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ON SOME MOTIVATED CONJECTURES AND PROBLEMS

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Dedicated to Professor Francesco Guglielmino on his 70th birthday

In this paper we propose some conjectures and problems, in different fields of analysis, presenting also their motivations.

The aim of this paper is to propose some conjectures and problems, in different fields of analysis, presenting also their motivations.

To state the first conjecture, we recall the following definition. Let $(E, \|\cdot\|)$ be a real normed space. A non-empty set $A \subseteq E$ is said to be antiproximinal with respect to $\|\cdot\|$ if, for every $x \in X \setminus A$ and every $y \in A$, one has

$$\|x - y\| > \inf_{z \in A} \|x - z\|.$$

Conjecture 1. *There exists a non-complete real normed space E with the following property: for every non-empty convex set $A \subseteq E$ which is antiproximinal with respect to each norm on E, the interior of the closure of A is non-empty.*

The motivation for Conjecture 1 comes from open mapping theory. In fact, using Theorem 4 of [2], it is possible to get the following

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Theorem 1. *Let X, E be two real vector spaces, C a non-empty convex subset of X, F a multifunction from C onto E, with non-empty values and convex graph.*

Then, for every non-empty convex set $A \subseteq C$ which is open with respect to the relativization to C of the strongest vector topology on X, the set $F(A)$ is antiproximinal with respect to each norm on E.

The second conjecture concerns the possibility of proving Schauder fixed point theorem in a completely new way.

Conjecture 2. *Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space, $X \subseteq H$ a non-empty compact convex set, $f : X \to X$ a continuous function, ε a positive real number. Denote by Λ_ε the set all continuous functions $\varphi : X \times H \to \mathbb{R}$ such that, for each $x \in X$, $\varphi(x, \cdot)$ is Lipschitzian in H, with Lipschitz constant less than or equal to ε. Consider Λ_ε equipped with the relativization of the strongest vector topology on the space $\mathbb{R}^{X \times H}$.*

Then, the set

$$\left\{(\varphi, x, y) \in \Lambda_\varepsilon \times X \times H : \langle f(x) - x, y \rangle = \varphi(x, y)\right\}$$

is disconnected.

Assume that one is able to prove Conjecture 2 without using any result based on Brouwer fixed point theorem. Then, the Schauder theorem would follow from the conjecture and from the following result which, in turn, does not depend on any result based on the Brouwer theorem.

Theorem 2 ([5], Theorem 22)**.** *Let X be a connected and locally connected topological space, E a real Banach space, $\Phi : X \to E^*$ a (strongly) continuous operator, with closed range. For each $\varepsilon > 0$, denote by Λ_ε the set of all continuous functions $\varphi : X \times E \to \mathbb{R}$ such that, for each $x \in X$, $\varphi(x, \cdot)$ is Lipschitzian in E, with Lipschitz constant less than or equal to ε. Consider Λ_ε equipped with the relativization of the strongest vector topology on the space $\mathbb{R}^{X \times E}$, and assume that the set*

$$\left\{(\varphi, x, y) \in \Lambda_\varepsilon \times X \times E : \langle \Phi(x), y \rangle = \varphi(x, y)\right\}$$

is disconnected.

Then, Φ vanishes at some point of X.

Now, let us consider the following

Problem 1. *Let X be a real Banach space, and let $J : X \to \mathbb{R}$ be a lipschitzian function, with Lipschitz constant L.*

Find conditions under which there exists a continuous linear functional Φ on E, with $\|\Phi\|_{E^*} > L$, such that, if we put

$$V = \{x \in X : \Phi(x) = J(x)\},$$

the restriction of J to V has a local minimum point.

The motivation for Problem 1 is as follows. Assume that we wish to minimize a given lipschitzian bounded below functional f on a real Banach space E. Moreover, suppose that there exists a sequence $\{\lambda_n\}$ in $]0, +\infty[$, with $\lim_{n\to+\infty} \lambda_n = +\infty$, such that, for every $x \in E$, one has

$$\lim_{n\to+\infty} \frac{f(\lambda_n x)}{\lambda_n} = 0.$$

Let L be the Lipschitz constant of f. Assume that there is some continuous linear functional Φ on $L^1([0, 1], E)$, whose norm is strictly greater than L, such that, if we put

$$V = \left\{u \in L^1([0, 1], E) : \Phi(u) = \int_0^1 f(u(t))\, dt\right\},$$

the restriction of Φ to V has a local minimum point, say u_0. By Theorem 3 of [1], u_0 is actually a global minimum point for the restriction of Φ to V. On the other hand, Theorem 2 of [6] ensures that

$$\inf_{u\in L^1([0,1],E)} \int_0^1 f(u(t))\, dt = \inf_{u\in V} \int_0^1 f(u(t))\, dt.$$

Consequently, u_0 is a global minimum point for the functional

$$u \to \int_0^1 f(u(t))\, dt$$

in $L^1([0, 1], E)$. This easily implies that f has a global minimum point in E.

The next problem we wish to propose is as follows

Problem 2. *Let Ω be a bounded open set in $\mathbb{R}^n$, and let $a \in L^\infty(\Omega)$. Consider on $H_0^1(\Omega)$ the norm*

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega (|\nabla u(x)|^2 + |u(x)|^2)\, dx \right)^{1/2}.$$

Find necessary and sufficient conditions in order that the quantity

$$\inf_{\|u\|_{H_0^1(\Omega)}=1} \left\{ \int_\Omega (|\nabla u(x)|^2 + a(x)|u(x)|^2)\, dx + \right.$$
$$\left. + \sup_{\|v\|_{H_0^1(\Omega)}=1} \left(\int_\Omega (\nabla u(x) \nabla v(x) + a(x) u(x) v(x))\, dx \right) \right\}$$

be strictly positive.

The motivation for Problem 2 comes from Theorem 3 below which improves Theorem 3.25 of [3].

Theorem 3 ([7], Theorem 1)**.** *Let $(H, \langle \cdot\, , \cdot \rangle)$ be a real Hilbert space and $A : H \to H$ a linear operator. If*

$$\inf_{\|x\|=1} (\langle A(x), x \rangle + \|A(x)\|) > 0,$$

then A is continuous and invertible.

To state the last problem, let us introduce some notation. Let $m, n \in \mathbb{N}$. Following [4], let us denote by $V(\mathbb{R}^n)$ the space of all functions $u \in C^\infty(\mathbb{R}^n)$ such that, for every non-empty bounded set $\Omega \subseteq \mathbb{R}^n$, one has

$$\sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in \Omega} |D^\alpha u(x)| < +\infty,$$

where $D^\alpha u = \partial^{\alpha_1 + \ldots + \alpha_n} u / \partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Now, for each $\alpha \in \mathbb{N}_0^n$, with $|\alpha| = \alpha_1 + \ldots + \alpha_n \leq m$, let $a_\alpha \in \mathbb{R}$ be given. Let $P : V(\mathbb{R}^n) \to V(\mathbb{R}^n)$ be the differential operator defined by putting

$$P(u) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha u$$

for all $u \in V(\mathbb{R}^n)$.

Problem 3. *Find necessary and sufficient conditions in order that*

$$P(V(\mathbb{R}^n)) = V(\mathbb{R}^n).$$

Let us recall that, at present, the only (very partial) answer to Problem 3 is provided by the following

Theorem 4 ([4], Theorem 4)**.** *Let* $a, b \in \mathbb{R} \setminus \{0\}$ *and* $h, k \in \mathbb{N}$. *For each* $u \in V(\mathbb{R}^2)$*, put*

$$P(u) = a\frac{\partial^h u}{\partial x^h} + b\frac{\partial^k u}{\partial y^k}.$$

Then, one has

$$P(V(\mathbb{R}^2)) = V(\mathbb{R}^2)$$

if and only if $|a| \neq |b|$.

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