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ESTIMATING THE RESOLVENT OF ELLIPTIC SECOND-ORDER PARTIAL DIFFERENTIAL OPERATORS

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Sharp estimates for the resolvent of a linear elliptic second-order partial differential operator under Dirichlet homogeneous boundary conditions are derived via a symmetrization technique.

1. Introduction.

In this paper we are concerned with partial differential equations having either of the following forms

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(b_i(x)u\big) + \Big(c(x)+\lambda\Big)u = f(x), \tag{1}$$

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + \Big(c(x)+\lambda\Big)u = f(x). \tag{2}$$

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We suppose the coefficients and the right-hand side are real-valued measurable functions defined in an open subset G of $\mathbb{R}^n$. We suppose λ is a positive constant parameter and the equations are uniformly elliptic, that is

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \sum_{i=1}^{n} \xi_i^2 \quad \text{for all } x \in G \text{ and for all } \xi \in \mathbb{R}^n \tag{3}$$

and

$$a_{ij} \in L^\infty(G). \tag{4}$$

Moreover, either

$$\left(\int_G \Big(\sum_{i=1}^{n} b_i^2\Big)^{k/2} dx\right)^{1/k} \leq B \tag{5}$$

or

$$\operatorname{ess\,sup}\Big(\sum_{i=1}^{n} b_i^2\Big)^{1/2} \leq B, \tag{6}$$

and

$$c \in L^{n/2}(G) \text{ and } c(x) \geq 0 \text{ for all } x \in G, \tag{7}$$

$$f \in L^p(G). \tag{8}$$

Here B is some nonnegative constant, $n < k < \infty$, and $p = \dfrac{2n}{n+2}$ or $p > 1$ according to whether $n > 2$ or $n = 2$.

We are interested in Dirichlet problems with zero boundary data; thus we look for functions u which satisfy either equation (1) or (2) and obey

$$u = 0 \quad \text{on the boundary } \partial G \text{ of } G. \tag{9}$$

We deal with weak solutions belonging to Sobolev space $W_0^{1,2}(G)$.

The present paper parallels earlier ones where estimates for solutions to boundary value problem for elliptic second-order partial differential equations in divergence form are obtained via inspection of level sets, isoperimetric inequality and rearrangements of functions. See for instance Alvino-Diaz-Lions-Trombetti [2], Alvino-Ferone-Kawohl-Trombetti [3], Alvino-Lions-Trombetti [4], [5], Alvino-Matarasso-Trombetti [6], Alvino-Trombetti [7], [8], Bandle [9], [10], Chiti [13], [12], Diaz [14], [15], [16], Maderna [18], Maderna-Pagani-Salsa [19], Maderna-Salsa [20], [21], [22], Pacella-Tricarico [24], Talenti [25], [27], [28], Trombetti-Vazquez [31], Weinberger [33] and the references quoted in [28].

It is closely related to [2], [4], [5], [6], [7], [8], [15], [25], [27] and [31]. We provide some variants and refinements. Comparison results for solutions to both problems (1)–(9) and (2)–(9) are established. Estimates are obtained as a corollary, where the dependence upon parameter λ is explicitly displayed. In other words, we obtain estimates of the resolvent operator – the main goal of this paper.

2. Notations.

$\mathbb{R}^n$	n-dimensional euclidean space
G	open subset of $\mathbb{R}^n$
∂G	boundary of G
$L^p(G)$	Lebesgue space
$W^{1,2}(G)$	Sobolev space
$W_0^{1,2}(G)$	closure of $C_0^\infty(G)$ in $W^{1,2}(G)$
$\lvert G\rvert$	n-dimensional measure of G
$G^\star$	ball centered at the origin having the same measure as G
μ_u	distribution function of u
u^*	decreasing rearrangement of u
$u^\star$	symmetric rerrangement of u
C_n	measure of the n-dimensional unit ball
κ_n	isoperimetric constant

Recall that μ_u is defined by

$$\mu_u(t) = |\{x \in G : |u(x)| > t\}|,$$

u^* is the right-continuous decreasing function from $[0, +\infty)$ into $[0, +\infty]$ equidistributed with u, and $u^\star$ is the function from $\mathbb{R}^n$ into $[0, +\infty]$ defined by $u^\star(x) = u^*(C_n|x|^n)$.

Finally recall

$$\kappa_n = \frac{1}{nC_n^{1/n}} \ ,$$

the smallest constant which makes the following isoperimetric inequality

$$(\text{measure of } E)^{1-1/n} \leq \kappa_n (\text{perimeter of } E)$$

true for every measurable subset E of $\mathbb{R}^n$ having finite measure.

For the properties of the objects defined above see [17], [25], [26], [27] and [30].

3. Main results.

Theorem 1. *Let u belong to $W_0^{1,2}(G)$ and satisfy equation* (1). *Suppose* (5) *holds. Then a measurable function g from* $[0, +\infty)$ *into* $[0, +\infty]$ *exists such that*

$$\left\{ nC_n \int_0^{n\kappa_n |G|^{1/n}} g(r)^k r^{n-1} dr \right\}^{1/k} \leq B, \tag{10}$$

and the solution w belonging to $W_0^{1,2}(G^\star)$ to the following problem

$$\begin{cases} -\Delta w - \dfrac{\partial \mathcal{F}}{\partial \Phi}(\Phi, \Psi) \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ \frac{x_i}{|x|} g(|x|) w \right\} + \\ \qquad + \dfrac{\partial \mathcal{F}}{\partial \Psi}(\Phi, \Psi) \Big(\lambda w - f^\star(|x|) \Big) = 0 & \text{in } G^\star \\ w = 0 & \text{on } \partial G^\star, \end{cases} \tag{11}$$

satisfies

$$\int_0^s u^*(t)\, dt \leq \int_0^s w^*(t)\, dt \tag{12}$$

for every s such that $0 \leq s \leq |G|$.

Here

$$\Phi = |x|^{n-1} g(|x|) w(x),$$

$$\Psi = -\lambda \int_0^{C_n |x|^n} w^*(t)\, dt + \int_0^{C_n |x|^n} f^*(t)\, dt,$$

$$\mathcal{F}(\Phi, \Psi) = \max\{\Phi/2, \Phi + \Psi\}.$$

Theorem 2. *Let u be as in Theorem 1. Suppose that* (6) *holds; then the solution w of the following problem*

$$(13)\qquad \begin{cases} -\Delta w - B\dfrac{\partial \mathcal{F}}{\partial \Phi}(\Phi,\Psi)\displaystyle\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left\{\frac{x_i}{|x|}w\right\}+ \\ \qquad\qquad + \dfrac{\partial \mathcal{F}}{\partial \Psi}(\Phi,\Psi)\Big(\lambda w - f^\star(|x|)\Big) = 0 & \text{in } G^\star \\ w = 0 & \text{on } \partial G^\star, \end{cases}$$

satisfies (12) *for every s such that* $0 \le s \le |G|$.

Here

$$\Phi = B|x|^{n-1}w(x)$$

and Ψ is as in Theorem 1, as well as $\mathcal{F}$.

Theorem 3. *Let u belong to $W_0^{1,2}(G)$ and satisfy equation* (2)*. Suppose that* (5) *holds and g is as in Theorem 1. Then the following problem*

$$(14)\qquad \begin{cases} -\Delta w + \displaystyle\sum_{i=1}^{n} g(|x|)\frac{x_i}{|x|}\frac{\partial w}{\partial x_i} + \lambda w = f^\star & \text{in } G^\star \\ w = 0 & \text{on } \partial G^\star, \end{cases}$$

has a solution in $W_0^{1,2}(G^\star)$; such solution satisfies

$$(15)\qquad \int_0^s \exp\Big(-\int_0^{n\kappa_n t^{1/n}} g(r)\,dr\Big)u^*(t)\,dt \le$$
$$\le \int_0^s \exp\Big(-\int_0^{n\kappa_n t^{1/n}} g(r)\,dr\Big)w^*(t)\,dt$$

for every s such that $0 \le s \le |G|$.

Theorem 4. *Let u be as in Theorem 3, and suppose that* (6) *holds; then the following problem*

$$(16)\qquad \begin{cases} -\Delta w + B\displaystyle\sum_{i=1}^{n}\frac{x_i}{|x|}\frac{\partial w}{\partial x_i} + \lambda w = f^\star & \text{in } G^\star \\ w = 0 & \text{on } \partial G^\star, \end{cases}$$

has a solution in $W_0^{1,2}(G^\star)$, and such solution satisfies

$$(17)\qquad \int_0^s \exp\Big(-n\kappa_n B t^{1/n}\Big)u^*(t)\,dt \le \int_0^s \exp\Big(-n\kappa_n B t^{1/n}\Big)w^*(t)\,dt$$

for every s such that $0 \le s \le |G|$.

Theorem 1 is demonstrated in Section 5. Theorem 2 can be proved quite in the same way as Theorem 1. Theorem 3 is a relative of Theorem 4, in the same way as Theorem 1 is a relative of Theorem 2. Theorem 3 can be proved by the same arguments involved in Theorem 1 plus Gronwall's Lemma. Theorem 4 is a special case of [4], Theorem 1. The proofs of Theorems 2, 3 and 4 will be omitted.

4. Some applications.

Theorem 5. *Consider the problem made up by equation* (2) *and boundary condition* (9) *under hypotheses* (3), (4), (6), (7) *and* (8). *Assume* $|G| < +\infty$ *and* $p > n/2$. *Then its weak solution obeys*

$$\text{ess sup}\,|u| \leq D\|f\|_{L^p(G)}. \tag{18}$$

Constant D *is defined as follows:*

$$a = \frac{n-1}{2}\Big(1 - \frac{B}{\sqrt{B^2+4\lambda}}\Big),$$

$$b = n\kappa_n |G|^{1/n}\sqrt{B^2+4\lambda},$$

$$\mathcal{I}(y) = \exp\Big[\Big(\frac{1}{2} - \frac{B}{\sqrt{B^2+4\lambda}}\Big)y\Big]\times \\ \times\Big(U(a,n-1,y) - \frac{U(a,n-1,b)}{M(a,n-1,b)}M(a,n-1,y)\Big),$$

$$p' = \frac{p}{p-1},$$

$$D = \frac{\Gamma(a)(B^2+4\lambda)^{n/(2p)-1}}{n^{1/p}C_n^{1/p}(n-2)!}\left\{\int_0^b \mathcal{I}(y)^{p'}y^{n-1}dy\right\}^{1/p'}. \tag{19}$$

Here U *and* M *are* Kummer's functions. D *is the smallest constant that makes* (18) *true under the assumptions above.*

Proof. The very definition of u^* gives

$$\text{ess sup}\,|u| = u^*(0), \tag{20}$$

and inequality (17) gives

$$u^*(0) \le w^*(0). \tag{21}$$

Observe that the solution w to problem (16) is radially symmetric and radially decreasing, in other words

$$w(x) = w^*(C_n|x|^n).$$

Therefore (16) reads:

$$\begin{cases} -\dfrac{d^2w}{d|x|^2} - \left(\dfrac{n-1}{|x|} - B\right)\dfrac{dw}{d|x|} + \lambda w = f^*(C_n|x|^n), \\ w(x) = 0 \quad \text{if } |x| = n\kappa_n|G|^{1/n}. \end{cases} \tag{22}$$

Two linearly independent solutions of the homogeneous equation

$$\frac{d^2w}{dr^2} - \left(\frac{n-1}{r} - B\right)\frac{dw}{dr} + \lambda w = 0,$$

whose wronskian is $-\dfrac{(n-2)!}{\Gamma(a)}\beta^{2-n}r^{1-n}e^{(\beta-2\alpha)r}$, are

$$w_1(r) = e^{-\alpha r}M(a, n-1, \beta r) \text{ and } w_2(r) = e^{-\alpha r}U(a, n-1, \beta r).$$

Here

$$\alpha = \frac{-B + \sqrt{B^2 + 4\lambda}}{2} \text{ and } \beta = \sqrt{B^2 + 4\lambda}.$$

Recall that $M(a, c, z)$ and $U(a, c, z)$ are the solutions of *Kummer's equation*

$$z\frac{d^2w}{dz^2} + (c - z)\frac{dw}{dz} - aw = 0$$

given by

$$M(a, c, z) = \frac{\Gamma(c)}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(c+k)}\frac{z^k}{k!},$$

if c is not a negative integer, and

$$U(a,c,z)=\frac{\pi}{\sin\pi c}\Big[\frac{M(a,c,z)}{\Gamma(c)\Gamma(a+1-c)}-z^{1-c}\frac{M(a+1-c,2-c,z)}{\Gamma(a)\Gamma(2-c)}\Big].$$

For more informations see [1] and [23].

The solution to problem (22) such that

$$\int_{G^\star}\Big(\frac{dw}{d|x|}\Big)^2 dx<+\infty$$

is then

$$w(x)=\big(a_1(|x|)+A\big)w_1(|x|)+a_2(|x|)w_2(|x|), \tag{23}$$

where

$$a_1(r)=\frac{\Gamma(a)}{(n-2)!}\beta^{n-2}\int_r^{n\kappa_n|G|^{1/n}} t^{n-1}e^{(\alpha-\beta)t}U(a,n-1,\beta t)f^*(C_nt^n)\,dt,$$

$$a_2(r)=\frac{\Gamma(a)}{(n-2)!}\beta^{n-2}\int_0^r t^{n-1}e^{(\alpha-\beta)t}M(a,n-1,\beta t)f^*(C_nt^n)\,dt,$$

$$A=-\frac{\Gamma(a)}{(n-2)!}\beta^{n-2}\gamma\int_0^{n\kappa_n|G|^{1/n}} t^{n-1}e^{(\alpha-\beta)t}M(a,n-1,\beta t)f^*(C_nt^n)\,dt$$

and

$$\gamma=\frac{U(a,n-1,b)}{M(a,n-1,b)}.$$

Thus

$$w(0)=\frac{\Gamma(a)}{(n-2)!}\beta^{n-2}\int_0^{n\kappa_n|G|^{1/n}} t^{n-1}e^{(\alpha-\beta)t}\big\{U(a,n-1,\beta t)-$$
$$-\gamma M(a,n-1,\beta t)\big\}f^*(C_nt^n)\,dt.$$

Hölder inequality and (20) and (21) give (18). □

Theorem 6. *Constant D given in Theorem 5 has the following properties:*

$$\frac{D}{\lambda^{n/(2p)-1}}\longrightarrow\frac{1}{n^{1/p}C_n^{1/p}2^{n/2-1}\Gamma(n/2)}\times \tag{24}$$
$$\times\Big(\int_0^\infty t^{(np-2n+2)/2p-2}[K_{n/2-1}(t)]^{p'}dt\Big)^{1/p'}\quad \textit{as } \lambda\to+\infty;$$

$$D=\lambda^{n/(2p)-1}\{\text{the r.h.s. of (24)}\}\quad \textit{if } |G|=+\infty \textit{ and } B=0.$$

Here $K_\nu(z)$ is the modified Bessel function of second kind and order ν *(see, e.g.,* [1], [23], [32]*).*

Proof. The proof consists of an inspection, based on formula (19) and formula [1], Chapter 13, Section 13.6, 13.6.21 relating Kummer and Bessel functions. □

Theorem 7. *Let u belong to $W_0^{1,2}(G)$ and satisfy* (1)*, let w belong to $W_0^{1,2}(G^\star)$ and satisfy* (11) *or* (13)*, according to whether* (5) *or* (6) *is in force. Then*

$$\|u\|_{L^q(G)} \leq \|w\|_{L^q(G^\star)} \tag{25}$$

for every q larger than or equal to 1.

Proof. Since a function and its decreasing rearrangement are equidistribuited, (25) follows from (12) and the lemma appearing in [29]. □

Theorem 8. *If u and w are as in the previous theorem, $m(G) = +\infty$ and $q > p > 1$, then*

$$\|u\|_{L(p,q)} \leq \|w\|_{L(p,q)}. \tag{26}$$

Here $L(p, q)$ stands for Lorentz space. For its definition we refer to [34].

Miscellaneous estimates, which easily follow from the previous theorems, are listed below. (Standard integrals of Bessel functions, and Bliss inequality – see [11] – are involved.)

- If $|G| = +\infty$, $B = 0$ and $p > n/2$, then

$$\|u\|_{L^p(G)} \leq \frac{1}{\lambda}\|f\|_{L^p(G)}.$$

- If $|G| = +\infty$, $B = 0$ and $q > p > 1$, then

$$\|u\|_{L(p,q)} \leq \frac{K^{1/q}}{\lambda}\|f\|_{L^p(G)},$$

where

$$K = \frac{p}{q(p-1)}\left[\frac{\Gamma(qp/(q-p))}{\Gamma(p/(q-p))\Gamma(p(q-1)/(q-p))}\right]^{q/p-1}.$$

5. Proof of Theorem 1.

We need the following lemmas.

Lemma 1. *For every measurable function f and for every measurable set E we have*

$$\int_E |f(x)|\,dx \leq \int_0^{|E|} f^*(f)\,ds. \tag{27}$$

Moreover if E equals a level set $\{x \in G : |f(x)| > t\}$ and t is positive, then equality holds in (27).

Lemma 2. *Let $u \in W_0^{1,2}(G)$ and let μ be the* distribution function *of u. Then*

$$\kappa_n^2 \mu(t)^{2/n-2}[-\mu'(t)]\left\{-\frac{d}{dt}\int_{\{x\in G:|u(x)|>t\}} |\nabla u|^2 dx\right\} \geq 1, \tag{28}$$

for almost every t from 0 *to* ess sup $|u|$. *As usual, ∇ stands for gradient, so that* $|\nabla u|^2 = \sum_{i=1}^{n} u_{x_i}^2$.

Lemma 1 is a special case of Hardy-Littlewood theorem, see [17], Theorem 378 or [26], Theorem 1.A. Lemma 2 is proved in [25], via Fleming-Rishel coarea formula and the isoperimetric inequality. An alternative form of Lemma 2 appears in [26], Lemma 1.E.

Proof of Theorem 1. Suppose u is a weak solution of problem (1) and (9), that is $u \in W_0^{1,2}(G)$ and

$$\int_G \left[\sum_{i,k=1}^{n} a_{ik} u_{x_i} \phi_{x_k} + \sum_{i=1}^{n} b_i u \phi_{x_i} + cu\phi\right] dx = \int_G f\phi\,dx \tag{29}$$

for every test function ϕ from $W_0^{1,2}(G)$. (All the integrals in (29) converge because of the hypothesis made on the coefficients, and Sobolev embedding theorem.)

We proceed as in [27]. We take $\phi = S \circ u$, where S is the continuous function defined as follows. Let $0 \leq s < t$; then $S(u) = 0$ if $|u| < s$, $S(u) = \operatorname{sign} u$ if $|u| > t$ and S is linear otherwise. Plugging such ϕ into (29) then passing to the limit as $s \uparrow t$ results in the following inequality

$$-\frac{d}{dt}\int_{\{x\in G:|u(x)|>t\}} |\nabla u|^2\,dx + \lambda \int_{\{x\in G:|u(x)|>t\}} |u|\,dx + \tag{30}$$

$$-t\left(-\frac{d}{dt}\int_{\{x\in G:|u(x)|>t\}}\sum_{1}^{n}b_i^2dx\right)^{1/2}\left(\frac{d}{dt}\int_{\{x\in G:|u(x)|>t\}}|\nabla u|^2dx\right)^{1/2}\leq$$
$$\leq\int_{\{x\in G:|u(x)|>t\}}|f|\,dx$$

for almost every positive t.

We use the following notations:

$$\xi=\Big(-\frac{d}{dt}\int_{\{x\in G:|u(x)|>t\}}|\nabla u|^2\,dx\Big)^{1/2},$$
$$\zeta=\frac{t}{2}\Big(-\frac{d}{dt}\int_{\{x\in G:|u(x)|>t\}}\sum_{i=1}^{n}b_i^2\,dx\Big)^{1/2}$$

and

$$h=\int_0^{\mu(t)}\Big(f^*(r)-\lambda u^*(r)\Big)\,dr.$$

Inequality (30) and Lemma 1 imply that $\xi(\xi-2\zeta)\leq h$. Hence

$$\xi\leq\zeta+\sqrt{\zeta^2+h}.$$

If we let $d=\kappa_n\mu(t)^{1/n-1}\sqrt{-\mu'(t)}$, Lemma 2 gives $\xi\geq 1/d$, so

$$\frac{1}{d}-\zeta\leq\sqrt{\zeta^2+h}. \tag{31}$$

The left-hand side of (31) can be positive or not. In the former case we obtain $1\leq 2\zeta d+hd^2$. In any case we have

$$1\leq\max\left\{\zeta d,\,2\zeta d+hd^2\right\}. \tag{32}$$

Thus we have proved that

$$1\leq\max\Bigg\{\frac{\kappa_n}{2}t\Big[-\frac{d}{dt}\int_{\{x\in G:|u(x)|>t\}}\sum_{i=1}^{n}b_i^2\,dx\Big]^{1/2}\mu(t)^{1/n-1}\sqrt{-\mu'(t)}\,,$$
$$\kappa_n t\Big[-\frac{d}{dt}\int_{\{x\in G:|u(x)|>t\}}\sum_{i=1}^{n}b_i^2\,dx\Big]^{1/2}\mu(t)^{1/n-1}\sqrt{-\mu'(t)}+$$
$$+\kappa_n^2\mu(t)^{2/n-2}[-\mu'(t)]\int_0^{\mu(t)}\Big(f^*(r)-\lambda u^*(r)\Big)\,dr\Bigg\} \tag{33}$$

for almost every t from $(0, \text{ess sup}\,|u|)$.

As shown in [27], the function g defined by

$$g(n\kappa_n s^{1/n}) = \left[\frac{d}{ds}\int_{\{x\in G:|u(x)|>u^*(s)\}} \sum_1^n b_i(x)^2\,dx\right]^{1/2} \tag{34}$$

is such that

$$\left[-\frac{d}{dt}\int_{\{x\in G:|u(x)|>t\}} \sum_1^n b_i^2\,dx\right]^{1/2} = \sqrt{-\mu'(t)}\,g\big(n\kappa_n\mu(t)^{1/n}\big) \tag{35}$$

for almost every t from $(0, \text{ess sup}\,|u|)$, and

$$\left\{nC_n\int_0^{n\kappa_n|G|^{1/n}} g(r)^k r^{n-1}\,dr\right\}^{1/k} \le \left\{\int_G \Big(\sum_i^n b_i(x)^2\Big)^{k/2}\,dx\right\}^{1/k}. \tag{36}$$

Thus (33) becomes

$$\begin{aligned} 1 \le [-\mu'(t)]\max\bigg\{&\frac{\kappa_n}{2}tg(n\kappa_n\mu(t)^{1/n})\mu(t)^{1/n-1}, \\ &\kappa_n tg(n\kappa_n\mu(t)^{1/n})\mu(t)^{1/n-1} - \lambda\kappa_n^2\mu(t)^{2/n-2}\int_0^{\mu(t)} u^*(r)\,dr + \\ &+\kappa_n^2\mu(t)^{2/n-2}\int_0^{\mu(t)} f^*(r)\,dr\bigg\} \end{aligned} \tag{37}$$

for almost every t from $(0, \text{ess sup}\,|u|)$.

We claim that

$$\begin{aligned} -\frac{du^*}{ds}(s) \le \max\bigg\{&\frac{\kappa_n}{2}s^{1/n-1}g(n\kappa_n s^{1/n})u^*(s), \\ &\kappa_n s^{1/n-1}g(n\kappa_n s^{1/n})u^*(s) - \lambda\kappa_n^2 s^{2/n-2}\int_0^s u^*(r)\,dr + \\ &+\kappa_n^2 s^{2/n-2}\int_0^s f^*(r)\,dr\bigg\} \end{aligned} \tag{38}$$

for almost every s from $(0, |G|)$.

Indeed, the very definition of u^* ensures that

$$u^* = \int_0^{+\infty} \chi_{[0,\mu(t)]}\,dt,$$

where χ stands for characteristic function. Hence

$$\int_0^{|G|} u^*(s)\phi'(s)ds = \int_0^{+\infty} \phi(\mu(t))dt$$

for every ϕ in $C_0^\infty\big((0, |G|)\big)$; consequently

$$\int_0^{|G|} \phi(s)\Big[-\frac{du^*}{ds}(s)\Big]\, ds = \int_0^{+\infty} \phi(\mu(t))\, dt,$$

since u^* is absolutely continuous (see [27]). Suppose $\phi \geq 0$. Then

$$\int_0^{|G|} \phi(s)\Big[-\frac{du^*}{ds}(s)\Big]\, ds \leq \int_0^{+\infty} \phi(\mu(t))\{\text{the braces in (37)}\}[-\mu'(t)]\, dt \leq$$

$$\leq \int_0^{+\infty} \phi(\mu(t))\{...\}[-d\mu(t)] = \int_0^{|G|} \phi(s)\{\text{the right-hand side of (38)}\}\, ds.$$

Inequality (38) follows, for ϕ is arbitrary.

If we put

$$U(s) = \int_0^s u^*(t)\, dt,$$

$$h(s) = \kappa_n^2 \int_0^s f^*(t)\, dt,$$

$$F(s, y, t) = \max\Big\{\frac{\kappa_n}{2} s^{1/n-1} g(n\kappa_n s^{1/n})t,\ \kappa_n s^{1/n-1} g(n\kappa_n s^{1/n})t - \lambda\kappa_n^2 s^{2/n-2} y + s^{2/n-2} h(s)\Big\}, \tag{39}$$

(38) gives

$$-U''(s) \leq F\Big(s, U(s), U'(s)\Big) \tag{40}$$

for almost every s belonging to $(0, |G|)$. Observe that U satisfies also

$$U(0) = 0, \qquad U'(|G|) = 0. \tag{41}$$

Let v be the solution to the following boundary value problem

$$\begin{cases} -v''(s) = F\big(s, v(s), v'(s)\big), \\ v(0) = 0,\ v'(|G|) = 0. \end{cases} \tag{42}$$

We want to show that $U(s) \le v(s)$ for every s between 0 and $|G|$.

If we let $\omega = v - U$, we have

$$\begin{cases} -\omega''(s) \ge F\big(s, v(s), v'(s)\big) - F\big(s, U(s), U'(s)\big), \\ \omega(0) = 0, \ \omega'(|G|) = 0. \end{cases}$$

But $F(s, \cdot, \cdot)$ is convex, thus

$$F(s, v, v') - F(s, U, U') \ge F_U(s, U, U')(v - U) + F_{U'}(s, U, U')(v' - U'),$$

where $\big(F_U(s, \cdot, \cdot), F_{U'}(s, \cdot, \cdot)\big)$ is a subgradient of F. Observe that $F_U \le 0$, as (39) shows.

In other words, ω satisfies

$$\begin{cases} -\omega''(s) \ge -\alpha(s)\omega'(s) - \beta(s)\omega(s), \\ \omega(0) = 0, \ \omega'(|G|) = 0, \end{cases} \tag{43}$$

where β is a nonnegative function.

Let's show that any sufficiently smooth solution ω to problem (43) is nonnegative.

Let us consider the function z defined by

$$z(s) = -\omega''(s) + \alpha(s)\omega'(s) + \beta(s)\omega(s),$$

which is nonnegative. If $y(s) = \exp\Big(-\int_0^s \alpha(r)\,dr\Big)$, ω minimizes the following functional

$$J(\phi) = \int_0^{|G|} y\{(\phi')^2 + \beta\phi^2 - 2z\phi\}ds$$

in the following function class $\big\{\phi \in W^{1,2}\big((0, |G|)\big) : \phi(0) = \phi'(|G|) = 0\big\}$. Observe that J is strictly convex, and $J(|\phi|) \le J(\phi)$ for every competing function ϕ. Consequently the minimizer is unique and nonnegative. We conclude that ω is nonnegative, as claimed.

A straightforward inspection shows that (11) and (12) imply

$$\int_0^s w^*(t)\,dt = v(s)$$

for every s belonging to $[0, |G|]$.

Theorem 1 is thus completely proved. $\square$

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