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A REMARK ON THE REGULARITY OF SOLUTIONS OF NON LINEAR ELLIPTIC EQUATIONS

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We prove regularity results for solutions of non linear elliptic equations containing a term which grows as $|u|^\sigma$, $\sigma > 0$, and satisfies a "sign condition". The source term is supposed to be in L^1 or in an intermediate space between L^1 and L^p, $p > 1$.

1. Introduction.

Let us consider the model problem:

$$\text{(1.1)} \qquad \begin{cases} -(a_{ij}u_{x_j})_{x_i} + c_0|u|^{\sigma-1}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω be an open bounded set of $\mathbb{R}^n$, $n \geq 3$, $\sigma > 0$, c_0 is a nonnegative constant, $\{a_{ij}(x)\}$ is a bounded matrix which satisfies, for a.e. $x \in \Omega$ and any $\xi \in \mathbb{R}^n$,

$$a_{ij}(x)\xi_i\xi_j \geq |\xi|^2,$$

and f belongs to $L^1(\Omega)$. The main motivation of this paper is the study of the regularity of solutions of problems having the structure of problem (1.1). The

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existence of a solution in the sense of distributions of (1.1) has been proved under various assumptions (see, for example, [16], when $c_0 = 0$, [6], [7], when $c_0 > 0$). In particular a solution is proved to exist in the class:

$$\mathcal{T}(\Omega) = \{w \in W_0^{1,1} : T_k(w) \in W_0^{1,2}(\Omega),\ \forall k \geq 0\}, \tag{1.2}$$

where:

$$T_k(\eta) = \begin{cases} k\text{sign}(\eta) & \text{if } |\eta| \geq k \\ \eta & \text{if } |\eta| < k. \end{cases}$$

As regards the regularity of such a solution the results can be summarized as follows: if $c_0 = 0$ then a solution of problem (1.1) exists in $W_0^{1,q}(\Omega)$, for every $q < n/(n-1)$ (see [16]), if $c_0 > 0$ then a solution of problem (1.1) exists in $W_0^{1,q}(\Omega)$, for every $q < \tilde{q} = \max\{n/(n-1), 2\sigma/(\sigma+1)\}$ (see [6], [7]). In particular the results described above show that the term "$c_0|u|^{\sigma-1}u$" has a regularizing effect when $\sigma > n/(n-2)$. We remark that the regularity $u \in W_0^{1,n/(n-1)}(\Omega)$, if $c_0 = 0$, or $u \in W_0^{1,\tilde{q}}(\Omega)$, if $c_0 > 0$, cannot be reached in general under the assumption that $f \in L^1(\Omega)$. In [11] it is proved that when $c_0 = 0$ the "limit regularity" $u \in W_0^{1,n/(n-1)}(\Omega)$ can be obtained assuming that f is slightly more summable. To be precise, if $c_0 = 0$ and f belongs to the Lorentz space $L(1, n/(n-1))$, then there exists a solution u of problem (1.1) which belongs to $W_0^{1,n/(n-1)}(\Omega)$ (see also Remark 4.2).

An application of the results proved in the present paper allows us to handle the case $c_0 > 0$. If we restrict our attention to the case $\sigma \geq n/(n-2)$, which is the interesting one, we prove an a priori estimate in $W_0^{1,\bar{q}}(\Omega)$, with $\bar{q} = 2\sigma/(\sigma+1)$, for solutions of (1.1) in the class $\mathcal{T}(\Omega)$, under the assumption that f belongs to the space $L(\log L)^{\sigma/(\sigma+1)}$ (see the definition in Section 2). Such a result improves a similar result contained in [9], where the limit regularity $u \in W_0^{1,\bar{q}}(\Omega)$ is obtained under the stronger hypothesis $f \in L(\log L)$ (see Remark 4.1).

Our proof is based on the fact that, using symmetrization techniques (see, for example, [17], [2]), one obtains estimates of the following type for solutions of (1.1) in the class $\mathcal{T}(\Omega)$:

$$\begin{cases} \displaystyle\int_{t<|u|\leq t+h} |Du|^2\,dx \leq h \int_{|u|>t} |f|\,dx, \\ \displaystyle c_0 \int_{|u|>t} |u|^\sigma\,dx \leq \int_{|u|>t} |f|\,dx, \end{cases} \tag{1.3}$$

for every positive real numbers t and h. Thus, in Section 3 we address the more general problem of studing the regularity of a function u satisfying (1.3) when

f belongs to $L^1(\Omega)$ or to an intermediate space between $L^1(\Omega)$ and $L^p(\Omega)$, $\forall p > 1$. Such results are finally applied in Section 4 to a problem which has the structure of (1.1), obtaining, on one side, the known results when $c_0 = 0$, and, on the other side, the improvement described above when $c_0 > 0$. In a forthcoming paper [1] we will show how the methods used in this paper can be extended to the more general case where the differential operator in (1.1) is like the p-laplacian.

2. Preliminary results.

First of all we recall the definition of decreasing rearrangement of a measurable function $w : \Omega \to \mathbb{R}$. If one denotes by $|E|$ the Lebesgue measure of a set E, one can define the distribution function $\mu_w(t)$ of w as:

$$\mu_w(t) = |\{x \in \Omega : |w(x)| > t\}|, \quad t \geq 0.$$

The function μ_w is decreasing and right continuous, and its generalized inverse function is the decreasing rearrangement w^* of w:

$$w^*(s) = \sup\{t \geq 0 : \mu_w(t) > s\}, \quad s \in (0, |\Omega|). \tag{2.1}$$

We recall that w and w^* are equimeasurable, i.e.,

$$\mu_w(t) = \mu_{w^*}(t), \quad t \geq 0.$$

This implies that for any Borel function ψ it holds:

$$\int_\Omega \psi(|w(x)|)\, dx = \int_0^{|\Omega|} \psi(w^*(s))\, ds, \tag{2.2}$$

and, in particular,

$$\|w^*\|_{L^p(0,|\Omega|)} = \|w\|_{L^p(\Omega)}, \quad 1 \leq p \leq \infty. \tag{2.3}$$

A simple consequence of (2.3) is that one can bound the L^p-norm of a function w by the corresponding norm of a function v every time one knows that $w^*(s) \leq v^*(s)$. We recall that such a result still holds under the weaker assumption (see, e.g., [2])

$$\int_0^s w^*(r)dr \leq \int_0^s v^*(r)dr, \quad s \in (0, |\Omega|). \tag{2.4}$$

Proposition 2.1. *Suppose that $w, v \in L^1(\Omega)$ satisfy (2.4). Then, for every non decreasing convex C^1 function $F : [0, +\infty) \to [0, +\infty)$, we have:*

$$\int_0^{|\Omega|} F(w^*(r))dr \leq \int_0^{|\Omega|} F(v^*(r))dr. \tag{2.5}$$

The theory of rearrangements is well known and exhanstive treatments of it can be found for example in [14], [13], [17], [2].

We finally introduce the Zygmund spaces $L^p(\log L)^\alpha$, $0 < p < \infty$, $-\infty < \alpha < +\infty$. We say (see, for example, [5]) that w belongs to $L^p(\log L)^\alpha$ if

$$\int_\Omega [|f(x)| \log^\alpha(2 + |f(x)|)]^p \, dx < \infty. \tag{2.6}$$

We remark that (2.2) implies that (2.6) is equivalent to the following condition involving rearrangements

$$\int_0^{|\Omega|} [f^*(s) \log^\alpha(2 + f^*(s))]^p \, ds < \infty.$$

3. Main result.

In this section we will consider functions u in the class $\mathcal{T}(\Omega)$ introduced in (1.2). For such functions it is possible to define, for any positive h, the function

$$\phi_h(t) = \int_{t<|u|\leq t+h} |Du|^2 \, dx, \quad t \geq 0. \tag{3.1}$$

Suppose that there exists a nonnegative function $g \in L^1(\Omega)$ such that the following inequality holds:

$$\phi_h(t) \leq h \int_{|u|>t} g \, dx, \quad \forall t \geq 0, \ h > 0. \tag{3.2}$$

A standard argument (see for example [11]) proves that (3.2) implies an improved summability of $|Du|$ on Ω.

Proposition 3.1. *Suppose that u belongs to the class $\mathcal{T}(\Omega)$ and that g is a nonnegative function in $L^1(\Omega)$. If (3.2) holds then $u \in W_0^{1,q}(\Omega)$, for any $1 \le q < \dfrac{n}{n-1}$.*

If, furthermore, g is such that

$$\int_0^{|\Omega|} \left(\int_0^s g^*(t)\, dt \right)^{\frac{n}{n-1}} \frac{ds}{s} < +\infty, \tag{3.3}$$

then (3.2) implies that $u \in W_0^{1,n/(n-1)}(\Omega)$.

Proof. We observe that the above statement is already contained in several papers, where it has been proved in different contests (see, for example, [17], [18], [8], [11]). We only sketch the proof in the case where g belongs to $L^1(\Omega)$.

Using Hölder's inequality in (3.2) we have, for $1 \le q < 2$,

$$\frac{1}{h} \int_{t<|u|\le t+h} |Du|^q \, dx \le \|g\|_1^{q/2} \left(\frac{\mu_u(t) - \mu_u(t+h)}{h} \right)^{1-q/2}. \tag{3.4}$$

For $q = 1$ we have:

$$\frac{1}{h} \int_{t<|u|\le t+h} |Du| \, dx \le \|g\|_1^{1/2} \left(\frac{\mu_u(t) - \mu_u(t+h)}{h} \right)^{1/2}.$$

Passing to the limit as $h \to 0^+$ and using Fleming-Rishel formula [12] and isoperimetric inequality [10], in a standard way (see, e.g., [17]) we get:

$$1 \le \|g\|_1 \frac{-\mu_u'(t)}{n^2 C_n^{2/n} \mu_u(t)^{2-2/n}}, \qquad \text{for a.e. } t > 0,$$

where C_n is the measure of the unit ball in $\mathbb{R}^n$. Integrating between 0 and $u^*(s)$ and using the properties of rearrangements we obtain:

$$u^*(s) \le \frac{\|g\|_1}{n(n-2)C_n^{2/n}} \frac{1}{s^{1-2/n}}, \qquad s \in (0, |\Omega|),$$

which gives:

$$\|u\|_r \le \frac{\|g\|_1}{n^{1-1/r}(n-2)(n-(n-2)r)^{1/r} C_n^{2/n}} |\Omega|^{\frac{1}{r} - \frac{n-2}{n}}, \tag{3.5}$$

for any $0 < r < n/(n-2)$.

Going back to (3.4) and passing to the limit as $h \to 0^+$ we have:

$$\frac{d}{dt}\int_{|u|\le t} |Du|^q\,dx \le \|g\|_1^{q/2}(-\mu_u'(t))^{1-q/2}, \quad \text{for a.e. } t > 0,$$

where $1 \le q < n/(n-1)$. Integrating the above inequality between 0 and $+\infty$ and using Hölder's inequality, it results:

$$(3.6)\quad \int_\Omega |Du|^q\,dx \le \|g\|_1^{q/2}\int_0^\infty (-\mu_u'(t))^{1-q/2}\,dt \le$$

$$\le \|g\|_1^{q/2}\|2+|u|\|_{2\alpha/(2-q)}^{\alpha}\left(\int_0^\infty \frac{1}{(2+t)^{2\alpha/q}}\,dt\right)^{q/2},$$

where α is chosen in such a way that $0 < 2\alpha/q < 1$ and $0 < 2\alpha/(2-q) < n/(n-2)$. Then (3.5) and (3.6) imply that $u \in W_0^{1,q}(\Omega)$, for any $1 \le q < \dfrac{n}{n-1}$. $\square$

Remark 3.1. The assumption (3.3) on g is equivalent to say that it belongs to the Lorentz space $L(1, n/(n-1))$ (see [4]). We point out that the following strict inclusions hold:

$$L^1 \supset L(1, n/(n-1)) \supset L^p, \quad \forall p > 1.$$

It should be clear that if one assumes more summability on g (for example $g \in L^p(\Omega)$, $p > 1$) then Du will be more regular than in Proposition 3.1. But here we are interested in showing how it is possible to get more regularity on Du by adding suitable assumptions on u, letting g belong only to an intermediate space between $L^1(\Omega)$ and $L^p(\Omega)$, with $p > 1$. More precisely we will assume that for $\sigma > 1$ the following inequality holds:

$$(3.7)\quad \int_{|u|>t} |u|^\sigma\,dx \le \int_{|u|>t} g\,dx, \quad \forall t \ge 0,$$

where g is a nonnegative function as in (3.2). We begin to show how (3.7) allows us to transfer any information on the summability of g to the summability of u.

Proposition 3.2. *Suppose $u \in L^\sigma(\Omega), \sigma \geq 1$, satisfies (3.7) under the assumption that $g \in L(\log L)^\alpha$, $\alpha \geq 0$. Then $u \in L^\sigma(\log L)^{\alpha/\sigma}$.*

Proof. First of all we show that inequality (3.7) implies:

$$U(r) \leq V(r), \quad r \in (0, |\Omega|), \tag{3.8}$$

where

$$U(r) = \int_0^r (u^*(s))^\sigma\, ds \text{ and } V(r) = \int_0^r g^*(s)\, ds. \tag{3.9}$$

Indeed, inequality (3.7) implies (3.8) for any $r \in (0, |\Omega|)$ such that $r = \mu_u(t)$, for some $t \geq 0$. Let now r_0 be a value which is not in the range of μ_u. Because of the monotonicity and the right continuity of μ_u, there exists t_0 such that

$$r_1 \equiv \mu_u(t_0) < r_0 \leq \mu_u(t_0^-) \equiv r_2,$$

and

$$u^*(r) = t_0, \quad s \in [r_1, r_2].$$

On the other hand the functions U and V defined in (3.9) are concave in $(0, |\Omega|)$ and furthermore U is linear in the interval $[r_1, r_2]$. Thus, once we observe that $U(r_1) \leq V(r_1)$ and $U(r_2) \leq V(r_2)$, we immediately have that $U(r_0) \leq V(r_0)$. So we have proved completely (3.8).

A straightforward calculation shows that the function

$$\psi(t) = t \log^\alpha(2 + t)$$

is a positive convex function on $(0, +\infty)$, for $\alpha \geq 0$. Using (3.8), Proposition 2.1 and the definition of $L^\sigma(\log L)^{\alpha/\sigma}$ we complete the proof. □

We are now in position to prove our main result.

Theorem 3.3. *Suppose that $u \in \mathcal{T}(\Omega) \cap L^\sigma(\Omega)$, $\sigma \geq n/(n-2)$, and that g is a nonnegative function in $L^1(\Omega)$ such that (3.2) and (3.7) hold. Then $u \in W_0^{1,q}(\Omega)$, for any $1 \leq q < \bar{q}$, where $\bar{q} = 2\sigma/(\sigma + 1)$.*

If, furthermore, $g \in L(\log L)^{\sigma/(\sigma+1)}$ then $u \in W_0^{1,\bar{q}}(\Omega)$.

Proof. The proof of the theorem will be obtained by using both the results of Propositions 3.1 and 3.2 and the arguments contained in their proofs. We will first consider the case $g \in L(\log L)^{\sigma/(\sigma+1)}$. Using assumption (3.2) we have:

$$\int_{t<|u|\le t+h} \frac{|Du|^2}{\psi(|u|)}\,dx \le \frac{1}{\psi(t)}\int_{t<|u|\le t+h} |Du|^2\,dx \le \le \frac{h}{\psi(t)}\int_{|u|>t} g\,dx, \tag{3.10}$$

where we have set $\psi(t) = (2+t)(\log(2+t))^{1/(\sigma+1)}$. Then Hölder's inequality and (3.10) imply:

$$\frac{1}{h}\int_{t<|u|\le t+h} |Du|^{\bar{q}}\,dx \le$$

$$\le \frac{1}{h}\left(\int_{t<|u|\le t+h} \frac{|Du|^2}{\psi(|u|)}\,dx\right)^{\frac{\sigma}{\sigma+1}}\left(\int_{t<|u|\le t+h} (\psi(|u|))^{\sigma}\,dx\right)^{\frac{1}{\sigma+1}} \le$$

$$\le \left(\frac{1}{\psi(t)}\int_{|u|>t} g\,dx\right)^{\frac{\sigma}{\sigma+1}}\left(\frac{1}{h}\int_{t<|u|\le t+h} (\psi(|u|))^{\sigma}\,dx\right)^{\frac{1}{\sigma+1}}.$$

As in the proof of Proposition 3.1 we can pass to the limit as $h \to 0^+$, obtaining,

$$\frac{d}{dt}\int_{|u|\le t} |Du|^{\bar{q}}\,dx \le \left(\frac{1}{\psi(t)}\int_0^{\mu_u(t)} g^*(s)\,ds\right)^{\frac{\sigma}{\sigma+1}} ((-\mu_u'(t))(\psi(t))^{\sigma})^{\frac{1}{\sigma+1}},$$

for a.e. $t > 0$. Integrating the above inequality between 0 and $+\infty$ and using Hölder's inequality it results:

$$\int_\Omega |Du|^{\bar{q}}\,dx \le \left(\int_0^{+\infty}\left(\int_0^{\mu_u(t)} g^*(s)\,ds\right)\frac{dt}{\psi(t)}\right)^{\frac{\sigma}{\sigma+1}} \cdot \left(\int_0^{+\infty}(-\mu_u'(t))(\psi(t))^{\sigma}\,dt\right)^{\frac{1}{\sigma+1}} \le \tag{3.11}$$

$$\le \left(\int_0^{+\infty}\left(\int_0^{\mu_u(t)} g^*(s)\,ds\right)\frac{dt}{\psi(t)}\right)^{\frac{\sigma}{\sigma+1}} \cdot \left(\int_\Omega (2+|u|)^{\sigma}(\log(2+|u|))^{\frac{\sigma}{\sigma+1}}\,dx\right)^{\frac{1}{\sigma+1}}.$$

We claim that

$$(3.12)\quad \int_0^{+\infty}\left(\int_0^{\mu_u(t)} g^*(s)\,ds\right)\frac{dt}{\psi(t)} = \\ = \int_0^{+\infty}\frac{1}{(2+t)(\log(2+t))^{1/(\sigma+1)}}\left(\int_0^{\mu_u(t)} g^*(s)\,ds\right)dt < +\infty.$$

We firstly prove that

$$(3.13)\qquad \int_0^{|\Omega|} g^*(s)(\log(2+u^*(s)))^{\sigma/(\sigma+1)}\,ds < +\infty.$$

A simple proof of (3.13) is based on the fact that, as in Proposition 3.2, assumption (3.7) implies that (3.8) holds true. Indeed, using the inequalities

$$(2+a)^\beta \le 3^{\beta-1}(2+a^\beta),\quad a\ge 0,\ \beta\ge 1,$$

and

$$(a+b)^\gamma \le a^\gamma + b^\gamma,\quad a,b\ge 0,\ 0\le\gamma\le 1,$$

we have:

$$(3.14)\quad \int_0^{|\Omega|} g^*(s)(\log(2+u^*(s)))^{\frac{\sigma}{\sigma+1}}\,ds \le \\ \le \frac{1}{\sigma^{\frac{\sigma}{\sigma+1}}}\int_0^{|\Omega|} g^*(s)((\sigma-1)\log 3+\log(2+(u^*(s))^\sigma))^{\frac{\sigma}{\sigma+1}}\,ds \le \\ \le c_1\|g\|_1 + c_2\int_0^{|\Omega|} g^*(s)(\log(2+(u^\sigma)^{**}(s)))^{\frac{\sigma}{\sigma+1}}\,ds,$$

where c_1, c_2 are constants which depend only on σ and, as usual, we have put:

$$(u^\sigma)^{**}(s) = \frac{1}{s}\int_0^s (u^\sigma)^*(r)dr \ge (u^*(s))^\sigma.$$

Now, using (3.8) in (3.14) we get:

$$(3.15)\qquad \int_0^{|\Omega|} g^*(s)(\log(2+u^*(s)))^{\frac{\sigma}{\sigma+1}}\,ds \le$$

$$\le c_1\|g\|_1 + c_2\int_0^{|\Omega|} g^*(s)(\log(2+\|g\|_1/s))^{\frac{\sigma}{\sigma+1}}\,ds = c_1\|g\|_1 +$$

$$+ c_2\left[\int_{\{s:g^*(s)>(\|g\|_1/s)^{1/2}\}} (\ldots)\,ds + \int_{\{s:g^*(s)\le(\|g\|_1/s)^{1/2}\}} (\ldots)\,ds\right] \le$$

$$\le c_1\|g\|_1 + 2^{\frac{\sigma}{\sigma+1}} c_2\int_0^{|\Omega|} g^*(s)(\log(2+g^*(s)))^{\frac{\sigma}{\sigma+1}}\,ds + c_3,$$

where c_3 is a constant which depends only on σ, $\|g\|_1$ and $|\Omega|$. Then (3.13) is proved.

Secondly we observe that (3.13) immediately implies (3.12) because the continuity of the function $(\psi(t))^{-1}$ and (3.13) allow us (see, for example, [15]) to perform an integration by parts:

$$\int_0^{+\infty}\left(\int_0^{\mu_u(t)} g^*(s)\,ds\right)\frac{dt}{\psi(t)} \le$$

$$\le \frac{\sigma+1}{\sigma}\int_0^{+\infty} g^*(\mu_u(t))(\log(2+t))^{\frac{\sigma}{\sigma+1}}(-d\mu_u(t)) =$$

$$= \frac{\sigma+1}{\sigma}\int_0^{|\Omega|} g^*(s)(\log(2+u^*(s)))^{\frac{\sigma}{\sigma+1}}\,ds.$$

Now, collecting (3.11), (3.12) and taking into account Proposition 3.2 we get the assertion.

The case $g \in L^1(\Omega)$ can be treated in a similar way. One has to put $\psi(t) = 1+t$ in (3.10) and then proceed as above (see also [8]). □

4. An application.

Let us consider the problem

$$(4.1)\qquad \begin{cases} -\mathrm{div}(a(x,Du)) + b(x,u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We assume that $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function which satisfy, for a.e. $x \in \Omega$ and any $\xi \in \mathbb{R}^n$,

$$|a(x,\xi)| \leq L(k(x) + |\xi|), \tag{4.2}$$

$$a(x,\xi)\xi \geq |\xi|^2, \tag{4.3}$$

where L is a positive constant and $k(x) \geq 0$ belongs to $L^2(\Omega)$. Furthermore $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying, for some $\sigma \geq 1$, the assumption:

$$b(x,\eta)\mathrm{sign}(\eta) \geq c_0|\eta|^\sigma, \quad \text{for a.e. } x \in \Omega, \ \forall \eta \in \mathbb{R}, \tag{4.4}$$

where c_0 is a positive constant.

We will say that u is a weak solution of (4.1) if it satisfies:

$$\begin{cases} \displaystyle\int_\Omega a(x,Du)D\varphi + \int_\Omega b(x,u)\varphi = \langle f, \varphi\rangle \quad \forall \varphi \in C_0^\infty(\Omega) \\ u \in W_0^{1,1}(\Omega), a(x,Du) \in L^1(\Omega), b(x,u) \in L^1(\Omega). \end{cases} \tag{4.5}$$

Under the further assumption that for a.e. $x \in \Omega$

$$(a(x,\xi) - a(x,\xi'))(\xi - \xi') > 0, \quad \forall \xi \neq \xi', \tag{4.6}$$

and

$$\sup\{|b(x,\eta)| : |\eta| \leq t\} \in L^1_{\mathrm{loc}}(\Omega), \quad \forall t \geq 0, \tag{4.7}$$

it has been proved that, if $f \in L^1(\Omega)$, then at least one weak solution u of (4.1) exists, such that $u \in W_0^{1,q}(\Omega)$ for any $q < \bar{q}$, $\bar{q} = 2\sigma/(\sigma+1)$ (see, for example, [8], [9]). It turns out that such a solution belongs to the class $\mathcal{T}(\Omega)$ defined in (1.2). As an application of the results contained in the previous section we will study the regularity of u when the summability of f is improved.

Theorem 4.1. *Suppose that $u \in \mathcal{T}(\Omega)$ is a weak solution of* (4.1), *that is, u satisfies* (4.5), *under the assumptions* (4.2), (4.3), (4.4), *with $\sigma \geq 2n/(n-2)$. We have:*

i) if $f \in L^1(\Omega)$, then $u \in W_0^{1,q}(\Omega)$, for any $q < \bar{q}$, with $\bar{q} = 2\sigma/(\sigma+1)$;
ii) if $f \in L(\log L)^{\sigma/(\sigma+1)}$, then $u \in W_0^{1,\bar{q}}(\Omega)$, where $\bar{q} = 2\sigma/(\sigma+1)$.

Proof. The proof is based on the fact that if $u \in \mathcal{T}(\Omega)$ satisfies (4.5), then one can use $T_k(u)$ as test function in (4.5), for any $k > 0$. Choosing $k = t$ and $k = t + h$, where $t, h > 0$, we get:

$$\int_{t<|u|\leq t+h} a(x, Du)Du\, dx + \int_{t<|u|\leq t+h} b(x, u)(|u| - t)\mathrm{sign}(u)\, dx +$$

$$+h\int_{|u|>t+h} b(x, u)\mathrm{sign}(u)\, dx =$$

$$= \int_{t<|u|\leq t+h} f(|u| - t)\mathrm{sign}(u)\, dx + h\int_{|u|>t+h} f\mathrm{sign}(u)\, dx.$$

Using assumptions (4.3) and (4.4) we have:

$$\int_{t<|u|\leq t+h} |Du|^2\, dx + hc_0\int_{|u|>t+h} |u|^\sigma\, dx \leq h\int_{|u|>t} |f|\, dx. \tag{4.8}$$

Inequality (4.8) implies:

$$\int_{t<|u|\leq t+h} |Du|^2\, dx \leq h\int_{|u|>t} |f|\, dx \tag{4.9}$$

and

$$c_0\int_{|u|>t+h} |u|^\sigma\, dx \leq \int_{|u|>t} |f|\, dx. \tag{4.10}$$

It is immediate to observe that (4.9) is exactly (3.2) and that letting h go to zero, (4.10) implies (3.7). Thus one obtains the theorem as an application of Theorem 3.3. □

Remark 4.1. As already observed, part i) of Theorem 4.1 recovers well known results contained for example in [8]. Part ii) of Theorem 4.1 improves a result contained in [9], where the existence of a weak solution of (4.1) belonging to $W_0^{1,\bar{q}}(\Omega)$ is proved under the assumptions (4.2), (4.3), (4.4), (4.6), (4.7) and $f \in L(\log L)$. As already observed in Section 2 the space $L(\log L)$ is strictly contained in $L(\log L)^{\frac{\sigma}{\sigma+1}}$.

Remark 4.2. By the same arguments used in Theorem 4.1 one can study the case when $b(x, \eta) \equiv 0$. This time one obtains only inequality (4.9) and then one can apply Proposition 3.1. It follows that if $f \in L^1(\Omega)$ then $u \in W_0^{1,q}(\Omega)$, for any $q < n/(n-1)$ (see also [6], [7]), while, if f satisfies (3.3) then $u \in W_0^{1,n/(n-1)}(\Omega)$ (see also [11]).

Remark 4.3. The previous remark explains why we assume $\sigma \geq n/(n-2)$. Indeed, if $1 \leq \sigma < n/(n-2)$, then $\bar{q} = 2\sigma/(\sigma+1) < n/(n-1)$. Therefore in such a case the presence of the term $b(x,u)$ in the equation does not give any improvement on the summability of Du. We point out that when $\sigma = n/(n-2)$ then $\bar{q} = n/(n-1)$ and the regularity $u \in W_0^{1,n/(n-1)}(\Omega)$ can be obtained under one of the following assumptions:

a) f belongs to $L(\log L)^{\frac{n}{2(n-1)}}$;
b) f satisfies (3.3).

A straightforward calculation proves that if $n \geq 4$ then condition a) is weaker than b).

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