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ABOUT THE MULTIPLICITY OF SOLUTIONS FOR CERTAIN CLASS OF FOURTH ORDER SEMILINEAR PROBLEMS

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Dedicated to Francesco Guglielmino on his 70th birthday

We consider the following problem:

$$(P) \qquad \begin{cases} \Delta^2 u + a^2 \Delta u = b\Big[(u+1)^+ - 1\Big] & \text{in } \Omega, \\ \Delta u = 0, \ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth open bounded set in $\mathbb{R}^N$, Δ^2 is the biharmonic operator, $u^+ = \max\{u, 0\}$, and a, b are constants. In this paper we study the problem (P) when $a^2 \geq \lambda_1$ and a^2 is close to λ_1 (here $(\lambda_k)_{k\geq 1}$ is the sequence of the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$). Moreover we replace the nonlinearity $(u+1)^+ - 1$ by a more general function g, by using a variational approach. Here we prove the existence of a nontrivial solution if either $b > \lambda_2(\lambda_2 - a^2)$ or $b < \lambda_1(\lambda_1 - a^2)$ and the existence of two nontrivial solutions when $b > \lambda_k(\lambda_k - a^2)$ and b is close to $\lambda_k(\lambda_k - a^2)$, for any $\lambda_k > \lambda_2$. Finally we show that if $a^2 = \lambda_1$ and $b < 0$ the problem (P) has only the trivial solution.

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Introduction.

Let Ω be a smooth open bounded set in $\mathbb{R}^N$. Let us consider the problem of the existence of nontrivial solutions of the following nonlinear equation:

$$(P) \qquad \begin{cases} \Delta^2 u + a^2 \Delta u = b\big[(u+1)^+ - 1\big] & \text{in} \quad \Omega, \\ \Delta u = 0, \ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Δ^2 is the biharmonic operator, $u^+ = \max\{u, 0\}$ and a, b are constants.

This fourth order semilinear elliptic problem has been pointed out by Lazer and McKenna in [4] as a possible model to study traveling waves in suspension bridges and in [5] they proved the existence of $2k - 1$ solutions when $\Omega \subset \mathbb{R}$ is an interval, $a^2 < \lambda_1$ and $b > \lambda_k(\lambda_k - a^2)$, by the global bifurcation method. (Here $(\lambda_k)_{k\geq 1}$ is the sequence of the eigenvalues of $-\Delta$ in H_0^1). Tarantello in [14] found a negative solution of (P) when $a^2 < \lambda_1$ and $b \geq \lambda_1(\lambda_1 - a^2)$, by a degree argument.

It is clear that the number of solutions of (P) depends on the position of a^2 and b with respect to λ_k and $\lambda_k(\lambda_k - a^2)$, respectively. We study the problem (P), when the nonlinearity $(u+1)^+ - 1$ is replaced by a more general function g (see (1.1)), as it has been suggested in [4] and [9]. It is our purpose to use a variational viewpoint.

In [10] by studying the geometry of the functional in the case $a^2 < \lambda_1$ we have the existence of two solutions if $b > \lambda_1(\lambda_1 - a^2)$ by a variation of linking theorem and the existence of three solutions if b is suitable close to $\lambda_k(\lambda_k - a^2)$ by a theorem of existence of three critical values. In [11] we study (P) when a^2 goes beyond λ_1 and we prove the existence of two solutions for b in a suitable position with respect to $\lambda_k(\lambda_k - a^2)$, by a different suitable use of a variation of linking theorem. Moreover in the case $g(s) = (s+1)^+ - 1$ we obtain some uniqueness result.

In this paper we study the case $a^2 \geq \lambda_1$ and a^2 close to λ_1. This is the "richest" case: problem (P) has a greater number of solutions than in the previous situation. The existence of a nontrivial solutions is proved when $b > \lambda_2(\lambda_2 - a^2)$ (see Theorem 2.12) and also when $b < \lambda_1(\lambda_1 - a^2)$ (see Theorem 4.7). Moreover the existence of two nontrivial solutions is proved when $b > \lambda_k(\lambda_k - a^2)$ and b is close to $\lambda_k(\lambda_k - a^2)$, for any $\lambda_k > \lambda_2$, (see Theorem 3.5).

1. The problem.

We consider the problem of the existence of solutions of the more general equation:

$$\begin{cases} \Delta^2 u + c\Delta u = bg(x,u) & \text{in } \Omega, \\ \Delta u = 0,\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth open bounded set in $\mathbb{R}^N$, $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Caratheodory's function and $b, c \in \mathbb{R}$. We study (1.1) by using a variational approach.

Definition 1.2. *Let $f_{bc} : H \longrightarrow \mathbb{R}$ be defined by:*

$$f_{bc}(u) = \frac{1}{2}\Big(\int (\Delta u)^2 - c\int |\nabla u|^2\Big) - b\int G(x,u),$$

where $G(x,s) = \int_0^s g(x,\sigma)d\sigma$. Let $H = H^2(\Omega) \cap H_0^1(\Omega)$ be the Hilbert space equipped with the inner product

$$\big(u, v\big)_H = \int \Delta u \Delta v + \int \nabla u \nabla v.$$

Remark 1.3. It is well known that if, for example, we assume:

$$(g) \quad |g(x,s)| \le a_0(x) + b_0|s|,\ \forall s \in \mathbb{R} \text{ and a.e. in } \Omega,$$
$$\text{where } a_0 \in L^2(\Omega) \text{ and } b_0 \in \mathbb{R}.$$

f_{bc} is a C^1 functional and its critical points are weak solutions of problem (1.1).

To use a variational approach it is necessary to study the Palais-Smale condition.

Definition 1.4. *We say that f_{bc} satisfies the Palais-Smale condition if for every sequence $(u_n)_{n\in\mathbb{N}}$ in H with $f_{bc}(u_n)$ bounded and $\lim_n \nabla f_{bc}(u_n) = 0$, there exists a convergent subsequence.*

Now we give a sufficient condition to obtain the Palais-Smale condition.

Proposition 1.5. *Assume (g) (see Remark 1.3) and:*

$$\begin{cases} (g_{+\infty}) \quad \lim\limits_{s\to+\infty} \dfrac{g(x,s)}{s} = 1 \text{ uniformly with respect to } x; \\ (G^*) \quad 2G(x,s) - g(x,s)s \ge \alpha_0(x)s^- - \alpha_1(x)\ \forall s \in \mathbb{R},\ a.e. in\ \Omega \\ \qquad \text{where } \alpha_0 \in L^\infty(\Omega),\ \alpha_0(x) > 0 \text{ a.e. in } \Omega \text{ and } \alpha_1 \in L^1(\Omega). \end{cases} \tag{1.6}$$

Then for any $c \in \mathbb{R}$, $b \neq \Lambda_1(c)$ and $b \neq 0$ the functional f_{bc} satisfies the Palais-Smale condition.

Proof. We give the proof (see [11]) for sake of completeness. First of all we observe that:

$$\nabla f_{bc}(u) = u + i^*\big((1+c)\Delta u - bg(x,u)\big), \tag{1.7}$$

where $i^* : L^2(\Omega) \longrightarrow H$ is a compact operator.

(i^* is the adjoint of the immersion $i : H \hookrightarrow L^2(\Omega)$).

Now let $(u_n)_{n\in\mathbb{N}}$ be a Palais-Smale sequence (see (1.4)). In particular:

$$\lim_n \nabla f_{bc}(u_n) = \lim_n \Big(u_n + i^*\big((1+c)\Delta u_n - bg(x,u_n)\big)\Big) = 0 \quad \text{strongly in } H. \tag{1.8}$$

It is enough to prove that $(\|u_n\|_H)_{n\in\mathbb{N}}$ is bounded, because of (1.7) and (g). By contradiction we suppose that $\lim_n \|u_n\|_H = +\infty$. Up to a subsequence we can assume that $\lim_n \frac{u_n}{\|u_n\|_H} = u$ weakly in H, strongly in $L^2(\Omega)$ and pointwise in Ω. By (1.8) we deduce:

$$\big(\nabla f_{bc}(u_n), \frac{u_n}{\|u_n\|_H}\big)_H = \frac{1}{\|u_n\|_H}\Big(\int |\Delta u_n|^2 - c\int |\nabla u_n|^2\Big) -$$
$$-b\int g(x,u_n)\frac{u_n}{\|u_n\|_H} = 2\frac{f_{bc}(u_n)}{\|u_n\|_H} + b\int \Big(2G(x,u_n) - g(x,u_n)u_n\Big)\frac{1}{\|u_n\|_H};$$

then passing to the limit, since $b \neq 0$:

$$\lim_n \int \Big(2G(x,u_n) - g(x,u_n)u_n\Big)\frac{1}{\|u_n\|_H} = 0.$$

Moreover by (G^*) of (1.6) we get:

$$\int \Big(2G(x,u_n) - g(x,u_n)u_n\Big)\frac{1}{\|u_n\|_H} \geq \int \alpha_0 \frac{(u_n)^-}{\|u_n\|_H} - \int \frac{\alpha_1(x)}{\|u_n\|_H}$$

and so passing to the limit:

$$0 \geq \int \alpha_0 u^-, \quad \text{which implies } u \geq 0 \text{ a.e. in } \Omega.$$

Then by $(g_{+\infty})$ of (1.6) and (g), using the Lebesgue's Theorem, we get:

$$\lim_n \frac{g(x,u_n)}{\|u_n\|_H} = u \quad \text{strongly in } L^2(\Omega). \tag{1.9}$$

On the other hand by (1.8) we get:

$$(1.10)\qquad 0=\lim_n \frac{\nabla f_{bc}(u_n)}{\|u_n\|_H}=$$

$$\lim_n \left\{\frac{u_n}{\|u_n\|_H}+i^*\Big[(1+c)\frac{\Delta u_n}{\|u_n\|_H}-b\frac{g(x,u_n)}{\|u_n\|_H}\Big]\right\} \qquad \text{strongly in } H.$$

Finally by (1.7), (1.9) and (1.10) we obtain:

$$\lim_n \frac{u_n}{\|u_n\|_H}=u \text{ strongly in } H \text{ and}$$

$$u\geq 0 \text{ is a non trivial solution of } \Delta^2 u+c\Delta u=bu.$$

(We recall that the sequence $\big(\frac{\Delta u_n}{\|u_n\|_H}\big)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, so it converges weakly in $L^2(\Omega)$ and $\big(i^*\frac{\Delta u_n}{\|u_n\|_H}\big)$ converges strongly in H). A contradiction arises, because $b\neq\Lambda_1(c)$.

We will use the following assumptions to build the geometric structures of the functional, which allow us to apply the variational principles of Section 4:

$$(1.11)\qquad \begin{cases} (G) & 0\leq 2G(x,s)\leq s^2 \text{ a.e. in } \Omega \text{ and } \forall s\in\mathbb{R};\\ (G_{-\infty}) & \lim_{s\to-\infty}\dfrac{2G(x,s)}{s^2}=0 \text{ uniformly with respect to } x;\\ (G_0) & \lim_{s\to 0}\dfrac{2G(x,s)}{s^2}=1 \text{ uniformly with respect to } x. \end{cases}$$

We note that if (G) and (G_0) hold then $g(\cdot\,,0)=0$ and (1.1) has the trivial solution.

Remark 1.12. We denote by λ_k the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ and by e_k the eigenfunction corresponding to λ_k normalized in $L^2(\Omega)$; we can choose $e_1>0$ in Ω. Let $\Lambda_k(c)=\lambda_k(\lambda_k-c)$. Set $H_k=\mathrm{span}(e_1,\dots,e_k)$ and $H_k^{\perp}=\{w\in H \mid (w,v)_H=0\ \forall v\in H\}$. We put $H_0=0$.

In the following we consider the case $\lambda_1\leq c<\lambda_2$.

2. A non trivial solution when c is close to λ_1 and $b \geq \Lambda_2(c)$.

We succeed to build a linking for the functional f_{bc} using a suitable vector. Hence we have a non trivial solution by the "variation of linking" Theorem 5.2.

We start with a technical lemma.

Lemma 2.1. *Assume* (G) *and* $(G_{-\infty})$ *(see* (1.11)*). Let* $b \geq 0$. *Then for any* $\varepsilon > 0$ *there exists* $h > 0$ *such that:*

$$f_{bc}(u) \geq \frac{1}{2}\Big(\int |\Delta u|^2 - c\int |\nabla u|^2\Big) - \frac{b}{2}\int (u^+)^2 - \varepsilon\int u^2 - h.$$

Proof. By definition of f_{bc}, and by (G) we get:

$$\begin{aligned} f_{bc}(u) &= \frac{1}{2}\Big(\int |\Delta u|^2 - c\int |\nabla u|^2\Big) - b\int G(x,u) = \\ &= \frac{1}{2}\Big(\int |\Delta u|^2 - c\int |\nabla u|^2\Big) - \frac{b}{2}\int (u^+)^2 + \\ &+ \frac{b}{2}\int_{\{x\in\Omega:u(x)\geq 0\}} \big(u^2 - 2G(x,u)\big) - \frac{b}{2}\int_{\{x\in\Omega:u(x)\leq 0\}} 2G(x,u) \geq \\ &\geq \frac{1}{2}\Big(\int |\Delta u|^2 - c\int |\nabla u|^2\Big) - \frac{b}{2}\int (u^+)^2 - b\int_{\{x\in\Omega:u(x)\leq 0\}} G(x,u). \end{aligned}$$

By $(G_{-\infty})$ and (G) we get that for any $\varepsilon > 0$ there exists $h > 0$ such that:

$$\int_{\{x\in\Omega:u(x)\leq 0\}} G(x,u) \leq \varepsilon\int_\Omega u^2 + h.$$

The claim follows. $\square$

Lemma 2.2. *Assume* (G) *(see* (1.11)*). If* $0 < b \leq \Lambda_{i+1}(c)$ *for* $i \geq 1$, *then:*

$$\inf_{w\in H_i^\perp} f_{bc}(w) \geq 0.$$

Proof. If $b > 0$ by (G) we obtain for any $w \in H_i^\perp$:

$$\begin{aligned} f_{bc}(w) &= \frac{1}{2}\Big(\int |\Delta w|^2 - c\int |\nabla w|^2\Big) - b\int G(x,w) \geq \\ &\geq \frac{1}{2}\Big(\int |\Delta w|^2 - c\int |\nabla w|^2\Big) - \frac{b}{2}\int w^2 \geq \\ &\geq \frac{1}{2}\Big(1 - \frac{b}{\Lambda_{i+1}(c)}\Big)\Big(\int |\Delta w|^2 - c\int |\nabla w|^2\Big) \geq 0, \end{aligned}$$

since $b \leq \Lambda_{i+1}(c)$. □

Lemma 2.3. *Assume* (G_0) *(see* (1.11)*). If* $\Lambda_i(c) < b$ *for* $i \geq 1$*, then there exists* $\rho > 0$ *such that:*

$$\sup_{\substack{v \in H_i \\ \|v\|_{L^2} = \rho}} f_{bc}(v) < 0.$$

Proof. By (G_0) we get for any $\varepsilon > 0$ there exists $\rho > 0$ such that if $|s| \leq \rho$ then $2G(x,s) \geq (1-\varepsilon)s^2$ a.e. in Ω. Thus if $v \in H_i$ with $\|v\|_{L^\infty} \leq \rho$ we have:

$$\begin{aligned} f_{bc}(v) &= \frac{1}{2}\Big(\int |\Delta v|^2 - c\int |\nabla v|^2\Big) - b\int G(x,v) \leq \\ &\leq \frac{1}{2}\Big(\int |\Delta v|^2 - c\int |\nabla v|^2\Big) - \frac{b}{2}(1-\varepsilon)\int v^2 \leq \\ &\leq \frac{1}{2}\Big(\Lambda_i(c) - b(1-\varepsilon)\Big)\int v^2, \end{aligned} \tag{2.4}$$

and so our claim follows ($\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^\infty}$ are equivalent, since $\dim H_i < +\infty$). □

Lemma 2.5. *Let* $h \geq 1$*. Set:*

$$\beta_{h+1}(c) = \max\left\{\int (z^+)^2 \mid z \in H_h^\perp,\ \int |\Delta z|^2 - c\int |\nabla z|^2 = 1\right\}. \tag{2.6}$$

Then:

$$\beta_{h+1}(c) < \frac{1}{\Lambda_{h+1}(c)}.$$

Proof. It is easy to see that $\beta_{h+1}(c) \le \frac{1}{\Lambda_{h+1}(c)}$. If $\beta_{h+1}(c) = \frac{1}{\Lambda_{h+1}(c)}$ then there exists a sequence $(z_n)_{n\in\mathbb{N}}$ in $H_h^\perp$ such that $\int |\Delta z_n|^2 - c\int |\nabla z_n|^2 = 1$ and $\lim_n \int (z_n^+)^2 = \frac{1}{\Lambda_{h+1}(c)}$. We point out that $\|z\|_H^2$ and $\int |\Delta z|^2 - c\int |\nabla z|^2$ are equivalent norms in $H_h^\perp$, since $c < \lambda_{h+1}$. So, up to a subsequence, we have $\lim_n z_n = z$ in $L^2(\Omega)$, so that $\int (z^+)^2 = \frac{1}{\Lambda_{h+1}(c)}$ and then $z \ne 0$. Moreover, since $z \in H_h^\perp \setminus \{0\}$ and $\int |\Delta z|^2 - c\int |\nabla z|^2 \le 1$, we have $0 \le \int (z^+)^2 + \int (z^-)^2 \le \frac{1}{\Lambda_{h+1}(c)}$; so $z^- = 0$. On the other hand we have $\int z e_1 = 0$, which implies $z^- \ne 0$. Then a contradiction arises. □

Lemma 2.7. *Set*

$$\lambda^* = \sup\Big\{\lambda \ge \lambda_1 \mid \exists e^* \in H_2 \setminus \{0\} \text{ s.t. } e^*(x) \le 0 \text{ in } \Omega \text{ and } \int |\Delta e^*|^2 - \lambda \int |\nabla e^*|^2 > 0\Big\}. \tag{2.8}$$

Then:

$$\lambda_1 < \lambda^* < \lambda_2.$$

Proof. It is easy to see that $\lambda^* < \lambda_2$. To get that $\lambda^* > \lambda_1$, it is enough to prove that:

$$\exists \delta > 0 \text{ s.t. } \forall c \in]\lambda_1, \lambda_1 + \delta[\ \exists e^* \in H_2,\ e^* \le 0 \text{ in } \Omega \text{ s.t.}$$
$$\int |\Delta e^*|^2 - c\int |\nabla e^*|^2 > 0.$$

We choose $e^*(x) = s e_2(x) - e_1(x)$ with $s \in \mathbb{R}$ and we take s so small that e^* is negative in Ω and c so close to λ_1 that:

$$\int |\Delta e^*|^2 - c\int |\nabla e^*|^2 = s^2\Lambda_2(c) - \Lambda_1(c) > 0.$$

That proves our statement. □

Lemma 2.9. *Assume* (G) *and* $(G_{-\infty})$ *(see* (1.11)*). Let* $\lambda_1 \le c < \lambda^*$ *(see* (2.8)*) and* $0 < b < \dfrac{1}{\beta_{h+1}(c)}$ *(see* (2.6)*) for some* $h \ge 2$*. Then there exist* $e^* \in H_h \setminus \{0\}$ *and* $R_0 > 0$ *such that for any* $R \ge R_0$:

$$\inf\Big\{f_{bc}(z) \mid z = w + \sigma e^*,\ w \in H_h^\perp,\ \sigma \ge 0,\ \int |\Delta z|^2 - c\int |\nabla z|^2 = R^2\Big\} > 0.$$

Proof. Since $c < \lambda^*$ by Lemma 2.7 there exists $e^* \in H_2 \subset H_h$, $e^* \leq 0$ in Ω such that:

$$\int |\Delta e^*|^2 - c\int |\nabla e^*|^2 > 0. \tag{2.10}$$

Now by Lemma 2.1, we get for any $w \in H_h^\perp$ and $\sigma \geq 0$, because of the negativity of e^*:

$$\begin{aligned} f_{bc}(w+\sigma e^*) \geq & \frac{1}{2}\Big(\int |\Delta w|^2 - c\int |\nabla w|^2\Big) + \\ & + \frac{1}{2}\sigma^2\Big(\int |\Delta e^*|^2 - c\int |\nabla e^*|^2\Big) - \\ & - \frac{b}{2}\int \big((w+\sigma e^*)^+\big)^2 - \varepsilon\int w^2 - \varepsilon\sigma^2 - h \geq \\ & \geq \frac{1}{2}\Big(\int |\Delta w|^2 - c\int |\nabla w|^2\Big) + \\ & + \frac{1}{2}\sigma^2\Big(\int |\Delta e^*|^2 - c\int |\nabla e^*|^2\Big) - \\ & - \frac{b}{2}\int (w^+)^2 - \varepsilon\int w^2 - \varepsilon\sigma^2 - h \geq \\ & \geq \frac{1}{2}\left(1 - b\beta_{h+1}(c) - \frac{2\varepsilon}{\Lambda_{h+1}(c)}\right)\Big(\int |\Delta w|^2 - c\int |\nabla w|^2\Big) + \\ & + \frac{1}{2}\sigma^2\Big(\int |\Delta e^*|^2 - c\int |\nabla e^*|^2 - 2\varepsilon\Big) - h. \end{aligned}$$

Thus the claim follows, since in virtue of (2.10) $\|w+\sigma e^*\|_H^2$ and

$$\int |\Delta w|^2 - c\int |\nabla w|^2 + \sigma^2\Big(\int |\Delta e^*|^2 - c\int |\nabla e^*|^2\Big)$$

are equivalent norms in the space span $(e^*) \oplus H_h^\perp$. $\square$

The following remark will be useful in the proof of Theorem 3.5.

Remark 2.11. Assume (G) and $(G_{-\infty})$ (see (1.11)). Let $\lambda_1 \leq c < \lambda^*$ (see (2.8)) and $0 < b < \frac{1}{\beta_{h+1}(c)}$ (see (2.6)) for some $h \geq 2$. Then there exists $R_0 > 0$ such that for any $R \geq R_0$:

$$\inf\Big\{ f_{bc}(z) \mid z = w + \sigma e_{h+1},\ w \in H_{h+1}^\perp,\ \sigma \geq 0, \\ \int |\Delta z|^2 - c\int |\nabla z|^2 = R^2 \Big\} > 0.$$

Theorem 2.12. *Assume* (g) *(see* (1.3)*),* (1.11) *and* (1.6). *If* $\lambda_1 \le c < \lambda^*$ *(see* (2.8)*) and* $b > \Lambda_2(c)$*, then the functional* f_{bc} *has at least two different critical values.*

Proof. By Lemmas 2.9, 2.2 and 2.3 it follows that, if $\Lambda_i(c) < b \le \Lambda_{i+1}(c) < \frac{1}{\beta_{i+1}(c)}$ (see (2.6)) for $i \ge 2$, there exist $e^* \in H_2 \setminus \{0\}$ and $R > \rho > 0$ such that:

$$\inf_{z \in \Sigma_R(e^*, H_i^\perp)} f_{bc}(z) > \sup_{\substack{v \in H_i \\ \|v\|_{L^2} = \rho}} f_{bc}(v),$$

where $\Sigma_R(e^*, H_i^\perp)$ is the boundary of the set $\{z = w + \sigma e^* \mid w \in H_i^\perp, \ \sigma \ge 0, \ \int |\Delta z|^2 - c \int |\nabla z|^2 \le R^2\}$ in $\operatorname{span}(e^*) \oplus H_i^\perp$. The claim follows by the variational statement 5.2. □

3. Two non trivial solutions when c is close to λ_1 and $b \ge \Lambda_2(c)$.

Now we build another linking for the functional $f_{b,c}$ in such a way as to use the "linking scale" Theorem 5.3.

Lemma 3.1. *Let* $k \ge 1$. *Set:*

$$l_k(b, c) = \inf_{w \in H_k^\perp} f_{bc}(w).$$

Assume (G) *and* $(G_{-\infty})$ *(see* (1.11)*). Then:*

(i) $0 \le b < \frac{1}{\beta_{k+1}(c)} \quad \Rightarrow \quad l_k(b, c) > -\infty$*, where:*

$$\beta_{k+1}(c) = \max \left\{ \int (w^+)^2 \mid w \in H_k^\perp, \ \int |\Delta w|^2 - c \int |\nabla w|^2 = 1 \right\} < \frac{1}{\Lambda_{k+1}(c)}$$

(see (2.6)*);*

(ii) $0 \le b \le \Lambda_{k+1}(c) \quad \Rightarrow \quad l_k(b, c) \ge 0$*;*

(iii) $\liminf_{b \to \Lambda_{k+1}(c)} l_k(b, c) \ge 0$.

Proof. First of all we denote by $\|w\|_c^2 = \int |\Delta w|^2 - c \int |\nabla w|^2$. Since $c < \lambda_{k+1}$, $\|\cdot\|_c$ and $\|\cdot\|_H$ are norms equivalent in the space $H_k^\perp$.

(i) If $w \in H_k^\perp$, by (2.1) we get:

$$f_{bc}(w) \ge \frac{1}{2}\|w\|_c^2 - \frac{b}{2} \int (w^+)^2 - \qquad (3.2)$$

$$- \varepsilon \int w^2 - h \ge \frac{1}{2}(1 - b\beta_{k+1}(c) - \frac{\varepsilon}{\Lambda_{k+1}(c)})\|w\|_c^2 - h.$$

Then it follows the existence of a minimum point of f_{bc} on $H_k^{\perp}$, because of the lower semicontinuity of f_{bc}.

(ii) If $0 \le b \le \Lambda_{k+1}(c)$ and $w \in H_k^{\perp}$, by (G) we get:

$$f_{bc}(w) = \frac{1}{2}\|w\|_c^2 - b\int G(x, w) \ge$$
$$\ge \frac{1}{2}\|w\|_c^2 - \frac{b}{2}\int w^2 \ge \frac{1}{2}(1 - \frac{b}{\Lambda_{k+1}(c)})\|w\|_c^2 \ge 0.$$

(iii) If $\lim_n b_n = \Lambda_{k+1}(c)$, we show that $\liminf_n l_k(b_n, c) \ge 0$. In *(i)* we have shown the existence of $w_n \in H_k^{\perp}$ such that:

$$\frac{1}{2}\|w_n\|_c^2 - b_n\int G(x, w_n) = l_k(b_n, c) \le \tag{3.3}$$
$$\le \frac{1}{2}\|w\|_c^2 - b_n\int G(x, w), \quad \forall w \in H_k^{\perp}.$$

Arguing by contradiction, we suppose $\lim_n \|w_n\|_c = +\infty$. Up a subsequence, we have $\lim_n \frac{w_n}{\|w_n\|_c} = w$ weakly in H, strongly in $L^2(\Omega)$ and a.e. in Ω, with $\|w\|_c \le 1$. Now we observe that by (2.1) we get:

$$l_k(b_n, c) \ge \frac{1}{2}\|w_n\|_c^2 - \frac{b_n}{2}\int (w_n{}^+)^2 - b_n \int_{\{x\in\Omega: w_n(x)\le 0\}} G(x, w_n).$$

As a result by this fact and by (3.3) it follows:

$$0 \ge \limsup_n \frac{l_k(b_n, c)}{\|w_n\|_c^2} \ge \liminf_n \frac{l_k(b_n, c)}{\|w_n\|_c^2} \ge$$
$$\ge \frac{1}{2}\Big(1 - \Lambda_{k+1}(c)\int (w^+)^2\Big) - \Lambda_{k+1}(c)\limsup_n \int_{\{x\in\Omega: w_n(x)\le 0\}} \frac{G(x, w_n)}{\|w_n\|_c^2}.$$

Moreover, by (G) and $(G_{-\infty})$, using Fatou's lemma, we get:

$$\limsup_n \int_{\{x\in\Omega: w_n(x)\le 0\}} \frac{G(x, w_n)}{\|w_n\|_c^2} \le 0;$$

then $1 - \Lambda_{k+1}(c)\int (w^+)^2 \le 0$.

By (2.5) a contradiction arises, since $\int (w^+)^2 \leq \beta_{k+1}(c)\|w\|_c^{\,2} \leq \beta_{k+1}(c)$ and $\beta_{k+1}(c) < \frac{1}{\Lambda_{k+1}(c)}$. Finally, since $(w_n)_{n\in\mathbb{N}}$ is bounded in H, up to a subsequence, we can suppose $\lim_n w_n = w_0$ weakly in H and strongly in $L^2(\Omega)$. By (3.3) we deduce:

$$\frac{1}{2}\|w_0\|_c^{\,2} - \Lambda_{k+1}(c)\int G(x, w_0) \leq \liminf_n l_k(b_n, c) \leq$$
$$\leq \frac{1}{2}\|w\|_c^{\,2} - \Lambda_{k+1}(c)\int G(x, w), \quad \forall w \in H_k^\perp,$$

then by (ii):

$$\liminf_n l_k(b_n, c) \geq l_k(\Lambda_{k+1}(c), c) = \frac{1}{2}\|w_0\|_c^{\,2} - \Lambda_{k+1}(c)\int G(x, w_0) \geq 0. \qquad \square$$

Lemma 3.4. *Let* $k \geq 1$. *Set:*

$$m_k(b, c; \rho) = \sup_{\substack{v\in H_k \\ \|v\|_{L^2}=\rho}} f_{bc}(v).$$

Assume (G_0) *(see* (1.11)*). Then:*

$$\limsup_{\rho\to 0} \frac{m_k(b, c; \rho)}{\rho^2} \leq \frac{1}{2}\big(\Lambda_k(c) - b\big).$$

Proof. By (2.4) it follows that for any $\varepsilon > 0$ and for ρ small enough:

$$\frac{m_k(b, c; \rho)}{\rho^2} \leq \frac{1}{2}\big(\Lambda_k(c) - b + \varepsilon b\big).$$

Then the claim follows. $\square$

Theorem 3.5. *Assume* (g) *(see* (1.3)), (1.11) *and* (1.6). *Let* $\lambda_1 \leq c < \lambda^*$ *(see* (2.8)*). For any* $\lambda_i > \lambda_2$ *there exists* $\varepsilon > 0$ *such that for any* $b \in \big]\Lambda_i(c), \Lambda_i(c) + \varepsilon\big[$ *the functional* f_{bc} *has at least three different critical values.*

Proof. Let $\lambda_1 \leq c < \lambda^* < \lambda_2 \leq \cdots \leq \lambda_k < \lambda_{k+1} = \cdots = \lambda_i < \lambda_{i+1}$. First of all since $c < \lambda_i < \lambda_{iH}$ and $\Lambda_{k+1}(c) = \Lambda_i(c) < b < \frac{1}{\beta_i(c)}$ by Lemmas 2.2, 2.3 and Remark 2.11 (where index $h+1$ is replaced by i) it follows that there exist $R_i > \rho_i > 0$ such that:

$$\inf_{z\in\Sigma_{R_i}(e_i, H_i^\perp)} f_{bc}(z) > \sup_{\substack{v\in H_i \\ \|v\|_{L^2}=\rho_i}} f_{bc}(v), \tag{3.6}$$

where: $\Sigma_{R_i}(e_i, H_i^\perp) = \{w \in H_i^\perp \mid \int |\Delta w|^2 - c\int |\nabla w|^2 \le R_i^2\} \cup \{z = w + \sigma e_i \mid w \in H_i^\perp,\ \sigma \ge 0,\ \int |\Delta z|^2 - c\int |\nabla z|^2 = R_i^2\}$.
Secondly by Lemmas 3.1 and 3.4 it follows that there exists $\varepsilon > 0$ such that for any $b \in \left[\Lambda_{k+1}(c), \Lambda_{k+1}(c) + \varepsilon\right[$ there exists $\rho_k > 0$ such that:

$$\inf_{w\in H_k^\perp} f_{bc}(w) = l_k(b,c) > m_k(b,c;\rho_k) = \sup_{\substack{v\in H_i \\ \|v\|_{L^2}=\rho_i}} f_{bc}(v). \tag{3.7}$$

Finally since $c < c^*$ and $0 < b < \frac{1}{\beta_{k+1}(c)}$ by Lemma 2.9 it follows that there exist $e^* \in H_k \setminus \{0\}$ and $R_k > \max\{R_i, \rho_k\}$ such that:

$$\inf_{z\in\Sigma_{R_k}(e^*, H_k^\perp)} f_{bc}(z) > \sup_{\substack{v\in H_i \\ \|v\|_{L^2}=\rho_i}} f_{bc}(v). \tag{3.8}$$

where: $\Sigma_{R_k}(e^*, H_k^\perp) = \{w \in H_k^\perp \mid \int |\Delta w|^2 - c\int |\nabla w|^2 \le R_k^2\} \cup \{z = w + \sigma e^* \mid w \in H_k^\perp,\ \sigma \ge 0,\ \int |\Delta z|^2 - c\int |\nabla z|^2 = R_k^2\}$. By (3.6), (3.7) and (3.8) using Theorem 5.3, we get the claim. □

4. A non trivial solution when $c > \lambda_1$ and $b \le \Lambda_1(c)$.

By the Mountain Pass Theorem we are able to prove that in this case the functional $f_{b,c}$ has a strictly positive critical value. We start with some technical lemmas.

Lemma 4.1. *Assume* (G) *and* (G_0) *(see* (1.11)*). Let* $b \le 0$. *Then for any* $\varepsilon > 0$ *there exists a function* $\theta : H \longrightarrow \mathbb{R}$ *such that:*

$$f_{b,c}(u) \ge \frac{1}{2}\Big(\int |\Delta u|^2 - c\int |\nabla u|^2\Big) - \\ - \frac{b}{2}(1-\varepsilon)\int u^2 - \|u\|_H u^2\theta(u) \ \ \textit{with} \ \ \lim_{u\to 0}\theta(u) = 0.$$

Proof. First of all, (G_0) implies that for any $\varepsilon > 0$ there exists $\rho > 0$ s.t. if $|s| \le \rho$ then $2G(x,s) \ge (1-\varepsilon)s^2$ a.e. in Ω. Then we can compute:

$$f_{b,c}(u) = \frac{1}{2}\Big(\int |\Delta u|^2 - c\int |\nabla u|^2\Big) - b\int_{\{x\in\Omega:|u(x)|\le\rho\}} G(x,u) - \\ - b\int_{\{x\in\Omega:|u(x)|\ge\rho\}} G(x,u) \ge \frac{1}{2}\Big(\int |\Delta u|^2 - c\int |\nabla u|^2\Big) - \tag{4.2}$$

$$-\frac{b}{2}(1-\varepsilon)\int u^2+\frac{b}{2}\int\limits_{\{x\in\Omega:|u(x)|\geq\rho\}}\big(-2G(x,u)+(1-\varepsilon)u^2\big)\geq$$

$$\geq\frac{1}{2}\Big(\int|\Delta u|^2-c\int|\nabla u|^2\Big)-\frac{b}{2}(1-\varepsilon)\int u^2+\frac{b}{2}\int\limits_{\{x\in\Omega:|u(x)|\geq\rho\}}u^2,$$

because of (G). On the other hand using Hölder inequality we get:

$$\int\limits_{\{x\in\Omega:|u(x)|\geq\rho\}}u^2\leq S\|u\|_H u^2\big(\text{meas}\{x\in\Omega:|u(x)|\geq\rho\}\big)^p, \tag{4.3}$$

for some positive constants S and p. By (4.2) and (4.3) the claim follows. □

Lemma 4.4. *Assume* (G) *and* (G_0) *(see* (1.11)*). If* $\lambda_k\leq c<\lambda_{k+1}$ *for* $k\geq 1$ *and* $b<\Lambda_1(c)$ *then there exists* $\rho>0$ *such that:*

$$\inf_{u\in\gamma_\rho(H)}f_{b,c}(u)>0,$$

where:

$$\gamma_\rho(H)=\big\{u=v+w\in H_k\oplus H_k^\perp\mid \int v^2+\Big(\int|\Delta w|^2-c\int|\nabla w|^2\Big)=\rho^2\big\}, \tag{4.5}$$

is homeomorphic to a sphere.

Proof. Let $u=v+w$ with $v\in H_k$ and $w\in H_k^\perp$. By Lemma 4.1, since $b<0$, we get:

$$f_{b,c}(v+w)\geq\frac{1}{2}\Big(\int|\Delta v|^2-c\int|\nabla v|^2\Big)+\frac{1}{2}\Big(\int|\Delta w|^2-c\int|\nabla w|^2\Big)-$$

$$-\frac{b}{2}(1-\varepsilon)\int v^2-\frac{b}{2}(1-\varepsilon)\int w^2-(\|v\|_H^2+\|w\|_H^2)\theta(v+w)\geq$$

$$\geq\frac{1}{2}\big(\Lambda_1(c)-b(1-\varepsilon)-a\theta(v+w)\big)\int v^2+\frac{1}{2}\big(1-a\theta(v+w)\big)\cdot$$

$$\cdot\Big(\int|\Delta w|^2-c\int|\nabla w|^2\Big),$$

where a is a positive constant. Now we point out that if $\|v+w\|_h^2$ and $\int v^2+\Big(\int|\Delta w|^2-c\int|\nabla w|^2\Big)$ are equivalent norms on the space H. Thus the claim follows, if $\rho>0$ is small enough. □

Lemma 4.6. *Assume* (G) *and* $(G_{-\infty})$ *(see* (1.11)*). If* $\lambda_1<c$ *and* $b<0$ *then:*

$$\lim_{s\to+\infty}f_{b,c}(-se_1)=-\infty.$$

Proof. We have:

$$f_{b,c}(-se_1) = s^2 \left(\Lambda_1(c) - b \int \frac{G(x, -se_1)}{s^2} \right);$$

moreover by (G) and $(G_{-\infty})$ we easily get:

$$\lim_{s \to +\infty} \int \frac{G(x, -se_1)}{s^2} = 0$$

and so the claim follows. □

Theorem 4.7. *Assume* (g) *(see* (1.3)*),* (1.11) *and* (1.6)*. Let* $\lambda_1 < c$ *and* $b < \Lambda_1(c)$*.*

Then the functional $f_{b,c}$ *has at least two different critical values.*

Proof. Let $\lambda_1 < \cdots \le \lambda_k \le c < \lambda_{k+1}$ for some $1 \le k$. Firstly, since $b < \Lambda_1(c)$, by Lemma 4.4 there exists a set:

$$\Gamma_\rho(H) = \left\{ v + w \in H_k \oplus H_k^\perp \mid \int v^2 + \big(\int |\Delta w|^2 - c \int |\nabla w|^2 \big) \le \rho \right\},$$

homeomorphic to a ball in H, whose boundary is the set $\gamma_\rho(H)$ (see (4.5)), such that:

$$\inf_{u \in \gamma_\rho(H)} f_{b,c}(u) > 0. \tag{4.8}$$

Moreover (G) implies $f_{b,c}(0) = 0$, with $0 \in \Gamma_\rho(H)$. Finally Lemma 4.6 ensures the existence of $s^* > 0$ such that $-s^* e_1 \notin \Gamma_\rho(H)$ and $f_{b,c}(-s^* e_1) < 0$. Thus the classical mountain pass theorem (see [3]) claims the existence of a critical value c_1 of $f_{b,c}$ such that:

$$c_1 \ge \inf_{u \in \gamma_\rho(H)} f_{b,c}(u) > 0.$$

It is evident that the trivial solution is the minimum of the functional $f_{b,c}$ on the set $\Gamma_\rho(H)$.

5. Variational setting.

In this section we recall two theorems (see [6], [7], [11] and [12]) of existence of critical points for a functional, which have been used in the previous sections.

Definition 5.1. *Let X be an Hilbert space, $Y \subset X$, $\rho > 0$ and $e \in X \setminus Y$, $e \neq 0$. Set:*

$$\begin{aligned} B_\rho(Y) &= \left\{ x \in Y \mid \|x\|_X \le \rho \right\}, \\ S_\rho(Y) &= \left\{ x \in Y \mid \|x\|_X = \rho \right\}, \\ \Delta_\rho(e, Y) &= \left\{ \sigma e + v \mid \sigma \ge 0,\ v \in Y,\ \|\sigma e + v\|_X \le \rho \right\}, \\ \Sigma_\rho(e, Y) &= \left\{ \sigma e + v \mid \sigma \ge 0,\ v \in Y,\ \|\sigma e + v\|_X = \rho \right\} \cup \\ &\qquad\qquad \cup \left\{ v \mid v \in Y,\ \|v\|_X \le \rho \right\}. \end{aligned}$$

First of all we recall a theorem of existence of two critical levels for a functional which is a variation of linking theorem (see Theorem 3.4 of [6] and [12]).

Theorem 5.2 ("a variation of linking")**.** *Let X be an Hilbert space, which is topological direct sum of the subspaces X_1 and X_2. Let $F \in C^1(X, \mathbb{R})$. Moreover assume:*

(a) $\dim X_1 < +\infty$*;*
(b) there exist $\rho > 0$, $R > 0$ and $e \in X_1$, $e \neq 0$ such that $\rho < R$ and $\sup_{S_\rho(X_1)} F < \inf_{\Sigma_R(e, X_2)} F$;
(c) $-\infty < a = \inf_{\Delta_R(e, X_2)} F$;
(d) (P.S.)$_c$ holds for any $c \in [a, b]$, where $b = \sup_{B_\rho(X_1)} F$.

Then there exist at least two critical levels c_1 and c_2 for the functional F such that:

$$\inf_{\Delta_R(e, X_2)} F \le c_1 \le \sup_{S_\rho(X_1)} F < \inf_{\Sigma_R(e, X_2)} F \le c_2 \le \sup_{B_\rho(X_1)} F.$$

Finally we recall a theorem of existence of three critical levels for a functional (see Theorem 8.4 of [7]).

Theorem 5.3 ("linking scale")**.** *Let X be an Hilbert space, which is topological direct sum of the four subspaces X_0, X_1, X_2 and X_3. Let $F \in C^1(X, \mathbb{R})$. Moreover assume:*

(a) $\dim X_i < +\infty$ *for* $i = 0, 1, 2$*;*
(b) there exist $\rho > 0$*,* $R > 0$ *and* $e \in X_2$*,* $e \neq 0$ *such that:*

$$\rho < R \quad \text{and} \quad \sup_{S_\rho(X_0 \oplus X_1 \oplus X_2)} F < \inf_{\Sigma_R(e, X_3)} F;$$

(c) there exist $\rho' > 0$*,* $R' > 0$ *and* $e' \in X_1$*,* $e' \neq 0$ *such that:*

$$\rho' < R' \quad \text{and} \quad \sup_{S'_\rho(X_0 \oplus X_1)} F < \inf_{\Sigma'_R(e', X_2 \oplus X_3)} F;$$

(d) $R \leq R' \quad \big(\Longrightarrow \quad \Delta_R(e, X_3) \subset \Sigma'_R(e', X_2 \oplus X_3)\big)$;
(e) $-\infty < a = \inf_{\Delta'_R(e, X_2 \oplus X_3)} F$*;*
(f) *(P.S.)*$_c$ *holds for any* $c \in [a, b]$*, where* $b = \sup_{B_\rho(X_0 \oplus X_1 \oplus X_2)} F$.

Then there exist three critical levels c_1*,* c_2 *and* c_3 *for the functional* F *such that:*

$$\begin{aligned} a \leq c_3 \leq \sup_{S'_\rho(X_0 \oplus X_1)} F < \inf_{\Sigma'_R(e', X_2 \oplus X_3)} F \leq \\ \leq \inf_{\Delta_R(e, X_3)} F \leq c_2 \leq \sup_{S_\rho(X_0 \oplus X_1 \oplus X_2)} F < \inf_{\Sigma_R(e, X_3)} F \leq c_1 \leq b. \end{aligned}$$

6. An uniqueness result when $c = \lambda_1$ and $b < 0$.

We will prove the following uniqueness result.

Proposition 6.1. *Let* $g : \mathbb{R} \longrightarrow \mathbb{R}$ *be such that:*

$$(6.2) \qquad \begin{cases} (i) & g \text{ is Lipschitz, is } C^1 \text{ except at a point } s_0 \text{ with } g(s_0) \neq 0 \\ & \text{and } g(0) = 0; \\ (ii) & g'(s) \geq 0 \ \forall s \in \mathbb{R} \setminus \{s_0\} \text{ and } g'(0) \neq 0. \end{cases}$$

Moreover assume:

$$(6.3) \qquad \begin{cases} (iii) & |g(s)| \leq a_0 + b_0|s|, \forall s \in \mathbb{R}, \text{ with } a_0, \ b_0 \in \mathbb{R}; \\ (iv) & \lim_{s \to +\infty} \dfrac{g(s)}{s} = 1; \\ (v) & 2G(s) - g(s)s \geq \alpha_0 s^- - \alpha_1 \ \forall s \in \mathbb{R}, \text{ with } \alpha_0, \ \alpha_1 \in \mathbb{R}^+; \\ (vi) & G(s) \geq 0 \ \forall s \in \mathbb{R}. \end{cases}$$

If $c = \lambda_1$ *and* $b < 0$*, then the functional* f_{b,λ_1} *has an unique trivial critical point, which is a local minimum point, so the problem* (1.1) *has only the trivial solution.*

Proof. First of all by (vi) of (6.3) we have $f_{b,\lambda_1}(0) = 0$ and $f_{b,\lambda_1}(u) \geq 0$ $\forall u \in H$.

Secondly we remark that critical points of $f_{b,\lambda_1}(u)$ are isolated. In fact if u_0 is a critical point of f_{b,λ_1} by *(iii)* of (6.3) using standard regularity results we have that $u_0 \in C_0(\Omega)$. Thus by (6.2)

$$(6.4) \quad f''_{b,\lambda_1}(u_0)(v)^2 = \int (\Delta v)^2 - \lambda_1 \int |\nabla v|^2 - b \int g'(u_0)v^2 \geq 0 \quad \forall v \in H.$$

If $f''_{b,\lambda_1}(u_0)(v)^2 = 0$ then by (6.4) and *(ii)* of (6.2) we get $\int (\Delta v)^2 - \lambda_1 \int |\nabla v|^2 = 0$, which implies $v = \sigma e_1$ for $\sigma \in \mathbb{R}$ and $\int g'(u_0)e_1^2 = 0$, which implies $g'(u_0) = 0$ in Ω. A contradiction arises since $u_0(x) = 0$ on $\partial\Omega$ and (6.2) holds. Then we have $f''_{b,\lambda_1}(u_0)(v)^2 > 0 \quad \forall v \in H \setminus \{0\}$. Therefore critical points of f_{b,λ_1} are isolated, since any critical point of f_{b,λ_1} is a strict local minimum point.

Finally if the functional f_{b,λ_1} has two different critical points, they are two local minima points. So by *(i)* of (6.2) and (6.3) using Theorem 6.5.3, page 354, of [2] we state the existence of a third critical point which is not a minimum point and a contradiction arises. □

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