le \liminf_h \Big\{ \int_\Omega f(\nabla u_h)\, dx + \int_\Omega f^\infty\Big(\frac{dD^s u_h}{d|D^s u_h|}\Big)\, d|D^s u_h| \Big\} \\ &\qquad \text{for every } \varepsilon \in]0, \operatorname{dist}(A, \partial\Omega)[. \end{aligned}$$

Finally again by Theorem 1.4 and by (1.20) we conclude that

$$\int_A f(\nabla u)\,dx + \int_A f^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)\,d|D^s u| = \int_A f(\nabla v)\,dx +$$
$$+ \int_A f^\infty\Big(\frac{dD^s v}{d|D^s v|}\Big)\,d|D^s v| \le \liminf_{\varepsilon\to 0+} \int_A f(\nabla v_\varepsilon)\,dx \le$$
$$\le \liminf_h \Big\{ \int_\Omega f(\nabla u_h)\,dx + \int_\Omega f^\infty\Big(\frac{dD^s u_h}{d|D^s u_h|}\Big)\,d|D^s u_h|\Big\},$$

from which the thesis follows letting A increase to Ω. $\square$

2. An abstract inner regularity result for increasing set functionals.

In the present section we prove a sufficient condition, that can be stated in an abstract setting, in order to deduce identity between a functional and its inner regular envelope.

Let U be a set of functions on $\mathbb{R}^n$ such that

$$u \in U,\ x_0 \in \mathbb{R}^n,\ t \in]0,1[\ \Rightarrow T[x_0]u \in U,\ O_t u \in U\,, \tag{2.1}$$

and let, for every bounded open set Ω of $\mathbb{R}^n$, $F(\Omega,\cdot) : U \to [0,+\infty]$ be a functional satisfying

(2.2) for every $u \in U\, F(\cdot, u)$ is increasing,

(2.3) $\liminf_{t\to 1-} F(\Omega, T[-x_0]O_t T[x_0]u) \ge F(\Omega, u)$ for every bounded open set Ω strongly star shaped with respect to x_0, $u \in U$

and

(2.4) $\limsup_{t\to 1+} F_-(x_0 + t(\Omega - x_0), T[-x_0]O_{1/t}T[x_0]u) \le F_-(\Omega, u)$ for every bounded open set Ω strongly star shaped with respect ro x_0, $u \in U$.

Proposition 2.1. *Let U be a set of functions on $\mathbb{R}^n$ verifying* (2.1) *and let, for every bounded open set Ω, $F(\Omega,\cdot) : U \to [0,+\infty]$ verifying* (2.2) $\div$ (2.4), *then*

(2.5) *$F(\Omega, u) = F_-(\Omega, u)$ for every strongly star shaped bounded open set Ω, $u \in U$.*

Proof. Let Ω, u be as in (2.5), $x_0 \in \Omega$ be such that Ω is strongly star shaped with respect to x_0 and $t \in]1, +\infty[$, then, since obviously $\Omega \subset\subset x_0 + t(\Omega - x_0)$, by (2.2) we have

$$(2.6) \quad F(\Omega, T[-x_0]O_{1/t}T[x_0]u) \leq F_-(x_0 + t(\Omega - x_0), T[-x_0]O_{1/t}T[x_0]u),$$

hence as t decreases to 1, by (2.6), (2.3), (2.4) and Remark 1.2 we deduce (2.5). □

3. Statement of the relaxation problem and elementary results.

Let g be a Borel function with

$$(3.1) \qquad g : z \in \mathbb{R}^n \mapsto g(z) \in [0, +\infty] .$$

In the present section we start the study, for every bounded open set Ω, of the relaxed functional in the $L^1(\Omega)$-topology of integral $G(\Omega, \cdot) : u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n) \mapsto \int_\Omega g(\nabla u)\, dx$ defined by

$$(3.2) \qquad \overline{G}(\Omega, \cdot) : u \in L^1(\Omega) \mapsto \inf \Big\{ \liminf_h \int_\Omega g(\nabla u_h)\, dx : \{u_h\} \subseteq W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n),\ u_h \to u \text{ in } L^1(\Omega) \Big\}.$$

Obviously

(3.3) for every bounded open set Ω, $\overline{G}(\Omega, \cdot)$ is $L^1(\Omega)$-lower semicontinuous

and (as usual here and in the sequel we assume that $\inf \emptyset = +\infty$)

$$(3.4) \quad \overline{G}(\Omega, u) = \min \Big\{ \liminf_h \int_\Omega g(\nabla u_h)\, dx : \{u_h\} \subseteq W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n), \text{ for every } h \in \mathbb{N}\ \nabla u_h(x) \in \operatorname{dom} g \text{ for a.e. } x \in \Omega,\ u_h \to u \text{ in } L^1(\Omega) \Big\}$$
$$\text{for every bounded open set } \Omega,\ u \in L^1(\Omega).$$

It is easy to see that $\overline{G}$ verifies the following properties:

(3.5) $\overline{G}(\Omega, u + c) = \overline{G}(\Omega, u)$ for every bounded open set Ω, $u \in L^1(\Omega)$, $c \in \mathbb{R}$,

(3.6) $\overline{G}(\Omega - x_0, T[x_0]u) = \overline{G}(\Omega, u)$ for every bounded open set Ω, $u \in L^1(\Omega)$, $x_0 \in \mathbb{R}^n$, (3.7) $\overline{G}(\Omega, O_t u) = \dfrac{1}{t^n}\overline{G}(t\Omega, u)$ for every bounded open set Ω, $t \in \mathbb{R}$, $u \in L^1(\Omega)$

and

(3.8) $\overline{G}(\Omega_2, u) \leq \overline{G}(\Omega_1, u)$ whenever Ω_1, Ω_2 are bounded open sets with $\Omega_1 \subseteq \Omega_2$, $|\Omega_2 \setminus \Omega_1| = 0$, $u \in L^1(\Omega_2)$.

Moreover we also have that

(3.9) $\overline{G}(\Omega_1, u) \leq \overline{G}(\Omega_2, u)$ whenever Ω_1, Ω_2 are bounded open sets with $\Omega \subseteq \Omega_2$, $u \in L^1(\Omega_2)$,

(3.10) $\overline{G}(\Omega_1, u) + \overline{G}(\Omega_2, u) \leq \overline{G}(\Omega_1 \cup \Omega_2, u)$ whenever Ω_1, Ω_2 are disjoint bounded open sets, $u \in L^1(\Omega_1 \cup \Omega_2)$.

In order to prove additional measure theoretic properties of $\overline{G}$ we need to assume further conditions on g, more precisely that

(3.11) $\operatorname{dom} g$ is convex,

(3.12) g is locally bounded on $\operatorname{ri}(\operatorname{dom} g)$, i.e. for every compact subset K of $\operatorname{ri}(\operatorname{dom} g)$ there exists $M_K > 0$ such that $g(z) \leq M_K$ for every $z \in K$

and that

(3.13) for every bounded subset L of $\operatorname{dom} g$ there exists $z_L \in \operatorname{ri}(\operatorname{dom} g)$ such that the function $t \in [0, 1] \mapsto g((1-t)z_L + tz)$ is upper semicontinuous at $t = 1$ uniformly as z varies in L, i.e. for every $\varepsilon > 0$ there exists $t_\varepsilon < 1$ such that $g((1-t)z_L + tz) \leq g(z) + \varepsilon$ for every $t \in]t_\varepsilon, 1]$ and $z \in L$.

Remark 3.1. Assumption (3.13) looks like a sort of uniform radial upper semicontinuity on bounded subsets of $\operatorname{dom} g$, nevertheless it does not imply in general (3.12) (think for example to the case in which $n = 2$, $g(z_1, z_2) = |z_2|/|z_1|$ if $|z_1|^2 + |z_2|^2 \leq 1$ and $z_1 z_2 \neq 0$, $g(z_1, z_2) = 0$ if $|z_1|^2 + |z_2|^2 \leq 1$ and $z_1 z_2 = 0$, $g(z_1, z_2) = +\infty$ otherwise in $\mathbb{R}^2$ and $z_L = (0, 0)$ independently on L). It is fulfilled if g is finite and continuous in $\mathbb{R}^n$ or if there exists $z_0 \in \operatorname{ri}(\operatorname{dom} g)$ such that the function $t \in [0, 1] \mapsto g((1-t)z_0 + tz)$ is increasing for every z in $\operatorname{dom} g$.

Lemma 3.2. *Let g be a Borel function as in* (3.1) *verifying* (3.11) *and $\overline{G}$ be given by* (3.2). *Let A be a bounded open set and $u \in W^{1,1}(A)$ be such that $\overline{G}(A, u) < +\infty$, then*

$$\nabla u(x) \in \overline{\operatorname{dom} g} \quad \textit{for a.e. } x \in A. \tag{3.14}$$

Proof. Since $\overline{G}(A,u) < +\infty$, by (3.4) there exists a sequence $\{u_h\} \subseteq W^{1,\infty}_{\rm loc}(\mathbb{R}^n)$ such that $u_h \to u$ in $L^1(A)$ and

(3.15) $$\text{for every } h \in \mathbb{N}\ \nabla u_h(x) \in \operatorname{dom} g \text{ for a.e. } x \in A\,.$$

We now observe that, being by (3.11) $\overline{\operatorname{dom} g}$ closed and convex there exist two families $\{a_\theta\}_{\theta\in T} \subseteq \mathbb{R}^n$ and $\{b_\theta\}_{\theta\in T} \subseteq \mathbb{R}$ such that $z \in \overline{\operatorname{dom} g}$ if and only if $a_\theta \cdot z + b_\theta \geq 0$ for every $\theta \in T$, therefore by (3.15) we obtain that

(3.16) $$a_\theta \frac{1}{|B|}\int_B \varphi \nabla u_h\, dx + b_\theta \geq 0 \text{ for every } h \in \mathbb{N},\ \theta \in T,$$
$$\text{every ball } B \subseteq A \text{ and every } \varphi \in C^1_0(B) \text{ with } \varphi \geq 0,\ \int_B \varphi\, dx = 1\,.$$

By (3.16), taking the limit as h diverges, we deduce that

$$\frac{1}{|B|}\int_B \varphi \nabla u\, dx \in \overline{\operatorname{dom} g} \text{ for every ball } B \subseteq A$$
$$\text{and every } \varphi \in C^1_0(B) \text{ with } \varphi \geq 0,\ \int_B \varphi\, dx = 1\,,$$

from which (3.14) follows. □

4. The case of bounded effective domain with nonempty interior.

Let g be a Borel function as in (3.1) and $\overline{G}$ be given by (3.2).

The integral representation result for $\overline{G}$ will be proved in some steps, in the first one, that is treated in the present section, we assume that

(4.1) $\overline{\operatorname{dom} g}$ is bounded,

(4.2) $(\operatorname{dom} g)^\circ \neq \emptyset$.

It is clear that, by (4.1) it results

(4.3) $\overline{G}(\Omega,u) = \inf\big\{\liminf_h \int_\Omega g(\nabla u_h)\,dx : \{u_h\} \subseteq W^{1,\infty}_{\rm loc}(\mathbb{R}^n)$, for every $h \in \mathbb{N}$ $\nabla u_h(x) \in \operatorname{dom} g$ for a.e. $x \in \Omega$, $u_h \to u$ in $w^*\text{-}W^{1,\infty}(\Omega)\big\}$ for every bounded open set Ω, $u \in L^1(\Omega)$.

Lemma 4.1. *Let g be a Borel function as in* (3.1) *verifying* (3.11) ÷ (3.13)*,* (4.1)*,* (4.2) *and let $\overline{G}$ be given by* (3.2)*, then*

(4.4) $$\overline{G}_-(\Omega_1 \cup \Omega_2, u) \leq \overline{G}_-(\Omega_1,u) + \overline{G}_-(\Omega_2,u)$$
whenever Ω_1, Ω_2 are bounded open sets, $u \in L^1(\Omega_1, \cup\Omega_2)$.

Proof. Let us preliminarily observe that, by (4.1), we can take $L = \operatorname{dom} g$ in (3.13) and that it is not restrictive to assume that $z_{\operatorname{dom} g} = 0$, otherwise we just have to consider the function $g' = g(z_{\operatorname{dom} g} + \cdot)$. In particular this, together with (4.2), yields that

(4.5) $\quad 0 \in (\operatorname{dom} g)^\circ$.

Let now Ω_1, Ω_2, u be as in (4.4), fix an open set A with $A \subset\subset \Omega$ and observe that there exist $A_1 \subset\subset \Omega_1$, $A_2 \subset\subset \Omega_2$ such that $A \subset\subset A_1 \cup A_2$. By virtue of this, in order to prove (4.4), it suffices to show that

(4.6) $\quad \overline{G}(A, u) \leq \overline{G}(A_1, u) + \overline{G}(A_2, u)$ whenever A, A_1, A_2 are bounded open sets with $A_1 \subset\subset \Omega_1$, $A_2 \subset\subset \Omega_2$ and $A \subset\subset A_1 \cup A_2$.

To do this we can obviously assume that the right-hand side of (4.6) is finite so that by (4.1) and (4.3) for $i = 1, 2$ there exists a sequence $\{u_h^i\} \subseteq W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ such that $u_h^i \to u$ in w^*-$W^{1,\infty}(A_i)$, for every $h \in \mathbb{N}$ $\nabla u_h^i(x) \in \operatorname{dom} g$ for a.e. $x \in A_i$ and

$$\overline{G}(A_i, u) = \lim_h \int_{A_i} g(\nabla u_h^i)\, dx\,. \tag{4.7}$$

Let B_1 be an open set with $B_1 \subset\subset A_1$ such that $A \subset\subset B_1 \cup A_2$, let $\varphi \in C_0^1(A_1)$ verifying

$$0 \leq \varphi \leq 1 \text{ in } \mathbb{R}^n, \quad \varphi = 1 \text{ in } B_1, \quad \|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)} \leq \frac{2}{\operatorname{dist}(B_1, \partial A_1)} \tag{4.8}$$

and set, for every $h \in \mathbb{N}$, $w_h = \varphi u_h^1 + (1 - \varphi)u_h^2$, then $w_h \to u$ in w^*-$W^{1,\infty}(A)$ and by (4.8) we have

$$\overline{G}(A, tu) \leq \liminf_h \int_A g(t\nabla w_h)\, dx \leq \limsup_h \int_{A\cap B_1} g(t\nabla u_h^1)\, dx + \tag{4.9}$$

$$+\limsup_h \int_{A_2} g(t\nabla u_h^2)\, dx + \limsup_h \int_{A\cap(A_1\setminus B_1)} g(t\nabla w_h)\, dx \quad \text{for every } t \in [0, 1[.$$

Let us fix now $t \in [0, 1[$, then, since for every $h \in \mathbb{N}$ $\nabla w_h = \varphi\nabla u_h^1 + (1 - \varphi)\nabla u_h^2 + (u_h^1 - u_h^2)\nabla\varphi$ and $\nabla u_h^i(x) \in \operatorname{dom} g$ for $i = 1, 2$ and a.e. $x \in A_i$, by (3.11) it results that for every $h \in \mathbb{N}$ $t\varphi(x)\nabla u_h^1(x) + t(1 - \varphi(x))\nabla u_h^2(x) \in t\operatorname{dom} g$ for a.e. $x \in A$. By virtue of this, once recalled that by (4.5) and (3.11) $\overline{t\operatorname{dom} g} \subseteq (\operatorname{dom} g)^\circ$ and that $u_h^i \to u$ in $L^\infty(A)$ for $i = 1, 2$, we obtain that there exist a compact subset K_t of $(\operatorname{dom} g)^\circ$ (depending only on t) and $h_{t,A_1,B_1} \in \mathbb{N}$

(depending on t, A_1 and B_1) such that for every $h \geq h_{t,A_1,B_1}$ $t\nabla w_h(x) \in K_t$ for a.e. $x \in A$ from which, together with (3.12), we conclude that

(4.10) there exist $M_t > 0$ and $h_{t,A_1,B_1} \in \mathbb{N}$ such that for every $h \geq h_{t,A_1,B_1}$ $g(t\nabla w_h(x)) \leq M_t$ for a.e. $x \in A$.

We now fix $\varepsilon > 0$, then by (3.13) we obtain the existence of $t_\varepsilon \in [0, 1[$ such that

$$\begin{aligned}(4.11)\qquad &\int_{A_1} g(t\nabla u_h^1)\,dx \leq \int_{A_1} g(\nabla u_h^1)\,dx + \varepsilon|A_1|,\\ &\int_{A_2} g(t\nabla u_h^2)\,dx \leq \int_{A_2} g(\nabla u_h^2)\,dx + \varepsilon|A_2|\\ &\qquad\qquad\text{for every } t \in]t_\varepsilon, 1[,\ h \in \mathbb{N},\end{aligned}$$

hence by (4.9) $\div$ (4.11) and (4.7) we deduce that

$$\begin{aligned}(4.12)\quad \overline{G}(A, tu) &\leq \limsup_h \int_{A_1} g(\nabla u_h^1)\,dx + \limsup_h \int_{A_2} g(\nabla u_h^2)\,dx +\\ &+ \varepsilon(|A_1| + |A_2|) + M_t|A \cap (A_1 \setminus B_1)| \leq \overline{G}(A_1, u) + \overline{G}(A_2, u) +\\ &+ \varepsilon(|A_1| + |A_2|) + M_t|A \cap (A_1 \setminus B_1)| \quad \text{for every } t \in]t_\varepsilon, 1[.\end{aligned}$$

As B_1 increases to A_1 and then t tends to 1^- we deduce by (4.12) and (3.3) that

$$\overline{G}(A, u) \leq \overline{G}(A_1, u) + \overline{G}(A_2, u) + \varepsilon(|A_1| + |A_2|)$$

from which inequality (4.6) follows as ε tends to zero $\square$

Lemma 4.2. *Let g be a Borel function as in* (3.1) *verifying* (3.11) $\div$ (3.13)*,* (4.1)*,* (4.2) *and let $\overline{G}$ be given by* (3.2)*, then*

$$(4.13)\qquad \overline{G}_-(\Omega, u) = \overline{G}(\Omega, u) \quad \textit{for every bounded open set } \Omega,\ u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n).$$

Proof. Let Ω, u be as in (4.13) then, since $\overline{G}(\cdot, u)$ is increasing on Ω, by Remark 1.2 we soon have that

$$(4.14)\qquad \overline{G}_-(\Omega, u) \leq \overline{G}(\Omega, u)\,.$$

In order to prove the reverse inequality in (4.14) we can obviously assume that $\overline{G}_-(\Omega, u) < +\infty$ so that $\overline{G}(A, u) < +\infty$ for every open set A with $A \subset\subset \Omega$ and, by Lemma 3.2, that

$$(4.15)\qquad \nabla u(x) \in \overline{\operatorname{dom} g} \quad \text{for a.e. } x \in \Omega\,.$$

Let now A, B be open sets with $A \subset\subset B \subset\subset \Omega$, then by (4.1) and (4.3) there exists $\{u_h\} \subseteq W^{1,\infty}_{\rm loc}(\mathbb{R}^n)$ such that $u_h \to u$ in w^*-$W^{1,\infty}(B)$ and $\overline{G}(B, u) = \lim_h \int_B g(\nabla u_h)\,dx$.

Let $\varphi \in C^1_0(B)$ be such that

$$(4.16) \qquad 0 \le \varphi \le 1 \ \text{ in } \ \mathbb{R}^n, \quad \varphi = 1 \ \text{ in } \ A, \quad \|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)} \le \frac{2}{\operatorname{dist}(A, \partial B)}$$

and define, for every $h \in \mathbb{N}$, $w_h = \varphi u_h + (1-\varphi)u$; then obviously $w_h \in W^{1,\infty}_{\rm loc}(\mathbb{R}^n)$ for every $h \in \mathbb{N}$ and $w_h \to u$ in w^*-$W^{1,\infty}(\Omega)$.

By (4.1), assuming as in Lemma 4.1 that $z_{\operatorname{dom} g}$ in (3.13) relatively to $L = \operatorname{dom} g$ is equal to 0 (and thus getting (4.5)), and by using (3.11) $\div$ (3.13), (4.5), (4.15), (4.16) and an argument similar to the one employed to get (4.10) we obtain that

(4.17) for every $t \in [0, 1[$ there exist $M_t > 0$ and $h_{t,B,A} \in \mathbb{N}$ such that for every $h \ge h_{t,B,A}$ $g(t\nabla w_h(x)) + g(t\nabla u(x)) \le M_t$ for a.e. $x \in \Omega$,

and that for fixed $\varepsilon > 0$ there exists $t_\varepsilon \in]0, 1[$ such that

$$(4.18) \qquad \int_B g(t\nabla u_h)\,dx \le \int_B g(\nabla u_h)\,dx + \varepsilon|B|, \quad \text{for every } t \in]t_\varepsilon, 1[,\ h \in \mathbb{N}.$$

By (4.16) $\div$ (4.18) we conclude that

$$\begin{aligned}(4.19) \quad \overline{G}(\Omega, tu) &\le \liminf_h \int_\Omega g(t\nabla w_h)\,dx \le \liminf_h \int_B g(t\nabla u_h)\,dx + \\ &+ \limsup_h \int_{B\setminus A} g(t\nabla w_h)\,dx + \int_{\Omega\setminus B} g(t\nabla u)\,dx \le \\ &\le \limsup_h \int_B g(\nabla u_h)\,dx + \varepsilon|B| + M_t|\Omega \setminus A| \le \\ &\le \overline{G}_-(\Omega, u) + \varepsilon|\Omega| + M_t|\Omega \setminus A| \quad \text{for every } t \in [0, 1[.\end{aligned}$$

As A increases to Ω and then t tends to 1^- we deduce by (4.19) and (3.3) that

$$(4.20) \qquad \overline{G}(\Omega, u) \le \overline{G}_-(\Omega, u) + \varepsilon|\Omega|,$$

hence as ε tends to zero by (4.20) and (3.15) equality (4.13) follows. □

Lemma 4.3. *Let g be a Borel function as in* (3.1) *and let $\overline{G}$ be given by* (3.2)*, then*

$$(4.21) \qquad \limsup_{r\to 0+} \frac{1}{r^n}\overline{G}(Q_r(x_0), u) \ge \overline{G}(Q_1, \nabla u(x_0)\cdot(\cdot))$$

for every $u \in W^{1,1}_{\rm loc}(\mathbb{R}^n)$, x_0 a.e. in $\mathbb{R}^n$.

Proof. Let $u \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$, then, see for example Theorem 3.4.2 in [32], we have

$$\lim_{r \to 0+} \int_{Q_1} |O_r T[x_0](u - u(x_0))(x) - \nabla u(x_0) \cdot x| \, dx = 0 \quad \text{for a.e. } x_0 \in \mathbb{R}^n, \tag{4.22}$$

and, by Lebesgue Differentiation Theorem,

$$\lim_{r \to 0+} \int_{Q_1} |\nabla(O_r T[x_0](u - u(x_0))) - \nabla u(x_0)| \, dx = 0 \quad \text{for a.e. } x_0 \in \mathbb{R}^n, \tag{4.23}$$

therefore by (4.22) and (4.23) we get

$$O_r T[x_0](u - u(x_0)) \to \nabla u(x_0) \cdot (\cdot) \text{ in } W^{1,1}(Q_1) \quad \text{as } r \to 0^+ \text{ for a.e. } x_0 \in \mathbb{R}^n. \tag{4.24}$$

By (4.24), (3.3), (3.7) and (3.5) we obtain

$$\overline{G}(Q_1, \nabla u(x_0) \cdot (\cdot)) \leq \liminf_{r \to 0+} \overline{G}(Q_1, O_r T[x_0](u - u(x_0))) = \\ = \limsup_{r \to 0+} \frac{1}{r^n} \overline{G}(Q_r(x_0), u),$$

that is condition (4.21). □

We are now in a position to prove a first integral representation result for $\overline{G}$.

Theorem 4.4. *Let g be a Borel function as in* (3.1) *verifying* (3.11) ÷ (3.13), (4.1), (4.2) *and let $\overline{G}$ be given by* (3.2), *then there exists $f : \mathbb{R}^n \to [0, +\infty]$ convex and lower semicontinuous such that*

$$\overline{G}(\Omega, u) = \int_\Omega f(\nabla u) \, dx \quad \textit{for every bounded open set } \Omega,\ u \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^n). \tag{4.25}$$

Proof. By (3.5), (3.6), (3.9), (3.10), Lemma 4.1, (3.7), Lemma 4.3, (3.3), (3.8) and Lemma 4.2 we get that the assumptions of Theorem 1.5 are fulfilled by the restrictions to $W^{1,\infty}_{\text{loc}}(\mathbb{R}^n)$ of the functionals $\overline{G}(\Omega, \cdot)$, Ω bounded open set, thus the thesis follows by Theorem 1.5. □

In the following result we specify the function f in (4.25).

Proposition 4.5. *Let g be a Borel function as in* (3.1) *verifying* (3.11) $\div$ (3.13), (4.1), (4.2) *and f the one appearing in* (4.25) *of Theorem 4.4, then $f = g^{**}$.*

Proof. Since $g \geq g^{**}$ we soon deduce by Theorem 4.4, by the convexity and the lower semicontinuity of g^{**} and by Theorem 1.4 that $f \geq g^{**}$; on the other side it is clear that $f \leq g$, therefore by using the properties of f and (1.2) we obtain that $f \leq g^{**}$ and the thesis. □

5. The case of bounded effective domain with empty interior.

We now want to consider the case in which assumption (4.2) is dropped.

For every $k \in \{1, \ldots, n\}$ we denote by 0_k the origin of $\mathbb{R}^k$, moreover we denote by $|\cdot|_k$ the k-dimensional Lebesgue measure on $\mathbb{R}^k$ and, for every open set A of $\mathbb{R}^k$ and u in $L^1(A)$, by $\tilde{u}$ the function on $A \times \mathbb{R}^{n-k}$ defined by $\tilde{u} : x = (x_1, \ldots, x_n) \in A \times \mathbb{R}^{n-k} \mapsto u(x_1, \ldots, x_k)$.

Lemma 5.1. *Let g be a Borel function as in* (3.1) *verifying* (3.11) $\div$ (3.13), (4.1) *and let $\overline{G}$ be given by* (3.2). *Assume that*

$$\text{(5.1)} \qquad \operatorname{aff}(\operatorname{dom} g) = \mathbb{R}^k \times \{0_{n-k}\} \quad \textit{for some } k \in \{1, \ldots, n-1\},$$

then there exists $f_p : \mathbb{R}^k \to [0, +\infty]$ convex and lower semicontinuous such that

$$\text{(5.2)} \quad \overline{G}(A \times I, \tilde{u}) = |I|_{n-k} \int_A f_p(\nabla u)\, dy \quad \textit{whenever } A \textit{ is a bounded open set of } \mathbb{R}^k,\ I \textit{ is a connected bounded open set of } \mathbb{R}^{n-k},\ u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^k).$$

Proof. Let us denote by g_p the function defined by $g_p : (z_1, \ldots, z_k) \in \mathbb{R}^k \mapsto g(z_1, \ldots, z_k, 0_{n-k}) \in [0, +\infty]$, define for every bounded open set A of $\mathbb{R}^k$ the functionals

$$G_p(A, \cdot) : u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^k) \mapsto \int_A g_p(\nabla u)\, dy\,,$$

$$\overline{G}_p(A, \cdot) : u \in L^1(A) \mapsto \inf\Big\{ \liminf_h \int_A g_p(\nabla u_h)\, dy : \{u_h\} \subseteq W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^k),\ \ u_h \to u \ \text{ in } \ L^1(A)\Big\}$$

and observe that obviously

$$\text{(5.3)} \quad \overline{G}_p(A, u) = \min\Big\{ \liminf_h \int_A g_p(\nabla u_h)\, dy : \{u_h\} \subseteq W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^k), \text{ for every } \ h \in \mathbb{N}\ \nabla u_h(y) \in \operatorname{dom} g_p \ \text{ for a.e. } \ y \in A,\ u_h \to u \ \text{ in } \ L^1(A)\Big\}$$
$$\text{for every bounded open set } A \text{ of } \mathbb{R}^k,\ u \in L^1(A).$$

The function g_p satisfies all the assumptions of Theorem 4.4 with $n = k$ and so by Theorem 4.4 we deduce the existence of $f_p : \mathbb{R}^k \to [0, +\infty]$ convex and lower semicontinuous such that

$$\overline{G}_p(A, u) = \int_A f_p(\nabla u)\, dy \quad \text{for every bounded open set } A \text{ of } \mathbb{R}^k,\ u \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^k). \tag{5.4}$$

Let now A, I, u be as in (5.2) and let us prove that

$$\overline{G}(A \times I, \tilde{u}) \leq |I|_{n-k} \int_A f_p(\nabla u)\, dy\,. \tag{5.5}$$

To do this we can assume that the right-hand side of (5.5) is finite so that by (5.3) and (5.4) there exists $\{u_h\} \subseteq W^{1,\infty}_{\text{loc}}(\mathbb{R}^k)$ such that for every $h \in \mathbb{N}$ $\nabla u_h(y) \in \operatorname{dom} g_p$ for a.e. $y \in A$, $u_h \to u$ in $L^1(A)$ and

$$\int_A f_p(\nabla u)\, dy = \liminf_h \int_A g(\nabla_1 u_h, \ldots, \nabla_k u_h, 0_{n-k})\, dy, \tag{5.6}$$

then obviously $\tilde{u}_h \to \tilde{u}$ in $L^1(A \times I)$, for every $h \in \mathbb{N}$ $\nabla \tilde{u}_h(x) \in \operatorname{dom} g$ for a.e. $x \in A \times I$ and, by (5.6),

$$\overline{G}(A \times I, \tilde{u}) \leq \liminf_h \int_{A\times I} g(\nabla \tilde{u}_h)\, dx =$$
$$= \liminf_h |I|_{n-k} \int_A g(\nabla_1 u_h, \ldots, \nabla_k u_h, 0_{n-k})\, dy = |I|_{n-k} \int_A f_p(\nabla u)\, dy\,,$$

that is (5.5).

In order to prove the opposite inequality to (5.5) we assume that $\overline{G}(A \times I, \tilde{u}) < +\infty$ so that there exists $\{v_h\} \subseteq W^{1,\infty}_{\text{loc}}(\mathbb{R}^n)$ such that for every $h \in \mathbb{N}$ $\nabla v_h(x) \in \operatorname{dom} g$ for a.e. $x \in A \times I$, $v_h \to \tilde{u}$ in $L^1(A \times I)$ and

$$+\infty > \overline{G}(A \times I, \tilde{u}) = \lim_h \int_{A\times I} g(\nabla v_h)\, dx\,, \tag{5.7}$$

then by (5.7) and (5.1) we have that for every $h \in \mathbb{N}$ $\nabla_{k+1} v_h = \ldots = \nabla_n v_h = 0$ a.e. in $A \times I$ from which, by taking into account the connectedness of I, we infer that v_h depends effectively only on its first k variables in $A \times I$ for every $h \in \mathbb{N}$.

By virtue of this we can assume that for every $h \in \mathbb{N}$ there exists $w_h \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^k)$ such that $v_h = \tilde{w}_h$, then $w_h \to u$ in $L^1(A)$ and by (5.7), (5.4) we have

$$\overline{G}(A \times I, \tilde{u}) = \lim_h \int_{A\times I} g(\nabla_1 w_h, \ldots, \nabla_k w_h, 0_{n-k})\, dx = \tag{5.8}$$

$$= |I|_{n-k} \lim_h \int_A g_p(\nabla w_h)\, dy \geq |I|_{n-k} \overline{G}_p(A, u) = |I|_{n-k} \int_A f_p(\nabla u)\, dy\,.$$

By (5.5) and (5.8) equality (5.2) follows. □

In order to extend (5.2) to a wider class of open sets we need to prove the following subadditivity result.

Lemma 5.2. *Let g be a Borel function as in* (3.1) *verifying* (3.11) $\div$ (3.13), (4.1), (5.1) *and let $\overline{G}$ be given by* (3.2), *then*

$$\overline{G}(\bigcup_{i=1}^{m}(A_i \times I_i), \tilde{u}) \leq \sum_{i=1}^{m} \overline{G}(A_i \times I_i, \tilde{u}) \quad \textit{whenever } A_1, \ldots, A_m \tag{5.9}$$

are pairwise disjoint bounded open sets of $\mathbb{R}^k$, $I_1, \ldots, I_m$ are connected bounded open sets of $\mathbb{R}^{n-k}$, $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^k)$.

Proof. Let $A_1, \ldots, A_m, I_1, \ldots, I_m, u$ be as in (5.9), obviously we can assume the right hand side of (5.9) to be finite so that by Lemma 3.2 we get

$$\nabla\tilde{u}(x) \in \overline{\operatorname{dom} g} \quad \text{for a.e. } x \in \bigcup_{i=1}^{m}(A_i \times I_i)\,, \tag{5.10}$$

moreover, by (4.1), it is not restrictive to assume that the point $z_{\operatorname{dom} g}$ in (3.13) is equal to the origin of $\mathbb{R}^n$, thus getting

$$0_n \in \operatorname{ri}(\operatorname{dom} g)\,. \tag{5.11}$$

By the finiteness of $\sum_{i=1}^{m} \overline{G}(A_i \times I_i, \tilde{u})$, (4.1) and (4.3) for every $i \in \{1, \ldots, m\}$ we deduce the existence of a sequence $\{u_h\} \subseteq W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ such that for every $h \in \mathbb{N}$ $\nabla u^i_h(x) \in \operatorname{dom} g$ for a.e. $x \in A_i \times I_i$, $u^i_h \to \tilde{u}$ in w^*-$W^{1,\infty}(A_i \times I_i)$ as h diverges and

$$\overline{G}(A_i \times I_i, \tilde{u}) = \lim_h \int_{A_i\times I_i} g(\nabla u^i_h)\, dx \quad \text{for every } i \in \{1, \ldots, m\}\,. \tag{5.12}$$

For every $i \in \{1, \dots, m\}$, by (5.1) and the connectedness of I_i, we obtain that for every $h \in \mathbb{N}$ the functions u_h^i depend effectively only on their first k variables in $A_i \times I_i$, because of this from now onwards we will think them as $W^{1,\infty}_{\text{loc}}(\mathbb{R}^k)$ functions.

For every $i \in \{1, \dots, m\}$ let B_i be an open set with $B_i \subset\subset A_i$ and let $\varphi_i \in C_0^1(A_i)$ verifying

$$(5.13)\quad 0 \le \varphi_i \le 1 \text{ in } \mathbb{R}^k, \quad \varphi_i = 1 \text{ in } B_i, \quad \|\nabla \varphi_i\|_{L^\infty(\mathbb{R}^k)} \le \frac{2}{\operatorname{dist}(B_i, \partial A_i)} .$$

For every $h \in \mathbb{N}$ we set $w_h = \sum_{i=1}^m \varphi_i u_h^i + (1 - \sum_{i=1}^m \varphi_i) u$, then obviously $w_h \to u$ in $w^*\text{-}W^{1,\infty}(\bigcup_{i=1}^m A_i)$ and $\tilde{w}_h \to \tilde{u}$ in $w^*\text{-}W^{1,\infty}(\bigcup_{i=1}^m (A_i \times I_i))$. Let us now observe that, being the sets $A_1, \dots, A_m$ pairwise disjoint, it turns out that the values $\phi_1(y), \dots, \phi_m(y)$ are all equal to zero except at most for one as y varies in $\bigcup_{i=1}^m A_i$, hence we have that

$$\nabla \tilde{w}_h = \sum_{i=1}^m \tilde{\varphi}_i \nabla \tilde{u}_h^i + (1 - \sum_{i=1}^m \tilde{\varphi}_i) \nabla \tilde{u} + \sum_{i=1}^m (\tilde{u}_h^i - \tilde{u}) \nabla \tilde{\varphi}_i ,$$

moreover, once recalled that $u_h^i \to u$ in $L^\infty(\operatorname{spt}(\varphi_i))$ for every $i \in \{1, \dots, m\}$, by arguing as in the proof of Lemma 4.1, we get by (3.11), (5.11), (5.10) and (5.13) that

(5.14) for every $t \in [0, 1[$ there exist a compact subset K_t of $\operatorname{ri}(\operatorname{dom} g)$ and $h_t \in \mathbb{N}$ such that for every $h \ge h_t$ $t\nabla \tilde{w}_h(x) \in K_t$ for a.e. $x \in \bigcup_{i=1}^m (A_i \times I_i)$.

By (5.14), being the sets $A_1, \dots, A_m$ pairwise disjoint, we have

$$(5.15)\qquad \overline{G}(\bigcup_{i=1}^m (A_i \times I_i), t\tilde{u}) \le \liminf_h \int_{\bigcup_{i=1}^m (A_i \times I_i)} g(t\nabla \tilde{w}_h)\, dx \le$$

$$\le \sum_{i=1}^m \limsup_h \int_{A_i \times I_i} g(t\nabla \tilde{w}_h)\, dx \le \sum_{i=1}^m \limsup_h \int_{A_i \times I_i} g(t\nabla \tilde{u}_h^i)\, dx +$$

$$+ \sum_{i=1}^m \limsup_h \int_{(A_i \setminus B_i) \times I_i} g(t\nabla \tilde{w}_h)\, dx .$$

Let us now fix $\varepsilon > 0$, then by (3.13) we obtain $t_\varepsilon \in]0,1[$ such that

$$\int_{A_i \times I_i} g(t\nabla \tilde{u}_h^i)\,dx \le \int_{A_i \times I_i} g(\nabla \tilde{u}_h^i)\,dx + \varepsilon |A_i|_k |I_i|_{n-k} \quad \text{for every } i \in \{1,\dots,m\},\ h \in \mathbb{N} \tag{5.16}$$

and by (5.14) and (3.12) that

(5.17) for every $t \in]0,1[$ there exists $M_t > 0$ such that for every $h \ge h_t$ $g(t\nabla \tilde{w}_h(x)) \le M_t$ for a.e. $x \in \bigcup_{i=1}^m (A_i \times I_i)$.

By (5.15) $\div$ (5.17) and (5.12) we conclude,

$$\overline{G}\Big(\bigcup_{i=1}^m (A_i \times I_i), t\tilde{u}\Big) \le \sum_{i=1}^m \overline{G}(A_i \times I_i, \tilde{u}) + \\ + \varepsilon \sum_{i=1}^m |A_i|_k |I_i|_{n-k} + M_t \sum_{i=1}^m |A_i \setminus B_i|_k |I_i|_{n-k}\,. \tag{5.18}$$

Letting first B_i increase to A_i for every $i \in \{1,\dots,m\}$, then t tend to 1^- and finally ε go to 0, we get inequality (5.9) by (5.18) and (3.3). □

We can now prove the representation result for $\overline{G}$ under assumption (4.1).

Theorem 5.3. *Let g be a Borel function as in* (3.1) *verifying* (3.11) $\div$ (3.13), (4.1) *and let $\overline{G}$ be given by* (3.2), *then there exists $f : \mathbb{R}^n \to [0,\infty]$ convex and lower semicontinuous such that*

$$\overline{G}(\Omega, u) = \int_\Omega f(\nabla u)\,dx \quad \textit{for every convex bounded open set } \Omega,\ u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n). \tag{5.19}$$

Proof. Let us assume for a moment that (5.1) holds.

Let Ω, u be as in (5.19) and assume that $\overline{G}(\Omega,u) < +\infty$, then by Lemma 3.2 we get that $\nabla u(x) \in \overline{\mathrm{dom}\, g}$ for a.e. $x \in \Omega$ and therefore, by taking into account (5.1) and the convexity of Ω, that u depends only on its first k variables in Ω. Let $v \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^k)$ be such that $u = \tilde{v}$ in Ω, then it is clear that

$$\overline{G}(\Omega, u) = \overline{G}(\Omega, \tilde{v})\,. \tag{5.20}$$

For every $\nu \in \mathbb{N}$, let R_ν be a partition of $\mathbb{R}^n$, up to a set of zero measure, made up by open cubes with faces parallel to the coordinate planes $A_i \times I_j$

$(i, j \in \mathbb{N})$, where for every $i, j \in \mathbb{N}$ A_i, is an open cube of $\mathbb{R}^k$, I_j is an open cube of $\mathbb{R}^{n-k}$ and let $S^\nu = \{(i, j) \in \mathbb{N} \times \mathbb{N} : A_i \times I_j \subset\subset \Omega\}$.

Let us fix $\nu \in \mathbb{N}$. By (5.20), (3.9), (3.10) and Lemma 5.1 we deduce the existence of $f_p : \mathbb{R}^k \to [0, +\infty]$ convex and lower semicontinuous such that

$$(5.21) \qquad \overline{G}(\Omega, u) \geq \overline{G}(\bigcup_{(i,j)\in S^\nu} A_i \times I_j, \tilde{v}) \geq$$
$$\geq \sum_{(i,j)\in S^\nu} \overline{G}(A_i \times I_j, \tilde{v}) = \sum_{(i,j)\in S^\nu} |I_j|_{n-k} \int_{A_i} f_p(\nabla v)\, dy\,.$$

At this point if we define the function f by

$$(5.22) \qquad f : (z_1, \dots, z_n) \in \mathbb{R}^n \mapsto \begin{cases} f_p(z_1, \dots, z_k) & \text{if } z_{k+1} = \dots = z_n = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

f turns out to be convex and lower semicontinuous, moreover by (5.21) we obtain

$$(5.23) \qquad \overline{G}(\Omega, u) \geq \sum_{(i,j)\in S^\nu} \int_{A_i \times I_j} f(\nabla u)\, dx = \int_{\bigcup_{(i,j)\in S^\nu} (A_i \times I_j)} f(\nabla u)\, dx\,.$$

As ν diverges we deduce by (5.23) that

$$(5.24) \qquad \overline{G}(\Omega, u) \geq \int_\Omega f(\nabla u)\, dx \quad \text{for every convex bounded open set } \Omega,\ u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n).$$

In order to prove the reverse inequality in (5.24) again when (5.1) holds let Ω, u be as in (5.19), obviously we can suppose that $\int_\Omega f(\nabla u)\, dx < +\infty$. By virtue of this and by the convexity of Ω we get that u depends effectively only on its first k variables in Ω and, as before, let $v \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^k)$ be such that $u = \tilde{v}$ in Ω, moreover for every $\nu \in \mathbb{N}$ let R_ν, S^ν be as above.

Let us fix $\nu \in \mathbb{N}$. For every $i \in \mathbb{N}$ let us define $S_i^\nu = \{j \in \mathbb{N} : (i, j) \in S^\nu\}$ and assume, for sake of simplicity, that $S_i^\nu \neq \emptyset$ if and only if $i \in \{1, \dots, m_\nu\}$.

For every $i \in \{1, \dots, m_\nu\}$ set $C_i = (\bigcup_{j\in S_i^\nu} \overline{I}_j)^\circ$, then, by using the convexity of Ω, it turns out that C_i is connected and $\bigcup_{i=j}^{m_\nu} (A_i \times C_i) \subset\subset \Omega$; moreover by (5.22), Lemma 5.1 and Lemma 5.2 we have

$$(5.25) \qquad \int_\Omega f(\nabla u)\, dx = \int_\Omega f_p(\nabla v)\, dx \geq \int_{\bigcup_{i=1}^{m_\nu} (A_i \times C_i)} f_p(\nabla v)\, dx =$$
$$= \sum_{i=1}^{m_\nu} |C_i|_{n-k} \int_{A_i} f_p(\nabla v)\, dy = \sum_{i=1}^{m_\nu} \overline{G}(A_i \times C_i, \tilde{v}) = \overline{G}(\bigcup_{i=1}^{m_\nu} (A_i \times C_i), \tilde{v})\,.$$

Let us now set $\Omega_\nu = \big(\overline{\bigcup_{i=1}^{m_\nu}(A_i \times C_i)}\big)^\circ$, then by (5.25) and (3.8) we deduce that

$$\int_\Omega f(\nabla u)\,dx \geq \overline{G}(\Omega_\nu, \tilde{v}) = \overline{G}(\Omega_\nu, u)\,, \tag{5.26}$$

therefore as ν diverges we obtain by (5.26) that

$$\int_\Omega f(\nabla u)\,dx \geq \overline{G}_-(\Omega, u) \quad \text{for every convex bounded open set } \Omega,\ u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n). \tag{5.27}$$

Finally by (3.9), (3.3), (3.6), (3.7), Proposition 2.1 applied with $U = W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ and $F = \overline{G}$, (1.6), (5.27) we infer that

$$\int_\Omega f(\nabla u)\,dx \geq \overline{G}(\Omega, u) \quad \text{for every convex bounded open set } \Omega,\ u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n). \tag{5.28}$$

By (5.28) and (5.24) we get (5.19) under assumption (5.1).

We now consider the general case, when (5.1) is not assumed.

If $\mathrm{aff}(\mathrm{dom}\, g) = \mathbb{R}^n$ the thesis follows by Theorem 4.4, hence we can assume that the dimension k of $\mathrm{aff}(\mathrm{dom}\, g)$ is strictly smaller than n.

If $k = 0$ $\mathrm{dom}\, g$ consists of a single point and (5.19) follows trivially, hence we can assume that $k \in \{1, \dots, n-1\}$.

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation such that, denoting by M_A the matrix associated to the linear part of A, $\det M_A = 1$, $A(\mathrm{aff}(\mathrm{dom}\, g)) = \mathbb{R}^k \times \{0_{n-k}\}$ and set $g_A : (z_1, \dots, z_n) \in \mathbb{R}^n \mapsto g(A^{-1}(z_1, \dots, z_n))$, then g_A verifies assumptions (3.11) $\div$ (3.13) and $\mathrm{aff}(\mathrm{dom}\, g_A) = \mathbb{R}^k \times \{0_{n-k}\}$.

Let $\overline{G}_A$ be the functional defined by (3.2) with $g = g_A$ and let us observe that for every convex bounded open set Ω the set $A(\Omega)$ is again convex bounded and open.

By the particular case considered above we get the existence of $f_A : \mathbb{R}^n \to [0, +\infty]$ convex and lower semicontinuous such that

$$\overline{G}_A(A^{-1}(\Omega), u^A) = \int_{A^{-1}(\Omega)} f_A(\nabla u^A)\,dy \quad \text{for every convex bounded open set } \Omega,\ u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n), \tag{5.29}$$

u^A being defined by $u^A : (y_1, \dots, y_n) \in \mathbb{R}^n \mapsto u(A(y_1, \dots, y_n))$.

Let us observe now that

$$\overline{G}_A(A^{-1}(\Omega), u^A) = \overline{G}(\Omega, u) \quad \text{for every bounded open set } \Omega,\ u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n) \tag{5.30}$$

and define the function f by $f : z \in \mathbb{R}^n \mapsto f_A(A(z))$, then obviously $f_A(z) = f(A^{-1}(z))$ for every $z \in \mathbb{R}^n$ and by (5.30) and (5.29) we get

$$\overline{G}(\Omega, u) = \overline{G}_A(A^{-1}(\Omega), u^A) = \int_{A^{-1}(\Omega)} f_A(\nabla_y u^A(y))\, dy =$$
$$= \int_{A^{-1}(\Omega)} f(A^{-1}(\nabla_y u^A(y)))\, dy = \int_{A^{-1}(\Omega)} f((\nabla_x u)(A(y)))\, dy =$$
$$= \int_\Omega f(\nabla_x u)\, dx \quad \text{for every bounded open set } \Omega,\ u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n),$$

that is the thesis. □

In the following result we specify the function f in (5.19).

Proposition 5.4. *Let g be a Borel function as in* (3.1) *verifying* (3.11) ÷ (3.13), (4.1) *and f the one appearing in* (5.19) *of Theorem 5.3, then $f = g^{**}$.*

Proof. Similar to the one of Proposition 4.5 but by using Theorem 5.3 in place of Theorem 4.4. □

6. A result on Lipschitz functions without boundedness assumptions of the effective domain.

Let g be a Borel function as in (3.1) and $\overline{G}$ be defined in (3.2).

The present section yields some preliminaries to the integral representation result for $\overline{G}$ when assumption (4.1) is dropped.

For every bounded open set Ω let us introduce the functional $G^{(\infty)}(\Omega, \cdot)$ as

$$G^{(\infty)}(\Omega, \cdot) : u \in W^{1,\infty}(\Omega) \mapsto \inf\Big\{ \liminf_h \int_\Omega g(\nabla u_h)\, dx : \{u_h\} \subseteq W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n),\ u_h \to u \text{ in } w^*\text{-}W^{1,\infty}(\Omega)\Big\} \tag{6.1}$$

and prove an integral representation result for $G^{(\infty)}$.

We observe explicitly that in general, for a given bounded open set Ω, $G^{(\infty)}(\Omega, \cdot)$ needs not to be sequentially w^*-$W^{1,\infty}(\Omega)$-lower semicontinuous and that

$$\overline{G}(\Omega, u) \le G^{(\infty)}(\Omega, u) \quad \text{whenever } \Omega \text{ is a bounded open set},\ u \in W^{1,\infty}(\Omega). \tag{6.2}$$

Theorem 6.1. *Let g be a Borel function as in* (3.1) *verifying* (3.11) $\div$ (3.13) *and let $G^{(\infty)}$ be given by* (6.1)*, then there exists $\phi : \mathbb{R}^n \to [0, +\infty]$ convex and Borel such that*

$$G^{(\infty)}(\Omega, u) \geq \int_\Omega \phi(\nabla u)\, dx \quad \text{for every convex bounded open set } \Omega,\ u \in W^{1,\infty}(\Omega), \tag{6.3}$$

$$G^{(\infty)}(\Omega, u) = \int_\Omega \phi(\nabla u)\, dx \quad \text{for every convex bounded open set } \Omega,\ u \in W^{1,\infty}(\Omega) \text{ such that } G^{(\infty)}(\Omega, u) < +\infty. \tag{6.4}$$

If in addition $(\operatorname{dom} g)^\circ \neq \emptyset$*, then*

$$G^{(\infty)}(\Omega, u) \geq \int_\Omega \phi(\nabla u)\, dx \quad \text{for every bounded open set } \Omega,\ u \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^n), \tag{6.5}$$

$$G^{(\infty)}(\Omega, u) = \int_\Omega \phi(\nabla u)\, dx \quad \text{for every bounded open set } \Omega,\ u \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^n) \text{ such that } G^{(\infty)}(\Omega, u) < +\infty. \tag{6.6}$$

Proof. Let us prove (6.3). For every $m \in \mathbb{N}$ let I_m be the indicator function of Q_m, set $g_m = g + I_m$ and define, for every bounded open set Ω, the functional $\overline{G}_m(\Omega, \cdot)$ by

$$\overline{G}_m(\Omega, \cdot) : u \in L^1(\Omega) \mapsto \inf\Big\{ \liminf_h \int_\Omega g_m(\nabla u_h)\, dx : \{u_h\} \subseteq W^{1,\infty}_{\text{loc}}(\mathbb{R}^n),\ u_h \to u \text{ in } L^1(\Omega)\Big\}.$$

It is clear that the sequence $\{g_m\}$ is decreasing, hence for every bounded open set Ω and u in $L^1(\Omega)$ so is also $\{\overline{G}_m(\Omega, u)\}$, moreover we also have that

$$G^{(\infty)}(\Omega, u) = \inf_{m \in \mathbb{N}} \overline{G}_m(\Omega, u) \quad \text{for every bounded open set } \Omega,\ u \in W^{1,\infty}(\Omega). \tag{6.7}$$

For fixed $m \in \mathbb{N}$ g_m verifies the assumptions of Theorem 5.3, therefore by this result we infer the existence of $f_m : \mathbb{R}^n \to [0, +\infty]$ convex and lower semicontinuous such that

$$(6.8) \qquad \overline{G}_m(\Omega, u) = \int_\Omega f_m(\nabla u)\, dx \quad \text{for every convex bounded open set } \Omega,\ u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n),\ m \in \mathbb{N}.$$

Since for every bounded open set Ω and u in $L^1(\Omega)$ $\{\overline{G}_m(\Omega, u)\}$ is decreasing, for every $z \in \mathbb{R}^n$ the sequence $\{f_m(z)\}$ too verifies the same property, therefore if we define ϕ by

$$(6.9) \qquad \phi : z \in \mathbb{R}^n \mapsto \inf_{m \in \mathbb{N}} f_m(z) \in [0, +\infty]\,,$$

we get that ϕ is convex and Borel and, by (6.7) and (6.8), that

$$(6.10) \qquad G^{(\infty)}(\Omega, u) = \inf_{m \in \mathbb{N}} \overline{G}_m(\Omega, u) = \inf_{m \in \mathbb{N}} \int_\Omega f_m(\nabla u)\, dx \geq$$
$$\geq \int_\Omega \phi(\nabla u)\, dx \quad \text{for every convex bounded open set } \Omega,\ u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n),$$

that is (6.3) once recalled that, being Ω convex, every element of $W^{1,\infty}(\Omega)$ can be extended to an element of $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$.

In order to prove (6.4) let us observe that $\phi(z) = \lim_m f_m(z)$ for every $z \in \mathbb{R}^n$ and that, if Ω is a convex bounded open set, $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ and $G^{(\infty)}(\Omega, u) < +\infty$, then (6.10) yields $\int_\Omega f_{m_0}(\nabla u)\, dx < +\infty$ for some $m_0 \in \mathbb{N}$, so that, by (6.10) and Lebesgue Dominated Convergence Theorem, we conclude that

$$(6.11) \qquad G^{(\infty)}(\Omega, u) = \lim_m \int_\Omega f_m(\nabla u)\, dx = \int_\Omega \phi(\nabla u)\, dx \quad \text{for every convex bounded open set } \Omega,\ u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n) \text{ such that } G^{(\infty)}(\Omega, u) < +\infty.$$

By (6.11) equality (6.4) follows once recalled that, being Ω convex, every element of $W^{1,\infty}(\Omega)$ can be extended to an element of $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$.

The proofs of (6.5) and (6.6) follow exactly as above but by using Theorem 4.4 in place of Theorem 5.3 □

Theorem 6.1 suggests to introduce, for every $g : \mathbb{R}^n \to [0, +\infty]$, the function $g^{(\infty)}$ by

$$(6.12) \qquad g^{(\infty)} : z \in \mathbb{R}^n \mapsto \inf_{m \in \mathbb{N}} (g + I_{Q_m})^{**}(z)\,.$$

In the following result we describe the function ϕ in Theorem 6.1.

Proposition 6.2. *Let g be a Borel function as in* (3.1) *verifying* (3.11) $\div$ (3.13), *$g^{(\infty)}$ be given by* (6.12) *and ϕ be the one appearing in Theorem 6.1, then $\phi = g^{(\infty)}$.*

Proof. Follows by (6.8), Proposition 5.4 and (6.9). □

The following result yields some properties of the function in (6.12).

Proposition 6.3. *Let g be as in* (3.1) *and $g^{(\infty)}$ be given by* (6.12), *then $g^{(\infty)}$ is convex, Borel and*

$$g^{**}(z) \leq g^{(\infty)}(z) \leq \operatorname{co} g(z) \quad \textit{for every } z \in \mathbb{R}^n, \tag{6.13}$$

moreover

$$\operatorname{ri}(\operatorname{dom} g^{**}) = \operatorname{ri}(\operatorname{dom} g^{(\infty)}) = \operatorname{ri}(\operatorname{dom}(\operatorname{co} g)),$$
$$\operatorname{rb}(\operatorname{dom} g^{**}) = \operatorname{rb}(\operatorname{dom} g^{(\infty)}) = \operatorname{rb}(\operatorname{dom}(\operatorname{co} g))$$

and

$$g^{**}(z) = g^{(\infty)}(z) = \operatorname{co} g(z) \quad \textit{for every } z \in \mathbb{R}^n \setminus \operatorname{rb}(\operatorname{dom} f).$$

Proof. It is clear that $g^{(\infty)}$ is convex and Borel.

Since obviously $g^{(\infty)} \leq \inf_{m \in \mathbb{N}} (g + I_{Q_m}) = g$ and $g^{(\infty)}$ is convex we soon obtain that

$$g^{(\infty)}(z) \leq \operatorname{co} g(z) \quad \text{for every } z \in \mathbb{R}^n. \tag{6.14}$$

On the other side, being for every $m \in \mathbb{N}$ $g \leq g + I_{Q_m}$, we have that $g^{**} \leq (g + I_{Q_m})^{**}$ and

$$g^{**}(z) \leq g^{(\infty)}(z) \quad \text{for every } z \in \mathbb{R}^n. \tag{6.15}$$

By (6.14) and (6.15) inequalities in (6.13) follow.

Finally the last parts of the thesis follow by (6.13) and Proposition 1.3 applied to $\operatorname{co} g$, once observed that (1.1) yields

$$g^{**} = (g^{**})^{**} \leq (\operatorname{co} g)^{**} \leq g^{**}. \qquad \square$$

7. A result on BV-functions.

Let g be a Borel function as in (3.1) and $\overline{G}$ be defined in (3.2). In the present section we prove the representation result for $\overline{G}$.

Lemma 7.1. *Let g be a Borel function as in* (3.1) *verifying* (3.11) ÷ (3.13) *and let $\overline{G}$ be given by* (3.2)*, then there exists $f : \mathbb{R}^n \to [0, +\infty]$ convex and lower semicontinuous such that*

$$\overline{G}_-(\Omega, u) = \int_\Omega f(\nabla u)\,dx + \int_\Omega f^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)\,d|D^s u| \quad \textit{for every convex bounded open set } \Omega,\ u \in BV(\Omega). \tag{7.1}$$

If in addition $(\operatorname{dom} g)^\circ \neq \emptyset$*, then*

$$\overline{G}_-(\Omega, u) = \int_\Omega f(\nabla u)\,dx + \int_\Omega f^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)\,d|D^s u| \quad \textit{for every bounded open set } \Omega,\ u \in BV(\Omega). \tag{7.2}$$

Proof. Let us prove (7.1). Let $G^{(\infty)}$ be the functional defined in (6.1), ϕ be the convex Borel function given by Theorem 6.1 and set $f = (\phi + I_{\operatorname{dom} g})^{**}$, then it is clear that f is convex, lower semicontinuous and that, being obviously $\phi \le g$, $f \le \phi + I_{\operatorname{dom} g} \le g$.

By virtue of this and of Proposition 1.7 we soon get

$$\overline{G}_-(\Omega, u) \ge \int_\Omega f(\nabla u)\,dx + \int_\Omega f^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)\,d|D^s u| \quad \text{for every bounded open set } \Omega,\ u \in BV(\Omega). \tag{7.3}$$

In order to prove the reverse inequality in (7.3) let us first observe that $I_{\operatorname{dom} g} \le \phi + I_{\operatorname{dom} g} \le g$ from which we conclude that $\operatorname{dom}(\phi + I_{\operatorname{dom} g}) = \operatorname{dom} g$ and, together with (3.11) and Proposition 1.3, that it results

$$\operatorname{ri}(\operatorname{dom} f) = \operatorname{ri}(\operatorname{dom}(\phi + I_{\operatorname{dom} g})) = \operatorname{ri}(\operatorname{dom} g)\,, \tag{7.4}$$

$$f(z) = \phi(z) + I_{\operatorname{dom} g}(z) = \phi(z) \quad \text{for every } z \in \operatorname{ri}(\operatorname{dom} f). \tag{7.5}$$

Let Ω be as in (7.1), $u \in C^\infty(\mathbb{R}^n)$, $z_1 \in \operatorname{ri}(\operatorname{dom} f)$, $t \in [0, 1[$ and observe that we can assume $\int_\Omega f(\nabla u)\,dx < +\infty$ so that $\nabla u(x) \in \operatorname{dom} f$ for every $x \in \Omega$ and there exists a compact subset K_t of $\operatorname{ri}(\operatorname{dom} f)$ such that $t\nabla u(x) + (1-t)z_1 \in K_t$ for every $x \in \Omega$. By (7.4) it follows that $K_t \subseteq \operatorname{ri}(\operatorname{dom} g)$ and hence, by using also (3.12), that

$$G^{(\infty)}(\Omega, tu + (1-t)u_{z_1}) \le \int_\Omega g(t\nabla u + (1-t)z_1)\,dx < +\infty\,.$$

This, together with (3.9), (6.2), Theorem 6.1, (7.5) and the convexity of f implies that

$$(7.6)\qquad \overline{G}_{-}(\Omega, tu+(1-t)u_{z_1}) \leq G^{(\infty)}(\Omega, tu+(1-t)u_{z_1}) = \\ = \int_\Omega \phi(t\nabla u+(1-t)z_1)\,dx = \int_\Omega f(t\nabla u+(1-t)z_1)\,dx \leq \\ \leq \int_\Omega f(\nabla u)\,dx + (1-t)f(z_1)|\Omega|,$$

hence, as t increases to 1, we obtain by (7.6) and (3.3) that

$$(7.7)\qquad \overline{G}_{-}(\Omega, u) \leq \int_\Omega f(\nabla u)\,dx \quad \text{for every convex} \\ \text{bounded open set } \Omega,\ u \in C^\infty(\mathbb{R}^n).$$

By (3.9), (3.3) and (7.7) the assumptions of Proposition 1.6 with $\mathcal{A}_0$ equal to the family of the convex bounded open sets, and, for every bounded open set Ω, $F(\Omega, \cdot) = \overline{G}(\Omega, \cdot)$ are fulfilled, hence by Proposition 1.6 we obtain

$$(7.8)\quad \overline{G}_{-}(\Omega, u) \leq \int_\Omega f(\nabla u)\,dx + \int_\Omega f^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)\,d|D^s u| \quad \text{for every} \\ \text{convex bounded open set } \Omega,\ u \in BV(\Omega).$$

By (7.8) and (7.3) equality (7.1) follows.

The proof of (7.2) follows exactly as above with the only difference that in this case (7.7) holds for every bounded open set and by taking $\mathcal{A}_0$ equal to the family of the bounded open sets in the application of Proposition 1.6. □

Theorem 7.2. *Let g be a Borel function as in* (3.1) *verifying* (3.11) $\div$ (3.13) *and let $\overline{G}$ be given by* (3.2)*, then there exists $f : \mathbb{R}^n \to [0, +\infty]$ convex and lower semicontinuous such that*

$$(7.9)\quad \overline{G}(\Omega, u) = \int_\Omega f(\nabla u)\,dx + \int_\Omega f^\infty\Big(\frac{dD^s u}{d|D^s u|}\Big)d\,|D^s u| \quad \textit{for every} \\ \textit{convex bounded open set } \Omega,\ u \in BV(\Omega).$$

Proof. Let f be given by Lemma 7.1, then by (3.9), (3.3), (3.6) and (3.7) Proposition 2.1 applies with $U = BV_{\text{loc}}(\mathbb{R}^n)$, $F = \overline{G}$ and by (1.6) and Lemma 7.1 we conclude that (7.9) holds. □

In the following proposition we identify the function f in Theorem 7.2.

Proposition 7.3. *Let g be a Borel functions as in* (3.1) *verifying* (3.11)÷(3.13) *and f the one appearing in Theorem 7.2, then $f = g^{**}$.*

Proof. By Proposition 6.3 we have

$$g^{**} = (g^{**})^{**} \leq (g^{(\infty)} + I_{\mathrm{dom}\, g})^{**} \leq (g + I_{\mathrm{dom}\, g})^{**} = g^{**}, \tag{7.10}$$

therefore by the definition of f in Lemma 7.1, Proposition 6.2 and (7.10) the thesis follows. □

By the above result we deduce the following corollary.

Corollary 7.4. *Let $g : \mathbb{R}^n \to [0, +\infty[$ be continuous and C be a convex subset of $\mathbb{R}^n$, then*

$$\begin{aligned} &\inf\Big\{ \liminf_h \int_\Omega g(\nabla u_h)\, dx : \{u_h\} \subseteq W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n), \\ &\textit{for every } \ h \in \mathbb{N}\ \nabla u_h(x) \in C \ \textit{ for a.e. } \ x \in \Omega,\ u_h \to u \ \textit{ in } \ L^1(\Omega)\Big\} = \\ = &\int_\Omega (g + I_C)^{**}(\nabla u)\, dx + \int_\Omega ((g + I_C)^{**})^{\infty}\Big(\frac{dD^s u}{d|D^s u|}\Big)\, d|D^s u| \quad \textit{for every} \\ &\hspace{6cm} \textit{convex bounded open set } \Omega,\ u \in BV(\Omega). \end{aligned}$$

Proof. Follows by Theorem 7.2 and Proposition 7.3. □

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