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A REMARK ON DIFFERENTIABILITY PROPERTIES OF SOLUTIONS OF SOME ELLIPTIC EQUATIONS

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A tribute to Professor Francesco Guglielmino in his 70th birthday

A class of nondivergence uniformly elliptic equations with measurable coefficients in a bounded smooth domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is studied. The oscillation of the coefficients near a Lebesgue point, measured in L^d-norm, is assumed to be controlled by an increasing function satisfying Dini's condition.

Under this assumption, slightly weaker than that of L.A. Caffarelli [2], a pointwise estimate for "good solutions" is established and related second order differentiability properties are pointed out.

1. Introduction and results.

Let Ω be a bounded smooth domain of $\mathbb{R}^d$, $d \geq 2$. In Ω let L be a second order linear uniformly elliptic nondivergence operator:

$$L = \sum_{i,j=1}^{d} a_{ij}(x) D_{ij}, \quad \text{where } D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, \tag{1.1}$$

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with measurable coefficients $a_{ij}(x)$ $(a_{ij} = a_{ji})$ satisfying for some constant $\nu \in (0; 1]$ the ellipticity condition:

$$\nu|\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \leq \nu^{-1}|\xi|^2, \tag{1.2}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^d$. We emphasize that no smoothness assumptions are made on the a_{ij}'s.

For operators of the form (1.1) the notion of classical solution of the equation

$$Lu = f \quad \text{in } \Omega, \tag{1.3}$$

with f a given function, turns out to be too limited. The results of N.V. Krylov and M.V. Safonov [12] have suggested that, in general, Hölder continuity is the optimal regularity for "solutions" of (1.3). So several notions of solutions have been built up by many mathematicians; see R.R. Jensen [11] for a through examination of the various definitions. Among the others recall the "viscosity solutions" of M.G. Crandall and P.L. Lions for which we refer to M.G. Crandall, H. Hishi and P.L. Lions [8], or the "good solutions" of M.C. Cerutti, L. Escauriaza and E.B. Fabes [4], [5]. Actually, as it has been shown by R.R. Jensen in [11], the notion of viscosity solution and that of good solution to (1.3) coincide.

In this paper we will deal with the latter notion. We recall that a function $u \in C^0(\overline{\Omega})$ is a "good solution" to the equation $Lu = f \in L^d$ in Ω if there exists a sequence of elliptic operators $\{L^n\}$, $L^n = \sum_{i,j=1}^{d} a_{ij}^n D_{ij}$ with smooth coefficients a_{ij}^n satisfying (1.2) for all n, $a_{ij}^n(x) \to a_{ij}(x)$ almost everywhere in Ω, and functions $u^n \in C^0(\overline{\Omega}) \cap W^{2,d}_{\mathrm{loc}}(\Omega)$ so that the sequence $\{u^n\}$ converges uniformly to u on $\overline{\Omega}$ and $L^n u^n = f$ in Ω.

Observe that good solutions to (1.3) do exist; for, recall that the results of N.V. Krylov and M.V. Safonov [12] imply that the sequence $\{u^n\}$ of smooth solutions to $L^n u^n = f$ admits a subsequence which converges uniformly on $\overline{\Omega}$.

As a matter of fact, within the class of solutions we are dealing with, a relevant question is that of the uniqueness for the Dirichlet problem associated to the operator L of the form (1.1):

$$Lu = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega. \tag{1.4}$$

Let me shortly, at the end of this sections, dwell upon this matter to just remind what is the state of art.

In this paper, actually, we will investigate second order differentiability properties of "good solutions" to elliptic equations of the type (1.3).

According to a result due to N.S. Nadirashivili [14] a good solution u possesses for almost every $x_0 \in \Omega$ a so-called derivative of order 2 in the sense that

$$u(x_0 + h) - P_{x_0}(h) = o(|h|^2) \quad \text{as } h \to 0 \tag{1.5}$$

with P_{x_0} a polynomial in h of degree 2.

Let us still denote by $D^2u = (D_{ij}u)$ the second derivatives of u, in the quadratic approximation sense (1.5).

In [6] M.C. Cerutti, E.B. Fabes and P. Manselli pointed out that, as a consequence of a result of L.A. Caffarelli [2], a good solution u has for almost every $x_0 \in \Omega$ first derivatives ∇u, defined by the linear approximation:

$$u(x_0 + h) = u(x_0) + \nabla u(x_0) \cdot h + o(|h|), \text{ as } h \to 0$$

and moreover its gradient ∇u is almost everywhere the pointwise limit of the gradients $\{\nabla u^n\}$ of the approximating sequence $\{u^n\}$.

The question that can be of some interest is the following: does the sequence of second derivatives $\{D^2u^n\}$ converge almost everywhere in Ω, having D^2u as its pointwise limit ?

We will give an affirmative answer (Theorem 1) to the above question provided the coefficients a_{ij} of L satisfy a suitable condition: roughly the oscillation of a_{ij} near x_0, measured in L^d-norm, is controlled by a function satisfying Dini's condition. Precisely, we will make the following assumption:

Assumption (A). Let $x_0 \in \Omega$ be a Lebesgue point for the a_{ij}'s and let $B_r = B_r(x_0)$ denote the ball with radius r centered at x_0.

There exists an increasing function $\omega(r)$ with $\omega(0) = 0$, satisfying the Dini's condition:

$$\int_0^1 s^{-1}\omega(s)\, ds < \infty, \tag{1.6}$$

such that for r sufficiently small:

$$\left(\fint_{B_r} |a_{ij}(x) - a_{ij}(x_0)|^d \right)^{1/d} \leq \omega(r). \tag{1.7}$$

This assumption is slightly weaker than that used by L.A. Caffarelli [2] to get $C^{2,\alpha}$ interior estimates for viscosity solutions.

We will be able (see Section 3) to prove the following result:

Theorem 1. *Assume that the assumption* (A) *on the coefficients a_{ij} of L holds. Let u be a good solution to $Lu = f \in L^d(\Omega)$. Let $\{u^n\}$ be a sequence of functions $C^0(\overline{\Omega})$ such that $L^n u^n = f$, with $\{L^n\}$ a sequence of smooth elliptic operators whose coefficients a_{ij}^n satisfy* (1.2) *for all n, $a_{ij}^n \to a_{ij}$ a.e. in Ω, and such that $u_n \to u$ uniformly on $\overline{\Omega}$.*

Then the sequence of the second derivatives $\{D_{ij}u^n\}$ $(i, j = 1, 2, \ldots, d)$ is convergent almost everywhere in Ω as $n \to \infty$ and for $i, j = 1, 2, \ldots, d$

$$D_{ij}u^n \to D_{ij}u \quad \textit{pointwise a.e. in } \Omega.$$

First of all, in Section 2, we will focus our attention to get a pointwise estimate for a good solution u, approximating it in L^∞ by a quadratic polynomial (Theorem 2). Such an estimate is the key, indeed, to establish Theorem 1.

Before concluding this section, let me say on uniqueness of solution to the Dirichlet problem (1.4).

If the dimension $d = 2$, uniqueness in $W^{2,2}$ holds-as it is well known-without any smoothness of the coefficients a_{ij} (we refer to G. Talenti [18]).

Let $d \geq 3$. Let E denote the set of points of Ω where the coefficients a_{ij} are allowed to be discontinuous. Uniqueness is, then, guaranteed if E is a sufficiently "small" set as in [4], [5], [13], [17]; to the best of my knowledge a somewhat more general case is that due to M.V. Safonov [17]: the set E has zero Hausdorff measure corresponding to the function $h(s) = s^\alpha$ for some positive α. And what is more, very recently, N.S. Nadirashivili [15] has been showing that – in general – there is no uniqueness for the Dirichlet problem: precisely, there exist a elliptic operator L of the form (1.1), defined in the unit ball $B_1 \subset \mathbb{R}^d$, $d \geq 3$, and a function $g \in C^2(\partial B_1)$ such that the Dirichlet problem $Lu = 0$ in B_1, $u = g$ on ∂B_1 has at least two "good solutions" u_1 and u_2 with $u_1(0) \neq u_2(0)$.

Later on a relationship between non weak uniqueness for elliptic equations and non existence of solutions in $W^{2,d}$ has been pointed out by C. Giannotti [10] in dealing with maximinimal operators, an extension of Pucci's extremal operators [16].

In the same context, recall the study of elliptic operators in $\mathbb{R}^d$ with coefficients depending on $(d-1)$ variables as in L. Escauriaza [9], O. Arena and P. Manselli [1] where $d = 3$, M. Cerutti, E.B. Fabes and P. Manselli [6]. Moreover, it is particularly worth mentioning the result due to F. Chiarenza,

M. Frasca and P. Longo [7]: the Dirichlet problem for elliptic operators with $L^\infty \cap VMO$ (vanishing mean oscillation) coefficients is uniquely solved in $W^{2,p}$.

At this stage a question must be cleared up: does the assumption (A) of our paper guarantee uniqueness of good solutions to Dirichlet problems of the form (1.4) ?

At any rate, further investigations are still worthy making to fully understand the theory of linear second order uniformly elliptic operators with merely measurable coefficients.

Acknowledgment. I have had the great opportunity to have been able to discuss about the questions of this paper, with Eugene Fabes, before his unexpected untimely passing away; I have fond memories of him.

2. A pointwise estimate.

In this section we will establish for good solutions of equation (1.3) a pointwise estimate, which gives a control for any second differential quotient. Precisely, we will prove the following theorem.

Theorem 2. *Let u be a bounded solution of $Lu = 0$ in the unit ball B_1. Assume that the assumption* (A) *holds, i.e. at Lebesgue points x_0, for r sufficiently small,*

$$\left(\fint_{B_r} |a_{ij}(x) - a_{ij}(x_0)|^d \, dx \right)^{1/d} \leq \omega(r)$$

with $\omega(r)$ increasing function, $\omega(0) = 0$ and such that $\int_0^1 s^{-1}\omega(s)\, ds < \infty$.

Then there exists a quadratic polynomial P such that:

$$\sup_{B_r} |u(x) - P(x)| \leq C\, r^2 \tilde{\eta}(r), \tag{2.1}$$

where the constant C depends on ν and $\|u\|_{L^\infty}$, and $\tilde{\eta} = \tilde{\eta}(r)$ is a function determined by $\omega(r)$ with $\tilde{\eta}(r) \to 0$ as $r \to 0$.

Remark 2.1. If u is a bounded solution of $Lu = f$, f continuous, the same conclusion holds provided that

$$\left(\fint_{B_r} |f(x) - f(x_0)|^d \, dx \right)^{1/d} \leq \varphi(r),$$

with $\varphi(r)$ increasing function, $\varphi(0) = 0$ and satisfying Dini condition $\int_0 \varphi(r) \frac{dr}{r} < \infty$. Moreover, one may assume $x_0 = 0$ and $f(0) = 0$.

Remark 2.2. The result of Theorem 2 is of the type of that due to L.A. Caffarelli [2], [3] for viscosity solutions. The assumption on the coefficients a_{ij} made in [2] is, here, relaxed to the assumption (A). The same also about the function f. In [2], actually, $\omega(r) = \varphi(r) = r^\alpha$ $(0 < \alpha < 1)$ and also $\tilde{\eta}(r) = r^\alpha$.

The proof of Theorem 2 is based essentially on an approximation argument for which next Lemma, due to L.A. Caffarelli [2] and below stated for reader's convenience, is crucial.

Lemma 2.1 (L.A. Caffarelli [2]). *Let u be a bounded solution of $Lu = f \in L^d$ in the unit ball B_1 with $\|u\|_{L^\infty} \leq 1$. Assume that:*

$$\int_{B_1} |a_{ij}(x) - \delta_{ij}|^d \, dx \leq \varepsilon^d.$$

Then there exists a harmonic function h such that:

$$\|u - h\|_{L^\infty(B_{1/2})} \leq C \left(\varepsilon^\gamma + \|f\|_{L^d(B_1)} \right)$$

for some positive γ, with a constant C depending on the ellipticity constant ν.

To go on now we need to make some considerations, starting from the assumptions (A) on the coefficients a_{ij}.

To begin with let us note that the Dini condition (1.6) satisfied by the function ω is equivalent to the convergence of the series $\sum_{k=0}^{\infty} \omega(2^{-k})$. Observe, then, there exists an increasing sequence $\{A_k\}$ such that:

$$\sum_{k=0}^{\infty} \omega(2^{-k}) A_k < \infty. \tag{2.2}$$

On the other hand, by rearranging the terms of the series (2.2), it turns out that there exists a decreasing sequence $\{a_k\}$ such that:

$$a_k \geq \omega(2^{-k}) A_k \quad \text{and} \quad \sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \omega(2^{-k}) A_k < \infty. \tag{2.3}$$

Hence, define for $t \in (0, 1]$:

$$\Phi(t) = A_k \quad \text{for} \quad 2^{-k-1} < t \leq 2^{-k} \tag{2.4}$$

and

$$\eta(t) = a_k \quad \text{for} \quad 2^{-k-1} < t \le 2^{-k}. \tag{2.5}$$

The functions Φ and η are respectively non-increasing and non-decreasing functions of t.

From (2.3), (2.4) and (2.5) it follows that:

$$\omega(t)\Phi(t) \le \eta(t), \quad t \in (0, 1]. \tag{2.6}$$

Moreover

$$\sum_{k=0}^{\infty} \eta(2^{-k}) < \infty. \tag{2.7}$$

Keeping in mind the above considerations, we are ready to get the pointwise estimate (2.1).

Proof of Theorem 2. First of all let us observe that the assumptions are invariant by dilation and notice that we may assume $\|u\|_{L^\infty} \le 1$.

To prove the theorem we start from the approximation result of Lemma 2.1 to construct a converging sequence of second order harmonic polynomials approximating the function u solution of $Lu = 0$.

To begin with, because of the assumption (A), for a sufficiently small value of r, we apply Lemma 2.1 to obtain a harmonic function h_1 and a positive γ

$$\|u - h_1\|_{L^\infty(B_{1/2})} \le C\omega^\gamma(r),$$

with a constant C depending on the ellipticity constant.

If $\bar{h}_1$ is the quadratic part of h_1 at the origin, then we have:

$$\|u - \bar{h}_1\|_{L^\infty(B_\lambda)} \le C\left(\omega^\gamma(r) + \lambda^3\right) \quad (\lambda \le 1/2).$$

Choose now a value of λ, call it λ_0, and a value of r, call it r_0, such that

$$\|u - \bar{h}_1\|_{L^\infty(B_{\lambda_0})} \le \lambda_0^2 \eta(\lambda_0) \tag{2.8}$$

where η is the function related to the function ω and introduced by (2.4).

We *claim* that there exists a sequence of second order harmonic polynomials

$$P^k(x) = a^k + b^k \cdot x + \frac{1}{2}x^t C^k x,$$

with b^k a d-dimensional vector and $C^k = (c^k_{ij})$ a dxd matrix, so that:

$$(\alpha) \qquad \|u - P^k\|_{L^\infty(B_{\lambda_0^k})} \leq \lambda_0^{2k} \max_{1\leq j\leq k} \eta(\lambda_0^j)\eta^{k-j}(\lambda_0)$$

$$(\beta) \qquad \sup_{i,j} |c^k_{ij}| \leq C\left(1 + \sum_{s=1}^{k-1} \eta(\lambda_0^s)\right)$$

for some suitable λ_0.

We will prove the claim by induction, starting with $P^1 \equiv \bar{h}_1$ for which (2.8) holds.

Assuming then the k-th step to be correct, let us show the validity of the $(1+k)$-th step.

To this end we consider the function

$$w_k = \frac{[u - P^k](\lambda_0^k x)}{\lambda_0^{2k} \max_{1\leq j\leq k} \eta(\lambda_0^j)\eta^{k-j}(\lambda_0)} \quad \text{for } x \in B_1.$$

Notice that $\|w_k\|_{L^\infty} \leq 1$. Moreover, it turns out that:

$$\left|L_{\lambda_0^k, r_0} w_k\right| \leq \frac{1}{\max_{1\leq j\leq k} \eta(\lambda_0^j)\eta^{k-j}(\lambda_0)} \sum_{i,j=1}^{d} |a_{ij}(\lambda_0^k r_0 x) - \delta_{ij}| \sup |c^k_{ij}|$$

where

$$L_{\lambda_0^k, r_0} = \sum_{i,j=1}^{d} a_{ij}(\lambda_0^k r_0 x) D_{ij}.$$

Hence:

$$(2.9) \qquad \|L_{\lambda_0^k, r_0} w_k\|_{L^d(B_1)} \leq \frac{\omega(\lambda_0^k r_0)}{\max_{1\leq j\leq k} \eta(\lambda_0^j)\eta^{k-j}(\lambda_0)} \sup_{i,j} |c^k_{ij}|.$$

Observe that, either $\max_{1\leq j\leq k} \eta(\lambda_0^j)\eta^{k-j}(\lambda_0) = \eta(\lambda_0^k)$ or it is not so, we have:

$$\frac{\omega(\lambda_0^k r_0)}{\max_{1\leq j\leq k} \eta(\lambda_0^j)\eta^{k-1}(\lambda_0)} \leq \frac{\omega(\lambda_0^k r_0)}{\eta(\lambda_0^k)}.$$

Therefore, since

$$\frac{\omega(\lambda_0^k r_0)}{\eta(\lambda_0^k)} \leq \frac{\omega(\lambda_0^k r_0)}{\eta(\lambda_0^k r_0)} \leq \frac{1}{\Phi(\lambda_0^k r_0)} \leq \frac{1}{\Phi(r_0)}$$

with Φ defined by (2.3), from (2.9) we get:

$$\|L_{\lambda_0^k, r_0} w_k\|_{L^d(B_1)} \leq \frac{1}{\Phi(r_0)} \sup_{i,j} |c_{ij}^k|. \tag{2.10}$$

We apply now Lemma 2.1 to w_k approximating it by a harmonic function h_k. If $\bar{h}_k$ denotes the quadratic part of h_k at the origin, using also (2.10), we obtain that:

$$\|w_k - \bar{h}_k\|_{L^\infty(B_{\lambda_0})} \leq C\left[\omega^\gamma(\lambda_0^k r_0) + \frac{1}{\Phi(r_0)} \sup_{i,j} |c_{ij}^k| + \lambda^3\right], \quad (\lambda_0 \leq 1/2).$$

Taking into account the induction assumption (β), now we choose λ_0 and r_0 so that

$$\|w_k - \bar{h}_k\|_{L^\infty(B_{\lambda_0})} \leq \lambda_0^2 \eta(\lambda_0).$$

Rescaling back we come up with

$$\left\| u - P^k - \lambda_0^{2k}\Big(\max_{1\leq j\leq k} \eta(\lambda_0^j)\eta^{k-j}(\lambda_0)\Big)\bar{h}_k(\lambda_0^{-k}x)\right\|_{L^\infty(B_{\lambda_0^{k+1}})} \leq \tag{2.11}$$

$$\leq \lambda_0^{2(k+1)} \max_{1\leq j\leq k+1} \eta(\lambda_0^j)\eta^{k+1-j}(\lambda_0)$$

and we set

$$P^{k+1} = P^k + \lambda_0^{2k}\Big(\max_{1\leq j\leq k+1} \eta(\lambda_0^j)\eta^{k-j}(\lambda_0)\Big)\bar{h}_k(\lambda_0^{-k}x). \tag{2.12}$$

Therefore the $(1+k)$-th step is clearly valid.

Moreover, it follows from (2.12) that:

$$\left.\begin{array}{r} |a^{k+1} - a^k| \\ \lambda_0^k |b^{k+1} - b^k| \\ \lambda_0^{2k} |C^{k+1} - C^k| \end{array}\right\} \leq C\lambda_0^{2k} \max_{1\leq j\leq k} \eta(\lambda_0^j)\eta^{k-j}(\lambda_0)\,. \tag{2.13}$$

To go through the convergence problem, let us now make the following remark. For fixed k, call j_k the value of j for which the maximum in (2.13) is

attained: $\max\limits_{1\le j\le k} \eta(\lambda_0^j)\eta^{k-j}(\lambda_0) = \eta(\lambda_0^{j_k})\eta^{k-j_k}(\lambda_0)$.
If $j_k \ge k/2$ then

$$\eta(\lambda_0^{j_k})\eta^{k-j_k}(\lambda_0) < \eta(\lambda_0^{k/2})$$

(recall $\eta < 1$) and $\sum\limits_k^\infty \eta(\lambda_0^{k/2}) = \sum\limits_k^\infty \eta(\tilde{\lambda}_0^k) < +\infty$.
If instead $j_k < k/2$ then

$$\eta(\lambda_0^{j_k})\eta^{k-j_k}(\lambda_0) < \eta^{k/2}(\lambda_0)$$

and $\sum\limits_k^\infty \eta^{k/2}(\lambda_0) = \sum\limits_k^\infty \tilde{\eta}^k < +\infty$, where $\tilde{\eta} = [\eta(\lambda_0)]^{1/2} < 1$.

Consequently, by (2.13) the sequence of polynomials P^k converges uniformly to a quadratic polynomial P.

Finally, if $r = \lambda_0^k$ then in B_r we have:

$$|u - P| \le |u - P^k| + \sum_{s=k+1}^{\infty} |P^{s+1} - P^s| \le$$

$$\le C\lambda_0^{2k} \max_{1\le j\le k} \eta(\lambda_0^j)\eta^{k-j}(\lambda_0) \le Cr^2\tilde{\eta}(r)$$

with $\tilde{\eta}(r) \to 0$ as $r \to 0$.

Observe that if

$$\max_{1\le j\le k} \eta(\lambda_0^j)\eta^{k-j}(\lambda_0) = \eta(\lambda_0^k)$$

then $\tilde{\eta}(r) = \eta(r)$. Otherwise $\tilde{\eta}(r) = [\eta(\lambda_0)]^{\frac{1}{2}\frac{\ln r}{\ln \lambda_0}}$.

The theorem is completely proven. $\square$

3. The proof of Theorem 1.

To prove the claimed convergence result for the second derivatives we will need the following proposition.

Lemma 3.1. *Let $A^n \equiv (\alpha_{ij}^n)_{i,j=1,2,\ldots,d}$ be a sequence of dxd symmetric matrices and let*

$$\langle A^n\theta, \theta\rangle = \sum \alpha_{ij}^n \theta_i\theta_j, \quad \theta \in \mathbb{R}^d,\ |\theta| = 1,$$

be the corresponding sequence of quadratic forms.
If

$$\lim_{n\to\infty} \langle A^n\theta, \theta\rangle = 0 \quad \text{for any } \theta \in \mathbb{R}^d,\ |\theta| = 1,$$

then

$$\lim_{n\to\infty} a_{ij}^n = 0 \quad \textit{for } i, j = 1, 2, \ldots, d.$$

Proof. Induction on the dimension d yields the result. □

In what follows the following remark is also needed.

Remark 3.1. Under the assumption (A) on the a_{ij}'s, the coefficients a_{ij}^n of the regularizing operators L^n satisfy too, for n sufficiently large, the assumption (A) at Lebesgue points $x_0 \in \Omega \setminus S$, with $|S| = 0$, with a possibly different increasing function ω such that $\omega(r) \to 0$ as $r \to 0$ and satisfying Dini's condition.

For, remember that if σ is any positive number, by Egorov theorem, there exists a set E_σ such that $|\Omega \setminus E_\sigma| < \sigma$ and the convergence of $a_{ij}^n(x)$ to $a_{ij}(x)$ is uniform on E_σ. Moreover the set of non density points of E_σ has measure zero.

Proof of Theorem 1. Without loss of generality we may prove the result at the point $x_0 = 0$.

We will show that the sequence $\{D_{ij}u^n(0)\}$ for $i, j = 1, 2, \ldots, d$ is a convergent sequence and its limit as $n \to \infty$ is $D_{ij}u(0)$.

To start with let us observe that, by Remark 3.1 and Theorem 2, the smooth solutions of $L^n u^n = f$ satisfy for $|x| \le r_0$ and n sufficiently large the estimate:

$$|u^n(x) - u^n(0) - \nabla u^n(0) \cdot x - \frac{1}{2} x^t D_{ij} u^n(0) x| \le C|x|^2 \tilde{\eta}(|x|), \tag{3.1}$$

with $\tilde{\eta}(|x|) \to 0$ as $|x| \to 0$.

Let $x = h\theta$, $h > 0$ and $\theta \in \mathbb{R}^d$, $|\theta| = 1$. We easily get the following estimate:

$$\begin{aligned}
&\frac{1}{2}\left|D_{ij}u^m(0)\theta_i\theta_j - D_{ij}u^n(0)\,\theta_i\theta_j\right| \le \\
&\le h^{-2}\left|\frac{1}{2}h^2 D_{ij}u^m(0)\,\theta_i\theta_j + \nabla u^m(0)\cdot h\theta + u^m(0) - u^m(h\theta)\right| + \\
&+ h^{-2}\left|\frac{1}{2}h^2 D_{ij}u^n(0)\,\theta_i\theta_j + \nabla u^n(0)\cdot h\theta + u^n(0) - u^n(h\theta)\right| + \\
&+ h^{-1}\left|\nabla u^m(0)\cdot\theta - \nabla u^n(0)\cdot\theta\right| + h^{-2}\left|u^m(h\theta) - u^n(h\theta)\right| + \\
&\qquad + h^{-2}\left|u^m(0) - u^n(0)\right|.
\end{aligned} \tag{3.2}$$

Letting $m, n \to \infty$, the last three terms of the right-hand side of (3.2) go to zero, since $\{u^n\}$ is uniformly convergent in $\overline{\Omega}$ and $\{\nabla u^n\}$ is convergent almost everywhere in Ω.

Consequently, owing to (3.1), from (3.2) we obtain that:

$$\overline{\lim_{m,n\to\infty}} |D_{ij}u^m(0)\,\theta_i\theta_j - D_{ij}u^n(0)\,\theta_i\theta_j| \le C\tilde{\eta}(h).$$

The arbitrariness of h and the vanishing of $\tilde{\eta}(h)$ as $h \to 0$ yield that:

$$|D_{ij}u^m(0)\,\theta_i\theta_j - D_{ij}u^n(0)\,\theta_i\theta_j| \to 0 \quad \text{as } m, n \to \infty.$$

Thus the sequence of quadratic forms

$$\langle H^n\theta, \theta\rangle \equiv \sum_{i,j=1}^{d} D_{ij}u^n(0)\,\theta_i\theta_j$$

is convergent as $n \to \infty$ to a quadratic form

$$\sum_{i,j=1}^{d} c_{ij}\ \theta_i\theta_j,$$

meaning that as $n \to \infty$

$$\sum_{i,j=1}^{d} \left\{D_{ij}u^n(0) - c_{ij}\right\}\theta_i\theta_j \to 0$$

for any $\theta \in \mathbb{R}^d$, $|\theta| = 1$.

Hence, by Lemma 3.1, the sequence $\{D_{ij}u^n(0)\}(i, j = 1, 2, \dots, d)$ is convergent as $n \to \infty$ and

$$\lim_{n\to\infty} D_{ij}u^n(0) = c_{ij} \quad (i, j = 1, 2, \dots, d).$$

Because of the twice differentiability of u^n and the existence of the second derivatives of u in the Nadirashivili's sense, since $u^n \to u$ uniformly in $\overline{\Omega}$ and $\nabla u^n \to \nabla u$ a.e. in Ω, we conclude that $c_{ij}(i, j = 1, 2, \dots, d)$ are nothing but $D_{ij}u(0)$. $\square$

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