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LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS INVOLVING THE GENERALIZED GAMMA FUNCTION

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By a simple approach, two classes of functions involving generalization Euler's gamma function and originating from certain problems are proved to be logarithmically completely monotonic and a class of functions involving the psi function is showed to be completely monotonic.

1. Introduction and Preliminaries

Euler's classical gamma function, defined for positive x by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt,$$

The logarithmic derivative of the gamma function is called the digamma function. It is know as the psi function and is denoted by $\psi(x)$.

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

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The following integral and series representations are valid (see [1]):

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} + \sum_{n\geq 1} \frac{x}{n(n+x)} \tag{1}$$

It's derivatives are given by

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-xt}}{1 - e^{-t}} dt \tag{2}$$

for $x > 0$ and $n \in \mathbb{N}$, where $\gamma = 0.57721566490153286...$ is the Euler-Mascheroni constant.

Euler, gave another equivalent definition for the $\Gamma(x)$ (see [7],[8],[13]),

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)\cdots(x+p)} = \frac{p^x}{x(1+\frac{x}{1})\cdots(1+\frac{x}{p})}, \quad x > 0, \tag{3}$$

where

$$\Gamma(x) = \lim_{p\to\infty} \Gamma_p(x). \tag{4}$$

The p-analogue of the psi function is defined as the logarithmic derivative of the Γ_p function (see [7]), that is

$$\psi_p(x) = \frac{d}{dx} \ln \Gamma_p(x) = \frac{\Gamma'_p(x)}{\Gamma_p(x)}. \tag{5}$$

The function ψ_p defined in (5) satisfies the following properties (see [7]). It has the following series representation

$$\psi_p(x) = \ln p - \sum_{k=0}^{p} \frac{1}{x+k}. \tag{6}$$

It is increasing on $(0,\infty)$ and it is strictly completely monotonic on $(0,\infty)$. It's derivatives are given by

$$\psi_p^{(n)}(x) = \sum_{k=0}^{p} \frac{(-1)^{n-1} \cdot n!}{(x+k)^{n+1}}. \tag{7}$$

Definition 1.1. A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0 \tag{8}$$

for $x \in I$ and $n \geq 0$.

Let C denote the set of completely monotonic functions.

Definition 1.2. A positive function f is said to be logarithmicaly completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^n[\ln f(x)]^{(n)} \geq 0 \tag{9}$$

for $x \in I$, and $n \geq 0$.

Let L on $(0,\infty)$ stand for the set of logarithmically completely monotonic functions. The notion "logarithmically completely monotonic function" was posed explicity in [10] and published formally in [9] and a much useful and meaningful relation $L \subset C$ between the completely monotonic functions and the logarithmically completely monotonic functions was proved in [9, 10].

Kershaw (see [5]) prove that for positive x and $0 \leq s \leq 1$,

$$\exp\left((1-s)\psi(x+s^{\frac{1}{2}})\right) \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq \exp\left((1-s)\psi\left(x+\frac{1+s}{2}\right)\right).$$

2. Main Results

Chen [3] proved that the function

$$x \longmapsto \frac{1}{[\Gamma(x+1)]^{\frac{1}{x}}}$$

is logarithmically completely monotonic in $(0,\infty)$. The following theorem is a generalized result of Chen [3].

Theorem 2.1. *The function*

$$f(x) = \frac{1}{[\Gamma_p(x+1)]^{\frac{1}{x}}} \tag{10}$$

is logarithmically completely monotonic in $(0,\infty)$.

Proof. Using Leibnitz rule

$$[u(x)v(x)]^{(n)} = \sum_{k=0}^{n}\binom{n}{k}u^{(k)}(x)v^{(n-k)}(x),$$

we obtain

$$\begin{aligned}[\ln f(x)]^{(n)} &= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{x}\right)^{(k)}\Big(-\ln\Gamma_p(x+1)\Big)^{(n-k)}\\ &= -\frac{1}{x^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k k! x^{n-k}\psi_p^{(n-k-1)}(x+1)\\ &\triangleq -\frac{1}{x^{n+1}}g(x)\end{aligned}$$

$$\begin{aligned}g^{'}(x) &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k k!(n-k)x^{n-k-1}\psi_p^{(n-k-1)}(x+1)+\\ &+\sum_{k=0}^{n}\binom{n}{k}(-1)^k k! x^{n-k}\psi_p^{(n-k)}(x+1)\\ &=\sum_{k=0}^{n-1}\binom{n}{k}(-1)^k k!(n-k)x^{n-k-1}\psi_p^{(n-k-1)}(x+1)+\\ &+x^n\psi_p^{(n)}(x+1)+\sum_{k=0}^{n}\binom{n}{k}(-1)^k k! x^{n-k}\psi_p^{(n-k)}(x+1)\\ &=\sum_{k=0}^{n-1}\binom{n}{k}(-1)^k k!(n-k)x^{n-k-1}\psi_p^{(n-k-1)}(x+1)+\\ &+x^n\psi_p^{(n)}(x+1)+\sum_{k=0}^{n-1}\binom{n}{k+1}(-1)^{k+1}(k+1)! x^{n-k-1}\psi_p^{(n-k-1)}(x+1)\\ &=\sum_{k=0}^{n-1}\left[\binom{n}{k}(n-k)-\binom{n}{k+1}(k+1)\right](-1)^k k! x^{n-k-1}\psi_p^{(n-k-1)}(x+1)\\ &+x^n\psi_p^{(n)}(x+1)=x^n\psi_p^{(n)}(x+1)\\ &=x^n(-1)^n\sum_{k=1}^{p+1}\frac{n!}{(x+k)^{n+1}}\end{aligned}$$

If n is odd, then for $x>0$,

$$g^{'}(x)>0\Rightarrow g(x)<g(0)=0\ \Rightarrow\ (\ln f(x))^{(n)}<0\ \Rightarrow\ \Rightarrow\ (-1)^n(\ln f(x))^{(n)}>0.$$

If n is even, then for $x>0$,

$$g^{'}(x)<0\ \Rightarrow\ g(x)<g(0)=0\ \Rightarrow\ (\ln f(x))^{(n)}>0\ \Rightarrow\ \Rightarrow\ (-1)^n(\ln f(x))^{(n)}>0.$$

Hence,

$$(-1)^n(\ln f(x))^{(n)} > 0$$

for all real $x \in (0,\infty)$ and all integers $n \geq 1$. The proof is completed. □

For the next result we will make use of the following Lemma.

Lemma 2.2. *Let f'' be completely monotonic on $(0,\infty)$, then for $0 \leq s \leq 1$, the functions*

$$x \mapsto \exp\Big(-f(x+1)-f(x+s)-(1-s)f'\Big(x+\frac{1+s}{2}\Big)\Big)$$

$$x \mapsto \exp\Big(f(x+1)-f(x+s)-\frac{1-s}{2}f'(x+1)+f'(x+s)\Big)$$

are logarithmically completely monotonic on $(0,\infty)$.

Proof. See [2]. □

The following Theorems are Γ_p analogues of the results from [2].

Theorem 2.3. *For $0 \leq s \leq 1$, the functions*

$$x \mapsto \frac{\Gamma_p(x+s)}{\Gamma_p(x+1)}\exp\Big((1-s)\psi_p\Big(x+\frac{1+s}{2}\Big)\Big)$$

and

$$x \mapsto \frac{\Gamma_p(x+1)}{\Gamma_p(x+s)}\exp\Big(-\frac{1-s}{2}\Big(\psi_p(x+1)+\psi_p(x+s)\Big)\Big)$$

are logarithmically completely monotonic.

Proof. Applying Lemma 2.2 to $f(x) = \log\Gamma_p(x)$, and using the fact that $f''(x) = \psi_p'(x)$ is completely monotonic on $(0,\infty)$ (see [7]), one obtains the proof. □

Theorem 2.4. *For positive x and $0 \leq s \leq 1$, then*

$$\exp\Big(\frac{1-s}{2}\Big(\psi_p(x+1)+\psi_p(x+s)\Big)\Big) \leq \frac{\Gamma_p(x+1)}{\Gamma_p(x+s)} \leq$$
$$\leq \exp\Big((1-s)\psi_p\Big(x+\frac{1+s}{2}\Big)\Big).$$

Proof. Let $f_p(x)=\frac{\Gamma_p(x+s)}{\Gamma_p(x+1)}\exp\Big((1-s)\psi_p\Big(x+\frac{1+s}{2}\Big)\Big)$ and

$$g_p(x)=\frac{\Gamma_p(x+1)}{\Gamma_p(x+s)}\exp\Big(-\frac{1-s}{2}\Big(\psi_p(x+1)+\psi_p(x+s)\Big)\Big).$$

Since,

$$\lim_{x\to\infty} f_p(x)=\lim_{x\to\infty} g_p(x)=1$$

and $f_p(x),g_p(x)$ are decreasing from theorem 2.3 we have

$$\exp\Big(\frac{1-s}{2}\Big(\psi_p(x+1)+\psi_p(x+s)\Big)\Big)\leq\frac{\Gamma_p(x+1)}{\Gamma_p(x+s)}\leq$$
$$\leq\exp\Big((1-s)\psi_p\Big(x+\frac{1+s}{2}\Big)\Big)$$

for $0\leq s\leq 1$. The proof is completed. $\square$

Let s and t be two real numbers with $s\neq t,\alpha=\min\{s,t\}$ and $\beta\geq-\alpha$, for $x\in(-\alpha,\alpha)$, define

$$h_{\beta,p}(x)=\begin{cases}\Big[\frac{\Gamma_p(\beta+t)}{\Gamma_p(\beta+s)}\cdot\frac{\Gamma_p(x+s)}{\Gamma_p(x+t)}\Big]^{\frac{1}{x-\beta}} & x\neq\beta\\ \exp[\psi_p(\beta+s)-\psi_p(\beta+t)] & x=\beta\end{cases}$$

for $p>0$.

The following theorem is a generalization of a result of [12].

Theorem 2.5. *The function $h_{\beta,p}(x)$ is logarithmically completely monotonic on $(-\alpha,+\infty)$ if $s>t$.*

Proof. For $x\neq\beta$, taking logarithm of the function $h_{\beta,p}(x)$ we have

$$\begin{aligned}
\ln h_{\beta,p}(x) &= \frac{1}{x-\beta}\Big[\ln\frac{\Gamma_p(\beta+t)}{\Gamma_p(\beta+s)} - \ln\frac{\Gamma_p(x+s)}{\Gamma_p(x+t)}\Big] \\
&= \frac{\ln\Gamma_p(x+s)-\ln\Gamma_p(\beta+s)}{x-\beta} - \frac{\ln\Gamma_p(x+t)-\ln\Gamma_p(\beta+t)}{x-\beta} \\
&= \frac{1}{x-\beta}\int_\beta^x \psi_p(u+s)du - \frac{1}{x-\beta}\int_\beta^x \psi_p(u+t)du \\
&= \frac{1}{x-\beta}\int_\beta^x [\psi_p(u+s)-\psi_p(u+t)]du \\
&= \frac{1}{x-\beta}\int_\beta^x\int_t^s \psi_p^{'}(u+v)dvdu \\
&= \frac{1}{x-\beta}\int_\beta^x \varphi_{p,s,t}(u)du \\
&= \int_0^1 \varphi_{p,s,t}((x-\beta)u+\beta)du,
\end{aligned}$$

and by differentiating $\ln h_{\beta,p}(x)$ with respect to x,

$$[\ln h_{\beta,p}(x)]^{(k)} = \int_0^1 u^k \varphi_{p,s,t}^{(k)}((x-\beta)u+\beta)du \tag{11}$$

If $x=\beta$ formula (11) is valid.

Since functions $\psi_p^{'}$ and $\varphi_{p,s,t}$ are completely monotonic in $(0,\infty)$ and $(-t,\infty)$ respectively, then $(-1)^i[\varphi_{p,s,t}(x)]^{(i)} \geq 0$ holds for $n\in(-t,\infty)$ for any nonnegative integer i.

Thus

$$(-1)^k[\ln h_{\beta,p}(x)]^{(k)} = \int_0^1 u^k(-1)^k\varphi_{p,s,t}^{(k)}((x-\beta)u+\beta)du \geq 0$$

in $(-t,\infty)$ for $k\in\mathbb{N}$. The proof is completed. □

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