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FRACTAL AND EUCLIDEAN INTERACTION IN SOME TRANSMISSION PROBLEMS

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In this talk some model examples of second order elliptic transmission problems with highly conductive layers will be described. Regularity and numerical results for solutions of transmission problems across fractal layers imbedded in Euclidean domains will be presented in the aim of better understanding the analytical problems which arise when fractal and Euclidean structures mutually interact.

1. Introduction

Within a research program with many friends and colleagues we have studied "unusual" elliptic and parabolic problems on domains with non-smooth boundary and fractal layers. For an exhaustive discussion of "unusual" problems on the classical setting of smooth domains I refer to the survey of D. E. Apushkinskaya and A. I. Nazarov (see [1]) where unusual boundary value problems are seen in the more general context of the so-called Venttsel problems which go back to the late 50's (see [34]).
More recently a new perspective arose in the theory of boundary value problems namely "large boundaries and small volumes" as pointed out by U. Mosco in a conference at the Accademia dei Lincei (2002) (see [26]). This interest emerges naturally in models of transmission problems of absorption or irrigation type

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where surface effects are enhanced. In this context fractal boundaries and fractal layers provide new, interesting settings adequate to this goal. Fractal analysis could provide appropriate frameworks to physical and biological problems with dominant surface effects as irrigation models, bronchial systems, root infiltration, tree foliage and other kinds of similar phenomena.
Here I will deal with second order transmission problems across fractal layers embedded in Euclidean domains. Second order transmission problems across smooth layers have been considered in the early seventies in connection with various applications.
Let me mention only the early work of J. R. Cannon and G. H. Meyer (see [4]) on the flow of oil from a ”reservoir” into a producing oil well and the contribution of H. Pham Huy and E. Sanchez-Palencia (see[31]) in which second order transmission problems provide models to describe the heat transfer through an infinitely conductive (smooth) layer.

From a mathematical point of view H. Attouch in his book (see [2]) chose a second order transmission problem across a plane layer as interesting example of singular homogenization.
Let me consider a model problem, (see Figure 1): Ω is the ”square” $[0,1] \times [-1/2, 1/2]$ and the thin layer Σ_ε is the neighborhood of the segment Σ of thickness ε i.e.

$$\Sigma = \{(x,0) \ : \ 0 < x < 1\} \tag{1.1}$$

$$\Sigma_\varepsilon = \{(x,y) \in \Sigma \ : \ |y| < \frac{\varepsilon}{2}\} \tag{1.2}$$

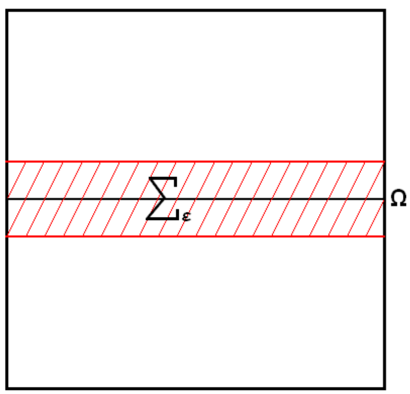

Figure 1:

the layer is supposed to have high conductivity: the conductivity coefficient is the inverse of the thickness:

$$a_\varepsilon(x,y) = \begin{cases} \frac{1}{\varepsilon} & \text{on} \quad \Sigma_\varepsilon \\ 1 & \text{on} \quad \Omega \setminus \Sigma_\varepsilon . \end{cases} \tag{1.3}$$

The corrisponding energy functional is $F_\varepsilon : H_0^1(\Omega) \to \mathbb{R}^+$

$$F_\varepsilon(u) = \int_\Omega a_\varepsilon(x,y)|\nabla u|^2 dxdy. \tag{1.4}$$

As ε goes to zero the sequence of functionals F_ε converges (in the weak topology of $H^1(\Omega)$) to the ”singular” functional $F : D_0(F) \to \mathbb{R}^+$

$$F = \int_\Omega |\nabla u|^2 dxdy + \int_\Sigma |u_x|^2 dx. \tag{1.5}$$

Remark 1.1. Let me stress the fact that the domain of the limit functional F is smaller than the common domain of the functionals F_ε in fact the domain of F_ε is $H_0^1(\Omega)$ while $D_0(F)$ is the subspace of the functions of $H_0^1(\Omega)$ having trace on $\overline{\Sigma}$ belonging to $H_0^1(\Sigma)$.

Coming back to the point of view of the boundary value problems we note for any choice of datum f in $L^2(\Omega)$ the corresponding minimizer u of the total energy functional satisfies some conditions: the limit layer Σ divides the domain Ω in two adjacent subdomains Ω^1 and Ω^2 and we have (see Figure 2)

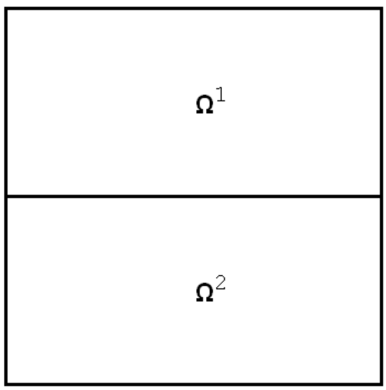

Figure 2:

$$
(T.P.)\begin{cases} i) & -\Delta u^j = f \qquad\qquad \text{on} \quad \Omega^j \\ \\ ii) & \begin{cases} u = 0 & \text{on} \quad \partial\Omega \\ u^1 = u^2 & \text{on} \quad \Sigma \\ u_{|_\Sigma} = 0 & \text{on} \quad \partial\Sigma \end{cases} \\ \\ iii) & \frac{\partial u^1}{\partial \nu^1} + \frac{\partial u^2}{\partial \nu^2} = \text{"}\Delta_\Sigma u\text{"} \qquad \text{on} \quad \Sigma. \end{cases}
$$

where $u^j = u_{|_{\Omega^j}}$, $j = 1,2$, $\frac{\partial}{\partial \nu^j}$ denotes the exterior normal derivative to the boundary of Ω^j and Δ_Σ the Laplace operator on Σ (with homogeneous boundary condition on $\partial\Sigma$). The layer is higher conductive than the surrounding medium and heat is absorbed by the layer and starts diffusing within it much more efficiently than in the surrounding volume. The normal derivatives from each side of the layer have a jump across the layer which acts as source term for the Laplace operator generating the in-layer diffusion. The resulting boundary condition iii) is thus of second order which is in some sense unusual for second order elliptic problems: moreover the condition has an implicit character since the source term of the layer equation: the jump of the normal derivatives is not among the data of the problems but depends on the solution itself.
The plan of this paper is the following: section 2 concerns singular homogenization results, section 3 deals with pre-fractal and fractal transmission problems; finally section 4 is devoted to some numerical results.

2. Singular homogenization results

Let Ω be the square $[0,1]\times[-\frac{1}{2},\frac{1}{2}]$ and let $\mathscr{D}$ be the polygonal sub-domain with vertices $A=(0,0)$, $B=(1,0)$, $C=(\frac{1}{2},\frac{1}{2\sqrt{3}})$ and $D=(\frac{1}{2},-\frac{1}{2\sqrt{3}})$, let K^0 denote the interval with and points A and B and K the Koch curve (with end points A and B) generated by the 4 contractive similitudes $\{\psi_1,\psi_2,\psi_3,\psi_4\}$

$$
\psi_1(z) = \frac{z}{3},\ \psi_2(z) = \frac{z}{3}\,e^{i\frac{\pi}{3}} + \frac{1}{3},
$$

$$
\psi_3(z) = \frac{z}{3}\,e^{-i\frac{\pi}{3}} + \frac{1}{2} + i\frac{\sqrt{3}}{2}\ \ \psi_4(z) = \frac{z+2}{3};\ (z\in\mathbb{C}).
$$

Denote $V^0=\{A,B\}$. For each integer $n>0$ we consider arbitrary $n-tuples$ of indices $i_{|_n}=(i_1,i_2,\dots,i_n)\in\{1,2,3,4\}^n$ and we define $\psi_{i_{|_n}}=\psi_{i_1}\circ\psi_{i_2}\circ\cdots\circ\psi_{i_n}$ and for any set $\mathscr{G}(\subseteq\mathbb{R}^2)$: $\mathscr{G}_{i_{|_n}}=\psi_{i_{|_n}}(\mathscr{G})$.
We consider the set of the vertices at the $n-$generation:

$$
V^n = \bigcup_{i_{|_n}} V^0_{i_{|_n}} \quad \text{and} \quad V^\infty = \bigcup_{n=0}^{+\infty} V^n \tag{2.1}
$$

(it holds $K = \overline{V^\infty}$ the closure in $\mathbb{R}^2$) and the polygonal curve at the $n-$generation

$$K^n = \bigcup_{i_{|n}} K^0_{i_{|n}}. \tag{2.2}$$

Having in mind the self similarity of the fractal we choose thin layer adapted to the contractive similitudes generating the Koch curve.
More precisely let $\mathscr{R}^\varepsilon$ the (open) rectangle of vertices P_1, P_2, P_3, P_4,

$$P_1 := (\frac{\varepsilon}{c_1}, \frac{\varepsilon}{2}),\ \ P_2 := (1-\frac{\varepsilon}{c_1}, \frac{\varepsilon}{2}),\ \ P_3 := (1-\frac{\varepsilon}{c_1}, -\frac{\varepsilon}{2}),\ \ P_4 := (\frac{\varepsilon}{c_1}, -\frac{\varepsilon}{2})$$

$\mathscr{T}_1^\varepsilon$ the (open) triangle of vertices A, P_1, P_4 and $\mathscr{T}_2^\varepsilon$ the (open) triangle of vertices P_2, B, P_3 the "conductive" layer Σ is the union $\overline{\Sigma}_\varepsilon = \overline{\mathscr{R}}^\varepsilon \cup \overline{\mathscr{T}}^\varepsilon_1 \cup \overline{\mathscr{T}}^\varepsilon_2$. Now we

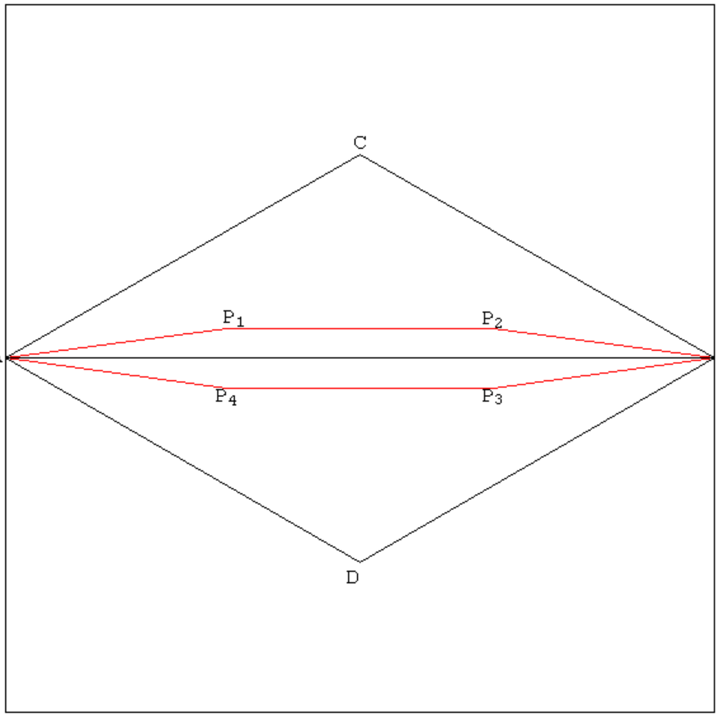

Figure 3: Geometry of the layer

make the maps ψ_i work and (at the n-th step of the iteration procedure) we construct the thin polygonal layers Σ^n_ε around the polygonal curve K^n: we put

$$\Sigma_\varepsilon^n = \bigcup_{i_{|n}} \Sigma_{\varepsilon\, i_{|n}} \tag{2.3}$$

Two different copies of set Σ_ε can not overlap each other they can share either a vertex or a whole side. This can be done by choosing the opening of the angles in Σ_ε according to the rotation angle of the similitudes ψ_i, that is we choose $c_1 = \tan\frac{\pi}{12}$ and $\varepsilon \leq \varepsilon_0 < \frac{c_1}{2}$.
For any (fixed) n we consider a family of functionals F_ε^n in $L^2(\Omega)$

$$F_\varepsilon^n(u) = \begin{cases} \int_\Omega a_\varepsilon^n |\nabla u|^2 d\xi d\eta & if \quad u \in H_0^1(\Omega, w_\varepsilon^n) \\ +\infty & if \quad u \in L^2(\Omega) \setminus H_0^1(\Omega, w_\varepsilon^n) \end{cases} \tag{2.4}$$

where the unbounded conductivity coefficient is

$$a_\varepsilon^n(\xi,\eta) = \begin{cases} w_\varepsilon^n(\xi,\eta) & if \quad (\xi,\eta) \in \Sigma_\varepsilon^n \\ 1 & if \quad (\xi,\eta) \notin \Sigma_\varepsilon^n. \end{cases} \tag{2.5}$$

The weight w_ε^n is defined in the following way: let P belong to $\partial(\Sigma_\varepsilon)_{i_{|n}}$ denote by $P^\perp$ its "orthogonal" projection on $K^0_{i_{|n}}$ and by $|P - P^\perp|$ the (Euclidean) distance between P and $P^\perp$ (in $\mathbb{R}^2$): then if $Q = (\xi,\eta)$ belongs to the segment of end points P and $P^\perp$ we put

$$w_\varepsilon^n(\xi,\eta) = \begin{cases} \frac{2+c_1^2}{4|P-P^\perp|}\sigma_n & if \quad (\xi,\eta) \in (\mathscr{T}_j^\varepsilon)_{i_{|n}} \quad j = 1,2 \\ \frac{1}{2|P-P^\perp|}\sigma_n & if \quad (\xi,\eta) \in (\mathscr{R}^\varepsilon)_{i_{|n}} \end{cases} \tag{2.6}$$

and

$$w_\varepsilon^n(\xi,\eta) = 1 \quad if \quad (\xi,\eta) \notin \Sigma_\varepsilon^n \tag{2.7}$$

where σ_n is a positive constant that here as n is fixed does not play any role a good choice will be essential in the asymptotic theory sketched in section 3. Let me stress the fact that, more delicate geometry, the conductive coefficient a_ε^n is not constant in the thin layers (for fixed n and ε) it is unbounded and belongs to the Muckenhoupt class $\mathscr{A}_2$.
The domain of the functional is a weighted Sobolev space:

$$H^1(\Omega, w_\varepsilon^n) = \{u \in L^2(\Omega) : \int_\Omega |\nabla u|^2 w_\varepsilon^n d\xi d\eta < +\infty\} \tag{2.8}$$

and $H_0^1(\Omega, w_\varepsilon^n)$ the completion of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{H^1(\Omega, w_\varepsilon^n)} := \{\int_\Omega |u|^2 d\xi d\eta + \int_\Omega |\nabla u|^2 w_\varepsilon^n d\xi d\eta\}^{\frac{1}{2}}.$$

Let me recall now the definition of convergence of functionals introduced in [23], denoted in the following M-convergence.

Definition 2.1. A family of functional F_ε $M-$converges to a functional F in $L^2(\Omega)$ if

(a) For every v_ε converging weakly to u in $L^2(\Omega)$

$$\underline{lim} F_\varepsilon(v_\varepsilon) \geq F(u), \quad as \quad \varepsilon \to 0.$$

(b) For every $u \in L^2(\Omega)$ there exists u^*_ε converging strongly in $L^2(\Omega)$ such that

$$\overline{lim} F_\varepsilon(u^*_\varepsilon) \leq F(u), \quad as \quad \varepsilon \to 0.$$

Remark 2.1. The family of functionals in (2.4) is asymptotically compact in $L^2(\Omega)$ according to Definition 2.3.1 in [24]. Hence the $M-$convergence is equivalent to the $\Gamma-$convergence (see Lemma 2.3.2 in [24]), thus we can take in *(a)* v_ε strongly converging to u in $L^2(\Omega)$.

The (pre-fractal) singular homogenization result is:

Theorem 2.2. *In the previous assumptions and n2007/01/01otations for any choice of (fixed) n the functionals F^n_ε (as $\varepsilon \to 0$) $M-$converge to the functional F^n where*

$$F^n(u) = \begin{cases} \int_\Omega |\nabla u|^2 d\xi d\eta + \sigma_n \int_{K^n} |\nabla_\tau u|^2 ds & if \quad u \in D_0(F^n) \\ +\infty & if \quad u \in L^2(\Omega) \setminus D_0(F^n) \end{cases} \tag{2.9}$$

where

$$D_0(F^n) = \{u \in H^1_0(\Omega) : u_{|_{K^n}} \in H^1_0(K^n)\}. \tag{2.10}$$

Here ∇_τ denotes the tangential derivative along the sides of the polygonal curve K^n and $H^1_0(K^n)$ denotes the Sobolev space on the polygonal curve K^n according to J. Nečas [30] (see also Remark 3.3). The proof of Theorem 2.1 is long and technical. I refer to [28] where more general Koch curves are considered.

Remark 2.3. Let me recall that the M-convergence can be characterized in terms of convergence of the resolvent operators, semigroups and spectral families associated with the forms allowing developments and applications (see Theorem 2.4.1, Corollaries 2.6.1 and 2.7.1 of [24]). However in this paper, I will not deal with these consequences of Theorem 2.1.

In the previous theorem the step n is supposed fixed a natural question is then what happens if n goes to infinity and ε goes to zero simultaneously; the following Theorem 2.2 answers to this question.
Actually when n increases the pre-fractal curves K^n develop increasing lengths up to reach the infinite length of the limit fractal: the "Euclidean pre-fractal

energies" (2.4) must be re-normalized and this amounts to take into account the "structural constants" of the limit fractal: the number of the similitudes, the contraction factor, the Hausdorff dimension and the scaling factor of the energy. Let me recall some basic results concerning our case.
The Koch K is a $d-$set with respect to the Hausdorff measure $\mathscr{H}^d$: $d = \frac{\lg 4}{\lg 3}$, on K is defined the energy form $E_K[u]$ limit of an increasing sequence of quadratic forms constructed by finite difference schemes

$$\begin{cases} E_K[u] = \lim_{n\to+\infty} E_K^{(n)}[u] \\ E_K^{(n)}[u] = 4^n \sum_{i_{|n}} (u(\psi_{i_{|n}}(A)) - u(\psi_{i_{|n}}(B)))^2 \end{cases} \tag{2.11}$$

the form $E_K[u]$ turns out to be a regular Dirichlet form on $L^2(K, \mathscr{H}^d)$ with (dense) domain $D_0(E_K)$ (see [7] and also [6] for definitions and details).
As previously let Σ_ε be the "ε-neighborhood" of K^0 union of the rectangle $\mathscr{R}_\varepsilon$ and the two triangles $\mathscr{T}_j^\varepsilon$: $j = 1, 2$, and let Σ_ε^n be as in (2.3) but now ε must depend on n: $\varepsilon = \varepsilon(n)$.
The "weighted" functional in $L^2(\Omega)$ is

$$E_n(u) = \begin{cases} \int_\Omega a_n(\xi,\eta)|\nabla u|^2 d\xi d\eta & if \quad u \in H_0^1(\Omega, w_n) \\ +\infty & if \quad u \in L^2(\Omega) \setminus H_0^1(\Omega, w2007/01/01_n) \end{cases} \tag{2.12}$$

where the unbounded conductivity coefficient is

$$a_n(\xi,\eta) = \begin{cases} \rho_n \ \ w_n(\xi,\eta) & if \quad (\xi,\eta) \in \Sigma_{\varepsilon(n)}^n \\ 1 & if \quad (\xi,\eta) \notin \Sigma_{\varepsilon(n)}^n. \end{cases} \tag{2.13}$$

The weight w_n is defined in the following way: let P belong to $\partial(\Sigma_{\varepsilon(n)})_{i_{|n}}$, denote by $P^\perp$ its "orthogonal" projection on $K^0_{i_{|n}}$ and by $|P - P^\perp|$ the (euclidean) distance between P and $P^\perp$ (in $\mathbb{R}^2$), if $(\xi,\eta) = Q$ belongs to the segment of end points P and $P^\perp$ then we set:

$$w_n(\xi,\eta) = \begin{cases} \frac{2+c_1^2}{4|P-P^\perp|}\sigma_0 & if \quad (\xi,\eta) \in (\mathscr{T}_j^{\varepsilon(n)})_{i_{|n}} \quad j = 1,2 \\ \frac{1}{2|P-P^\perp|}\sigma_0 & if \quad (\xi,\eta) \in (\mathscr{R}_{\varepsilon(n)})_{i_{|n}} \end{cases} \tag{2.14}$$

and

$$w_n(\xi,\eta) = 1 \qquad if \quad (\xi,\eta) \notin \Sigma_{\varepsilon(n)}^n \tag{2.15}$$

where σ_0 is a fixed positive constant.
The relative Sobolev spaces are

$$H^1(\Omega, w_n) = \{u \in L^2(\Omega) : \int_\Omega |\nabla u|^2 w_n d\xi d\eta < +\infty\} \tag{2.16}$$

and $H_0^1(\Omega, w_n)$ the completion of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{H^1(\Omega,w_n)} := \{\int_\Omega |u|^2 d\xi d\eta + \int_\Omega |\nabla u|^2 w_n d\xi d\eta\}^{\frac{1}{2}}.$$

We are now in position to enounce:

Theorem 2.4. *In the previous assumptions and notations let $\rho_n = (3^{d-1})^n$ and let $\varepsilon(n) \to 0$ as $n \to +\infty$. Then the sequence of the functionals E_n defined in (2.12), $M-$converges to the functional E:*

$$E(u) = \begin{cases} \int_\Omega |\nabla u|^2 d\xi d\eta + \sigma_0 E_K[u] & if \quad u \in D_0(E) \\ +\infty & if \quad u \in L^2(\Omega) \setminus D_0(E) \end{cases} \quad (2.17)$$

where

$$D_0(E) = \{v \in H_0^1(\Omega) : v_{|_K} \in D_0(E_K)\}. \quad (2.18)$$

I refer to [29] for proofs and details.
The geometry is "delicate" in Theorem 2.1 and the conductivity coefficient a_n is unbounded and belongs to the Muckenhaupt class $\mathscr{A}_2$ moreover here a new term appears the re-normalizing factor ρ_n that takes into account the Hausdorff dimension d of the Koch curve, the number and the contraction factor of the similitudes and the scaling factor of the energies (see (2.11)).

3. Transmission problems

The study of the fractal transmission problem formally written in Section 1 is delicate: the rigorous definition of the operators and functional spaces in condition i)–iii) is one of the main technical difficulty of this type of problem: however the condition in (T.P.) can be seen as the Euler conditions satisfied by the minimizer of a suitable energy functional. Hence a natural approach is to prove existence and uniqueness of the weak solution by variational principles and then to establish regularity results in order to state rigorously the strong formulation (T.P.). Moreover an asymptotic "constructive" theory for the pre-fractal approximation is an important step toward the numerical analysis of the problem and in this spirit the functionals F^n in (2.9) and the functional E in (2.17) can be related by choosing in an appropriate way, the constants σ_n in (2.6). More precisely the following results hold:

Proposition 3.1. *The spaces $D_0(F^n)$ and $D_0(E)$ given in Section 2 are Hilbert spaces under the intrinsic norms:*

$$\left.\begin{aligned} \|u\|_{D_0(F^n)} &= (F^n(u))^{\frac{1}{2}} \\ \|u\|_{D_0(E)} &= (E(u))^{\frac{1}{2}} \end{aligned}\right\} \quad (3.1)$$

Moreover, the bilinear forms $\alpha_n(\cdot,\cdot)$ and $\alpha(\cdot,\cdot)$ associated with the functionals F^n and E, with domains $D_0(F^n)$ and $D_0(E)$ are regular, strongly local Dirichlet forms in $L^2(\Omega)$.

(see [6] and also [16] and [20]). As consequence we have:

Corollary 3.2. *Given $f \in L^2(\Omega)$, there exist a unique $u_n \in D_0(F^n)$ and a unique $u \in D_0(E)$ such that*

$$\alpha_n(u_n, v) = \int_\Omega f v dx dy, \quad \textit{for every} \quad v \in D_0(F^n). \tag{3.2}$$

$$\alpha(u, v) = \int_\Omega f v dx dy, \quad \textit{for every} \quad v \in D_0(E). \tag{3.3}$$

We recall that the solutions u_n of (3.2) and (3.3) attain the minimum in the corresponding functionals.

Remark 3.3. Also in the classical case of [31], the regularity of the normal derivative for problems of this type is not obvious at all. Actually the solution v of the following Dirichlet problem, in an open regular set $\mathscr{G}$ of $\mathbb{R}^2$,

$$\begin{cases} -\Delta v = f & \text{in } \mathscr{G} \\ v = h & \text{on } \mathscr{G} \end{cases}$$

with $f \in L^2(\mathscr{G})$ and $h \in H^1(\partial\mathscr{G})$, is in $H^{3/2}(\mathscr{G})$; the additional information, that the normal derivative $\frac{\partial v}{\partial \nu}$ belongs to $L^2(\mathscr{G})$, can not be deduced directly from the fact that $v \in H^{3/2}(\mathscr{G})$ by using the classical trace theorem.

In [24] the following regularity results have been proved, remember that K^n divides Ω in two adjacent sub-domains Ω_n^j, $j = 1, 2$.

Theorem 3.4. *Let u_n be the variational solution (in Corollary 3.1) then we have that*

$$u_n \in C(\overline{\Omega}) \tag{3.4}$$

$$u_n^1 \in H^{\frac{8}{5}-\varepsilon}(\Omega_n^1),\, u_n^2 \in H^{\frac{7}{4}-\varepsilon}(\Omega_n^2) \tag{3.5}$$

$$\frac{\partial u_n^j}{\partial \nu_j} \in L^2(K^n),\, j = 1, 2 \tag{3.6}$$

in particular conditions ii), are satisfied pointwise, i) and iii) almost everywhere and

$$\Delta_{K^n} = \sigma_n D_\ell^2. \tag{3.7}$$

Here u_n^j is the restriction of u_n to Ω_n^j, $\frac{\partial u_n^j}{\partial \nu_j}$ the outward "normal derivatives", D_ℓ^2 the "piecewise" second order tangential derivative along the sides of K^n.

Remark 3.5. Let me note that the discrepancy between the Sobolev regularity exponents for u_n^1 and u_n^2 is due to the geometry of the polygonal domains Ω_n^1 and Ω_n^2 which have different (largest) angles $\left(\frac{5}{3}\right)\pi$ and $\left(\frac{4}{3}\right)\pi$ respectively. As it is know from the regularity theory, the regularity of the solutions improves if the opening of the inner angles becomes smaller. This effect holds on, despite the implicit character of the equations, and the dependence of the regularity exponent on the angle remains unperturbed. In [20] we have proved for the "classical" case ($n=0$) that both the restrictions u_0^j belong to the Sobolev space $H^2(\Omega_0^j)$ $j=1,2$.

For the present application, in my opinion, the more convenient definition of Sobolev spaces on polygonal boundaries is that given by Brezzi and Gilardi in [3] (definition 2.27). If $0 \le s \le \frac{3}{2}$ the Sobolev space $H^s(K^n)$ defined in [3] coincides, with equivalent norms, with the Sobolev space defined in Nečas [30] by local Lipschitz charts. (see Theorem 2.23 in [3]). In particular the Sobolev space $H_0^1(K^n)$ coincides with the space $\{v \in C_0(K^n) : \ v_{|_M} \in H^1(M)\, \forall M \in K^n\}$,more generally, for $s \ge 1$, we have $H^s(K^n) = \{u \in H^1(K^n), \quad u_{|_M} \in H^s(M)\, \forall M \in K^n\}$; where M denotes a segment at the n-generation. Roughly speaking in the definition of the Sobolev space, natural in this context, compatibility conditions on the vertices are required for the function and not for the derivatives.
Finally the Lions-Magenes space $H_{0,0}^{\frac{1}{2}}(K^n)$ is defined as: $\{u \in L^2(K^n) : \ \exists v \in H_0^1(\Omega) : \ v_{|_{K^n}} = u\}$ equipped with the quotient norm

$$\|u\|_{H_{0,0}^{\frac{1}{2}}(K^n)} = \inf_{v \in H_0^1(\Omega), v_{|_{K^n}}=0} \|v\|_{H^1(\Omega)}.$$

Also in the fractal case, the variational solution u satisfies a second order transmission condition which is obtained via integration by parts and Green formulas in each sub-domains Ω^j. The normal derivatives have to be intended in a suitable sense, that is, a dual sense, namely they belong to the dual of the subspace $B_{\beta,0}^{2,2}$ of the Besov space $B_\beta^{2,2}$ where β is equal to $\frac{d}{2}$ (we recall that d is the Hausdorff dimension of K). Roughly speaking $B_{\beta,0}^{2,2}$ is the fractal analogous of the Lions-Magenes space $H_{0,0}^{\frac{1}{2}}(\Gamma)$ that is defined when Γ is a Lipschitz surface. Then the variational solution u satisfies the transmission condition in a dual sense: that is the sense of the dual of the domain $D_0(E_K)$. More precisely the following result has been proved in [16]:

Theorem 3.6. *Let u be the variational solution of Corollary 3.1 then we have* *that*

$$u^j \in H^2_{loc}(Q^i) \tag{3.8}$$

$$\frac{\partial u^j}{\partial \nu_j} \in \left(B^{2,2}_{\beta,0}(K)\right)', \ \beta = \frac{d}{2}, \ j = 1,2 \tag{3.9}$$

and the transmission condition iii) holds in $(D_0(E_K))'$ *that is*

$$\left\langle \Delta_K u_{|_K}, z \right\rangle_{(D_0(E_K))', D_0(E_K)} = \left\langle \left[\frac{\partial u}{\partial \nu}\right]_K, z \right\rangle_{(D_0(E_K))', D_0(E_K)} \tag{3.10}$$

where $\left(B^{2,2}_{\beta,0}(K)\right)'$ *is the dual of* $B^{2,2}_{\beta,0}(K)$, $(D_0(E_K))'$ *is the dual of* $D_0(E_K)$, Δ_K *is the variational operator from* $D_0(E_K) \to (D_0(E_K))'$ *and* $< \cdot, \cdot >_{(D_0(E_K))', D_0(E_K)}$ *is the duality pairing between* $(D_0(E_K))'$ *and* $D_0(E_K)$.

Here u^j is the restriction to Ω^j, $\frac{\partial u^j}{\partial \nu_j}$, $j = 1,2$ the outward "normal derivative" and $\left[\frac{\partial u}{\partial \nu}\right] = \frac{\partial u^1}{\partial \nu_1} + \frac{\partial u^2}{\partial \nu_2}$ the jump of the normal derivative.
We now study the convergence property of problem (3.2) as $n \to +\infty$. Here a correct choice of σ_n is essential.

Theorem 3.7. *Let u be the variational solution of the fractal transmission problem in the domain* Ω*, with layer K the (fractal) Koch curve. For every integer* $n \geq 1$*, let* u_n *be the variational solution of the transmission problem in* Ω *with pre-fractal layer* K^n*. If we scale the energy functionals (2.9), by taking* $\sigma_n = \sigma_0 (3^{d-1})^n$ *then as* $n \to \infty$ *we find:*

$$u_n \to u \text{ strongly in } H^1_0(\Omega) \tag{3.11}$$

$$\int_{K^n} \frac{\partial u_n^j}{\partial \nu^j} \phi \, d\ell \to \left\langle \frac{\partial u^j}{\partial \nu^j}, \phi \right\rangle_{(B^{2,2}_{\beta,0}(K))', B^{2,2}_{\beta,0}(K)} \quad \forall \phi \in H^1_0(\Omega) \tag{3.12}$$

$$\beta = d/2, \ j = 1,2$$

$$\int_{K^n} \Delta_{K^n} u_n \phi d\ell \to \langle \Delta_K u, \phi \rangle_{(D_0(E_K)', D_0(E_K)} \ \forall \phi \in D_0(E). \tag{3.13}$$

where Δ_{K^n} is defined in (3.7) and Δ_K is the variational Laplacean as in Theorem 3.2.
A stronger result is indeed proved in [21]: the *M-convergence* of the pre-fractal energies F^n defined in (2.9) to the fractal energy E defined in (2.17).

Remark 3.8. From the probabilistic point of view, Brownian motions penetrating fractal sets – a probabilistic counterpart of the analytic variational approach adopted here – have been constructed by Lindstrøm [22] and Kumagai [14], however without reference to transmission problems and related transmission conditions.

4. Numerical results

Let me conclude this paper with some numerical results obtained by E. Vacca in her PHD Thesis (see [33]).

Consider the pre-fractal transmission problems:the pre-fractal Koch curve K^n that divides Ω in two adjacent sub-domains Ω_n^1 and Ω_n^2 (see Figure 4). The

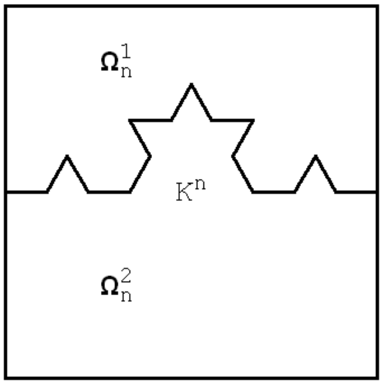

Figure 4:

variational solution u_n enjoys some regularity properties (see [33] and also Theorem 3.1). In particular we have:

Theorem 4.1. *Let u_n be the variational solution (as in Corollary 3.1). Then*

$$r_j^{\mu_j} D^\alpha u_n^j \in L^2(\Omega_n^j), \ |\alpha| = 2, \ j = 1,2, \ \mu_1 > \frac{2}{5}, \ \mu_2 > \frac{1}{4} \tag{4.1}$$

$$u_n \in H^2(K^n). \tag{4.2}$$

Where $r_j = r_j(P)$ denotes the distance of the point P from the set of the vertices of the "reentrant" corners of Ω^j.

Remark 4.2. There is a strict relation between the weights in Theorem 4.1 and the smoothness "exponent" in the fractional Sobolev spaces (see Theorem 3.1); let me stress the fact that the value of weights μ_j plays an important role in the error estimate as we will see in Theorem 4.2 (following).

The pre-fractal problems are approximated by the "h-version" of Galerkin finite element method. Let me recall the main definitions:

Definition 4.1. A triangulation $\mathscr{T}^j = \mathscr{T}^j(h)$ of Ω^j is regular and conformal if

- $\overline{\Omega}^j = \bigcup_{T \in \mathscr{T}^j} T$ forever:
- $\dot{T} \neq \emptyset, \forall \;\; T \in \mathscr{T}^j$
- $\dot{T}_1 \cap \dot{T}_2 = \emptyset, \quad \forall \;\; T_1, T_2 \in \mathscr{T}^j : \quad T_1 \neq T_2$
- $T_1 \cap T_2 \neq \emptyset, \quad T_1 \neq T_2 \quad \Rightarrow \quad T_1 \cap T_2 =$ edge or vertex
- $\exists \; \sigma > 0$ such that $\max_{T \in \mathscr{T}^i} \left(\frac{h_T}{\eta_T} \right) \leq \sigma$

where $h_T = diam(T)$ and $\eta_T = \sup\{diam(B) : \; B \; ball \subset T \; \}$.

Here $h = \sup\left\{h_T : \;\; T \in \mathscr{T}^j, \; j = 1,2\right\}$ denotes the size of the triangulation $\mathscr{T}^j$.

The choice of an appropriate triangulation is a crucial point in order to obtain more precise discrete solutions and better error estimates. From now on n denotes the step of iteration in the pre-fractal curve K^n: hence as n goes to infinity the "size" of the triangulation $h = h(n)$ goes to zero.

Definition 4.2. The family of triangulations $\mathscr{T}^j_{n,h}$, $h \in \mathbb{R}$, $n \in \mathbb{N}$, $j = 1,2$, is "adapted" to our problem if

- the vertices of the pre-fractal curves K^n are nodes of the triangulations
- the meshes are conformal and regular according to Definition 4.1
- $\exists \;\; \sigma > 0$ such that:

$$\begin{cases} h_T \leq \sigma h^{\frac{1}{1-\mu_j}} & \forall T \in \mathscr{T}^j_{n,h} : T \cap K^n \neq \emptyset \\ h_T \leq \sigma h \cdot \inf_T r_j^{\mu_j} & \forall T \in \mathscr{T}^j_{n,h} : T \cap K^n = \emptyset \end{cases}$$

where $\mu_1 = \frac{2}{5} + \varepsilon, \; \mu_2 = \frac{1}{4} + \varepsilon, \; 0 < \varepsilon$ "*small*".
Here $h = h(n) = \sup\left\{diam(T), T \in \mathscr{T}^j_{n,h}, \; j = 1,2\right\}$ is the size of the triangulation and $r_j = r_j(P)$ denotes the distance of the point P from the set of the vertices of the "reentrant" corners of Ω^j.

Let $V_n^{1,h}$ denote the "discrete" space:

$$V_n^{1,h} = \left\{v \in C^0\left(\bar{\Omega}\right), \; v = 0 \;\text{ on } \partial\Omega, \; v_{|_T} \text{ polynomial of degree } 1\right\}$$

that is a subspace of the domain $D_0(F^n)$, (see 2.10), hence there exist a unique "discrete" solution in $V_n^{1,h}$ that minimizes the total energy:

$$u_{h,n} = \text{argmin}_{V_n^{1,h}} \left\{ \frac{1}{2} F^n[u] - \int_Q f u \, dQ \right\} \tag{4.3}$$

(see also Corollary 3.1).

The following estimates hold:

Theorem 4.3. *Let u_n be the variational solution (see Corollary 3.1) and $u_{h,n}$ the "discrete solution" in $V_n^{1,h}$ (see (4.3)).*

Then we have:

$$\|u_n - u_{h,n}\|_{D_0(F^n)} \leq Ch\left\{ \sum_{j=1}^{2} \sum_{|\alpha|=2} \|r_j^{\mu_j} D^\alpha u_n^j\|_{L^2\left(\Omega_n^j\right)} + \|u_n\|_{H^2(K)} + \|u_n\|_{H_0^1(\Omega)} \right\} \tag{4.4}$$

where C is independent from h and n, $h = h(n) = \sup\{diam(T),\ T \in \mathcal{T}_{n,h}^j\}$ and $r_j = r_j(P)$ denotes the distance of the point P from the set of the vertices of the "reentrant" corners of Ω^j.

Remark 4.4. In the previous assumptions and notations, using (ordinary) fractional Sobolev spaces we would obtain a worse estimate: i.e.:

$$\|u_n - u_{h,n}\|_{D_0(F^n)} \leq Ch^{\frac{3}{5}-\varepsilon} \left\{ \|u_n^1\|_{H^{\frac{8}{5}-\varepsilon}(\Omega^1)} + \|u_n^2\|_{H^{\frac{7}{4}-\varepsilon}(\Omega^2)} + \|u_n\|_{H^2(K^n)} \right\}. \tag{4.5}$$

REFERENCES

[1] D.E. Apushkinskaya - A.I. Nazarov, *A survey of results on nonlinear Venttsel problems*, Appl. Math. **45** (1) (2000), 69-80.

[2] H. Attouch, *Variational Convergence for Functions and Operators*, Eds. Pitman Advanced Publishing Program, London, 1984.

[3] F. Brezzi - G. Gilardi, *Finite Elements Mathematics*, in Finite Element Handbook, Eds. H. Kardestuncer, D. H. Noriie, McGraw-Hill Book Co. New York, 1987.

[4] J.R. Cannon - G.H. Meyer, *On a diffusion in a fractured medium*, SIAM J. Appl. Math. **3** (1971), 434-448.

[5] K. Falconer, *The Geometry of Fractal Sets*, 2nd ed. Cambridge Univ. Press, 1990.

[6] M. Fukushima, *Dirichlet forms, diffusion processes and spectral dimensions for nested fractals*, S. Albeverio et al. (eds): Ideas and Methods in Mathematical Analysis, Stochastics, and Applications, Cambridge Univ. Press, Cambridge, (1992), 151-161.

[7] M. Fukushima - Y. Oshima - M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter Studies in Mathematics, Vol. 19, W. de Gruyter, Berlin, 1994.

[8] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman, Boston, 1985.

[9] J. E. Hutchinson, *Fractals and selfsimilarity*, Indiana Univ. Math. J. **30** (1981), 713-747.

[10] D. Jerison - C.E. Kenig, *The Neumann Problem in Lipschitz domains*, Bull. Amer. Math. Soc. **4** (1981), 71-88.

[11] D. Jerison - C.E. Kenig, *The inhomogeneous Dirichlet problem in Lipschitz domains*, J. Funct. Anal. **130** (1995), 161-213.

[12] A. Jonsson, *Dirichlet forms and Brownian motion penetrating fractals*, Potential Analysis **13** (2000), 69-80.

[13] A. Jonsson - H. Wallin, *Function Spaces on Subsets of* $\mathbb{R}^n$, Part 1 (Math. Reports Vol. 2) Harwood Acad. Publ. London, 1984.

[14] T. Kumagai, *Brownian motion penetrating fractals: an application of the trace theorem of Besov spaces*, J.Funct. Anal. **170**, (2000), pp. 69-92.

[15] S. Kusuoka, *Diffusion processes in nested fractals*, L. N. M. 1567, Springer, 1993.

[16] M.R. Lancia, *A transmission problem with a fractal interface*, Z. Anal. und Ihre Anwend. **21** (2002), 113-133.

[17] M.R. Lancia, *Second order transmission problems across a fractal surface*, Rend. Acc. Naz. XL, Mem. Mat. **121** Vol. XXVII (2003), 191-213.

[18] M.R. Lancia - P. Vernole, *Convergence results for parabolic transmission problems across highly conductive layers with small capacity*, Adv. Math. Sc. Appl. (2) **16** (2006), to appear.

[19] M.R. Lancia - M.A. Vivaldi, *Lipschitz spaces and Besov traces on self-similar fractals*, Rend. Acc. Naz. Sci. XL Mem. Mat. Appl. (5) **23** (1999), 101-106.

[20] M.R. Lancia - M.A. Vivaldi, *On the regularity of the solutions for transmission problems*, Adv. Math. Sc. Appl. **12** (2002), 455-466.

[21] M.R. Lancia - M.A. Vivaldi, *Asymptotic convergence of transmission energy forms*, Adv. Math. Sc. Appl. **13** (2003), 315-341.

[22] T. Lindstrøm , *Brownian motion penetrating the Sierpinski gasket, Asymptotic problems in probability theory: stochastics models and diffusions on fractals*, K. D. Elworthy, N. Ikeda Eds., Longman, 1993.

[23] U. Mosco, *Convergence of convex sets and of solutions of variational inequalities*, Adv. in Math. **3** (1969), 510-585.

[24] U. Mosco, *Composite media and asymptotic Dirichlet forms*, J. Funct. Anal. **123** (1994), 368-421.

[25] U. Mosco, *Energy functionals on certain fractal structures*, J. Convex Anal. **9** 2 (2002), 581-600.

[26] U. Mosco, *Highly conductive fractal layers*, Proceeding of the Conference ”Whence the boundary conditions in modern continuum physics?”, Roma Accademia dei Lincei October 14-16, 2002. *Atti Convegni Lincei, Accademia Nazionale dei Lincei* 2005.

[27] U. Mosco - M.A. Vivaldi, *Variational problems with fractal layers*, Rend. Acc. Naz. XL, Mem. Mat. 121 Vol. XXVII (2003), 237-251.

[28] U. Mosco - M.A. Vivaldi, *Regularity and asymptotic results for transmission problems*, (In preparation) Preprint (2006).

[29] U. Mosco - M.A. Vivaldi, *An exemple of fractal homogenization*, Submitted for the pubblication.

[30] J. Nečăs, *Les méthodes directes en théorie des équationes elliptiques*, Masson, Paris. 1967.

[31] H. Pham Huy - E. Sanchez-Palencia, *Phénomènes des transmission à travers des couches minces de conductivité élevée*, J. Math. Anal. Appl. **47** (1974), 284-309.

[32] H. Triebel, *Fractals and Spectra. Related to Fourier Analysis and Function Spaces*, (Monographs in Mathematics: Vol. 91), Birkhäuser Verlag, Basel, 1997.

[33] E. Vacca, *Galeakin approximation for Highly Conductive Layers*, PHD Thesis Dipartimento MeMoMat (2006).

[34] A.D. Venttsel, *On boundary conditions for multidimensional diffusion processes*, Teor. Veroyatnost. i Primenen. **4** (1959), 172-185; English translation: Theor. Probability Appl. **4** (1959), 164-177.

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