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ON THE DEPENDENCY LOCI OF SECTIONS OF SEMIAMPLE VECTOR BUNDLES ON PROJECTIVE VARIETIES

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To the memory of Umberto Gasapina

Here we study the topology of the dependency locus of sections of semi-ample vector bundles on complex projective varieties. The main motivation came from the study of subvarieties of the Grassmannians.

0. Introduction.

In this paper we are interested in the topology of the dependency loci of sections of vector bundles which are rather "positive" i.e. in Lefschetz type results for vector bundles. The results we use are rather standard (see Lemma 1.4 i.e. essentially [8] Theorem 1.1 of Part II, p. 152), but their use has for us at least two different motivations. The first motivation is the study of the connectedness of the 0-locus of a section or of the dependency locus of a vector space of sections of a spanned (but not ample) vector bundle i.e. the study of the 0-loci (or dependency loci) of sections of the universal quotient bundle of subvarieties of Grassmannians; see Corollaries 1.8 and 1.10 for the first interesting case: therefolds contained in the Grassmannian $G(2, 5)$ of lines of $\mathbf{P}^4$. The second motivation came from the study of "positive" vector bundles with a section with

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as 0-locus a very particular variety (e.g. a projective space or a quadric or a quadric or a scroll over a curve of genus > 0) (see 2.3, 2.4, 2.6 and 2.8), extending (under more restrictive numerical assumptions) the results proved in [13], [14], [15] for ample vector bundles. Our main assumption of positivity of a vector bundle E on a projective variety X is that E is semiample, i.e. that there is $m > 0$ such that $\mathbf{O}_{\mathbf{P}(E)}(m)$ is spanned, i.e. that $\mathbf{O}_{\mathbf{P}_E}(1)$ is semiample. A source of inspiration was the notion of k-ampleness introduced by Sommese ([19]) and the connectivity results proved by him (see [19], n. 1 or [16], p. 36). However, the use of a finer measure of non-ampleness (see Definition 1.2) is essential for non trivial applications to subvarieties of Grassmannians. At the end of Section 1 we introduce what is (we believe) a useful definition of sectional genus for a vector bundle E on a projective manifold X when $\text{rank}(E) \geq \dim(X) \geq 3$ (see Definition 1.11 and eq. (1)).

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1. Subvarietie of Grassmannians.

Recall ([8], p. 25) that a proper morphism $\pi : A \to B$ of complex spaces with A of pure dimension is said to be semismall if for every integer $k > 0$ the set $F(\pi, k) := \{x \in A : \dim(\pi^{-1}(\pi(x)) \geq k)\}$ has codimension $\geq 2k$ in A. By definition the deviation $D(\pi)$ of semismallness of π is 0 if π semismall, and it is $\sup_{k>0}\{2k - \text{Codim}(F(\pi, k))\}$ otherwise.

Remark 1.1. Let U be a purely n-dimensional algebraic variety and $\pi : U \to \mathbf{C}^N$ be a proper morphism. Let $D(\pi)$ be the deviation of semismallness of π. Then $H_i(U; \mathbf{Z}) = 0$ for $i > n + D(\pi)$ ([8], p. 25).

For the next key Lemma 1.4 the main point is to prove parts (a) and (b). We will use a method of Sommese (see [19], n. 1) related to k-ample bundles in the sense of [19], n. 1. This method was popularized in [16], pp. 35-36, and used again in [13], Theorem 1.3. However the use of the same method for the deviation from semismallness of a bundle instead of the "defect of ampleness" of E is crucial for the application to subvarieties of Grassmannians (see Corollaries 1.8 and 1.10 for the case of 3-folds in $G(2, 5)$).

Definition 1.2. *Let X be a projective variety and E a vector bundle on E. Set $A = \mathbf{P}(E^*)$ and let $\mathbf{O}_A(1)$ the tautological quotient line bundle on A. Assume that there is an integer $m > 0$ such that $\mathbf{O}_A(m)$ is spanned, i.e. assume that $\mathbf{O}_A(1)$ is semiample; we will say in this case that E is semiample. Let*

$u : A \to B$ be the morphism induced by $H^0(A, \mathbf{O}_a(m)) \cong H^0(X, S^m(E))$. E is said to be semismall if the morphism u is semismall. We define the deviation $D[E]$ of semismallness of E as the deviation of semismallness $D(u)$ of u. Note that this definition does not depend on the choice of the integer m.

Remark 1.3. Instead of the morphism induced by

$$H^0(A, \mathbf{O}_A(m)) \cong H^0(X, S^m(E))$$

in the Definition 1.2 of $D[E]$ we may take the morphism $A \to \mathbf{P}(V)$ associated to any vector space $V \subseteq H^0(A, \mathbf{O}_A(m))$ such that V spans $\mathbf{O}_A(m)$.

The key parts of the next lemma are parts (a) and (b), because the remaining assertions will follow as in the case of ample vector bundles.

Lemma 1.4. *Let X be a connected projective manifold of pure dimension n and E a rank r semiample vector bundle on X. Let $D[E]$ be the deviation of semismallness of E. Fix $s \in H^0(X, E)$ and let Z be its 0-locus. Then we have:*

- *(a) for every integer $i \geq n + D[E] + r$ we have $H_i(X \setminus Z; Z) = 0$;*
- *(b) $H_{n+r+D[E]-1}(X \setminus Z; \mathbf{Z})$ is finitely generated and torsion-free;*
- *(c) for every integer $i \leq n - D[E] + r$ we have $H^i(X, Z; \mathbf{Z}) = 0$;*
- *(d) the restriction map $H^i(X; \mathbf{Z}) \to H^i(Z; \mathbf{Z})$ is an isomorphism if $i \leq n + D[E] + r - 1$ and it is injective with torsion-free cokernel if $i = n - D[E] + r$;*
- *(e) assume Z smooth; then the restriction map $H^i(X, \mathbf{O}_X) \to H^i(Z, \mathbf{O}_Z)$ is an isomorphism if $i \leq n - r - 1 - D[E]$ and it is injective if $i = n - r - D[E]$;*
- *(f) assume Z smooth; fix integers $p \geq 0$, $q \geq 0$; then the restriction map $H^q(X, \Omega_{X^p}) \to H^q(Z, \Omega_{Z^p})$ os an isomorphism if $p + q \leq n - r - 1 - D[E]$ and it is injective if $p + q = n - r - D[E]$;*
- *(g) assume Z smooth and $n \geq r + D[E] + 2$; then the induced map $Alb(Z) \to Alb(X)$ is an isomorphism;*
- *(h) assume Z smooth and $n = r + D[E] + 1$; then the induced map $Alb(Z) \to Alb(X)$ is surjective;*
- *(i) assume Z smooth and $n \geq r + D[E] + 3$; then the induced map $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is an isomorphism;*
- *(j) assume Z smooth and $n = r + D[E] + 2$; then the induced map $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is injective with discrete and torsion-free cokernel.*

Proof. Proof of part (a): Look at the construction in [19], n 1 or [16], pp. 35-36 or [13], proof of Theorem 1.3. Set $A := \mathbf{P}(E^*)$ and ley $\mathbf{O}_A(1)$ be the tautological quotient line bundle on A. Let $u : A \to B$ be the morphism induced by $H^0(A, \mathbf{O}_A(m)) \cong H^0(X, S^m(E))$ for a large integer m such that $\mathbf{O}_A(m)$ is spanned. Let $f : A \to A'$ be the Stein factorization of u (hence B is normal, f has connected fibers and $A' \to B$ is finite). Note that s^m induces a section $s^* \in H^0(A, \mathbf{O}_A(m))$. Call M (resp. M') the hypersurface of A (resp. A') with $M = f^{-1}(M')$ and M 0-locus of s^*. Since $A' \setminus M'$ is affine of dimension $n + r - 1$, by Remark 1.1 and the definition of $D[E]$ we have $H_i(A' \setminus M'; \mathbf{Z}) = 0$ for $i \geq n + r + D[E]$. It was checked in [18], n 1, or [15], p. 36 that the projection $A \setminus m \to X \setminus Z$ is a is an affine $\mathbf{C}^{r-1}$-bundle. Hence $A \setminus M$ and $X \setminus Z$ are homotopy equivalent, proving part (a).
Proof of part (b): With the notations of the proof of part (a), $A \setminus M$ has the homotopy type of a CW-complex of dimension $\leq n + r - 1 + D[E]$ by [8], Theorem 1.1* of Part II, p. 152. Hence part (b) follows from [1], p. 717.
Part (c) follows from (a) and Alexander-Lefschetz duality. The injectivity statement of part (d) follows from part (c) and the long cohomology exact sequence of the pair (X, Z). The torsion-freeness in part (d) follows using also part (b). Parts (e) and (f) follow from part (c), the injectivity statement of part (d) and the fact the restriction map commutes with the Hodge decomposition. Parts (f) and (h) follow from part (c) with $i = 2$ and part (e) with $i = 1, 2$. Parts (g) and (i) follow from part (c). Part (j) follows from part (b).

We stress the following remark because for the connectedness results for subvarieties of Grassmannians we will use only part (a) of Lemma 1.4.

Remark 1.5. In the proof of Lemma 1.4, parts (a) and (b) we used only that $X \setminus Z$ is smooth and that the pair (X, Z) satisfies Poincaré-Alexander-Lefschetz duality for a suitable range of integers over $\mathbf{Q}$ for part (a), over $\mathbf{Z}$ for part (b). For instance to apply part (a) of Lemma 1.4 it is sufficient to assume that X is a $\mathbf{Q}$-homology manifold (e.g. that X has only quotient singularities). For general spaces there is Verdier's duality in the derived category or Kaup's duality complex (see [10], [11]) which measures the failure of the corresponding duality statement. Hence it is trivial to use this approach for a singular space X to obtain connectivity theorems in suitable range of integers.

Let Y be an integral subvariety of the Grassmannian $G(r, N + 1)$ of all $\mathbf{P}^{r-1}$'s into $\mathbf{P}^N$. Recall that a fundamental $\mathbf{P}^k$ of Y, $0 \leq k \leq r-2$, is a projective subspace A of $\mathbf{P}^N$, $\dim(A) = k$, contained into infinitely many of the $\mathbf{P}^{r-1}$'s of Y. We make the following general definition.

Definition 1.6. Let Y be an integral subvariety of the Grassmannian $G(r, N +$

1) of all $\mathbf{P}^{r-1}$'s into $\mathbf{P}^N$. Let A be a linear subspaces of $\mathbf{P}^N$. Set $t := \dim(U(A, Y))$. A is said to be a fundamental t-linear subspace of weight $u(A, Y)$ if $u(A, Y) > 0$. Define the defect $\Delta(y)$ of Y in the following way. Let $u(Y)$ be the supremum for all integers t with $0 \leq t \leq r-1$ and all fundamental t-linear spaces A of the integer $u(A, Y)$. Set $\Delta(Y) := \max\{0, 2u(Y) - \dim(Y)\}$.

Recall that the 0-loci of the universal quotient bundles on the Grassmannians have a geometric interpretation. Hence by the Definitions 1.6 and 1.2 and Lemma 1.4 we obtain the following result.

Proposition 1.7. *Let Y be an integral subvariety of the Grassmannian $G(r, N+1)$ of all $\mathbf{P}^{r-1}$'s into $\mathbf{P}^n$ and let Q be the tautological quotient sheaf of Y. Then $D[Q] = \Delta(Y)$. In particular if Y is smooth and $N + 2 - r + \Delta(Y) \leq \dim(Y)$, then a general section of Q has connected 0-locus.*

Corollary 1.8. *Let $X \subset G(2, 5)$ be a smooth connected 3-fold and Q the universal rank 2 quotient bundle on X. Assume that not all lines of X pass through a fundamental point of X. Let C be the 0-locus of a general section of Q. Then C is connected.*

Proof. By Lemma 1.4 (d) and Remark 1.3 it is sufficient to check that $D[Q] = 0$. Let $u : \mathbf{P}(Q^*) \to \mathbf{P}^4$ be the morphism associated to the embedding on X into $G(2, 5)$. By assumptions u has at most a one-dimensional family of fibers of positive dimension and at most finitely many fibers of dimension 2. Since $\dim(\mathbf{P}(Q^*)) = 4$, we have $D(u) = 0$, as wanted.

A main feature of our approach is that using Remark 1.9 below we may obtain a similar statement (see 1.10) for the dependency locus of the dual of the tautological subbundle of a 3-fold in $G(2, 5)$.

Remark 1.9. Let X be a smooth projective variety, E a vector bundle on X and $V \subseteq H^0(X, E)$ a finite dimensional vector space spanning E. Set $n := \dim(X)$ and $r := \operatorname{rank}(E)$ and assume $r \geq n \geq 3$. Let $U \subseteq V$, $U' \subset U$ be general subspaces with $\dim(U) = \dim(U') + 1 = r - n$. Set $w := V/U$, $W' := V/U'$ and let B (resp B') be the subsheaf of E spanned by U (resp. U'). Set $F := E/B$ and $F' := E/B'$. By Bertini's theorem F' is a rank n vector bundle on X with $c_i(F') = c_i(E)$ for every i. By [4], n. 1, f is a smooth sheaf in the sense of [4] and in particular it is a reflexive sheaf with $c_i(F) = c_i(E)$ for every i and which is not locally free exactly at $c_n(E)$ points of X. By [4], Theorem 1, or [9], Remark 4.1.1, (stated only for the case $n = 3$ but the proof given there works for arbitrary n) a general section of W vanishes exactly on a smooth curve, C, with sectional genus $g(E)$ defined by the formula

(1) $$2g(E) - 2 = (\omega_X + c_1(E)) \cdot c_{n-1}(E) + c_n(E)$$

(see [9], Theorem 4.1 for $n = 3$ and $X = \mathbf{P}^3$ and hence $\deg(K_X) = -4$, but the proof given there works in general) and passing through the $c_n(F)$ points $Sing(F)$. C is the dependency locus of W, i.e. the dependency locus of a general $(r-n+1)$-dimensional subspace of V. if $r = n$, then a local calculation contained in [4] (or see [3], n. 1) shows that $\mathbf{P}(E)$ is smooth, i.e. E is a Banica sheaf in the sense of [3]. Hence if $r = n$ we may apply Lemma 1.4 even if $F' := F = E/B$ is not locally free. For a discussion of the integer $g(e)$, see Definition 1.11 below.

Corollary 1.10. *Let $X \subset G(3,5)$ be smooth connected 3-fold, Q the universal rank 3 quotient bundle on X and c the dependency locus of two general sections of Q. Assume that for a general $P \in \mathbf{P}^4$ there is no 3-dimensional family of lines of X passing through P. Then C is smooth and connected.*

Proof. Let $V \subseteq H^0(X, Q)$, $\dim(V) = 5$, be the subspace corresponding to the embedding of X into $G(3,5)$. Fix a general $s \in V$ and set $W := V/\mathbf{K}_s$ as quotient vector space. By the generality of s. $Q' := Q/s(\mathbf{O}_x)$ is a rank 2 reflexive sheaf with exactly $c_3(Q)$ non-locally free pits and which is smooth in the sense of [4]. Q' is curvilinear in the sense of [9]. W spans Q'. C is smooth (see e.g. [9], Remark 4.1.1 for the case $n = 3$ but the proof given there works in general). C is the 0-locus of a general section of W. Hence the smoothness of C follows from [4], Theorem 1) (see the proof of Remark 1.9) and the connectedness of C follows from the last part of the discussion of Remark 1.9.

We are unable to deduce directly 1.10 from 1.8 using the duality isomorphism $G(2,5) \cong G(3,5)$.
Motivated by Remark 1.9 we introduce the following integer $g(E)$ which will be called the sectional genus of E.

Definition 1.11. Let X be a Gorenstein projective variety with $\dim(X) = n \geq 3$ and E a vector bundle on X with $\text{rank}(E) \geq n$. The sectional genus $g(E)$ of E is defined by the formula (1).

Lemma 1.12. *Assume X Gorenstein with isolated singularities and $r \geq n \geq 3$. Then $g(E)$ is an integer.*

Proof. Fix an ample $H \in \text{Pic}(X)$. Then for large t there are $r - n + 1$ sections of $E(tH)$ whose dependency locus is a smooth curve $Z(t) \subset X_{reg}$. We have $g(E(tH)) = p_a(Z(t))$. Hence $g(E(tH))$ is an integer. Now use the transformation formulas (see e.g. [9], Lemma 2.1, for the case $X = \mathbf{P}^n$ and an hint of the general formulas) for $c_i(E(tH))$. These formulas are "universally true" since the Chern classes of a vector bundle are cohomology classes, i.e.

operators. Thus $2g(E(x,h)) - 2$ is a polynomial with rational coefficients, say $2p(x)$, in x which takes even values for large t. Hence $p(x)$ has integers values for all x. Thus $g(E) = p(0) + 1 \in \mathbf{Z}$.

2. Bundles with a section with very special 0-locus.

We recall the following definition

Definition 2.1. Let X be an integral projective variety and E a vector bundle on X. Let $\pi : \mathbf{P}(E^*) \to X$ be the projection. Assume that E is semiample, i.e. assume that $\mathbf{O}_{\mathbf{P}(E^*)}(1)$ is semiample and fix an integer $m > 0$ such that $\mathbf{O}_{\mathbf{P}(E^*)}(m)$ is spanned. Let $u : \mathbf{P}(E^*) \to \mathbf{P}^N$ be the morphism associated to $H^0(\mathbf{P}(E^*).\ \mathbf{O}_{\mathbf{P}(E^*)}(m))$ and set $A := \{x \in \mathbf{P}(E^*) : \dim_x(u^{-1}(u(x))) > 0\}$. Set $\mathrm{Disamp}(E) := \pi(a) \subseteq X$ and call $\mathrm{Disamp}(E)$ the disamplitude locus of e. The definition of $\mathrm{disamp}(E)$ does not depend on the choice of the integer $m > 0$ such that $\mathbf{O}_{\mathbf{p}(E^*)}(m)$ is spanned.

Lemma 2.2. *Let X be an integral quasi-projective variety and $C \subset X_{\mathrm{reg}}$ a smooth complete rational curve. Let $\{a_i\}_{1 \le i \le n-1}$ the degree of the rank 1 direct summands of the normal bundle $N_{C/X}$ of C in X. Let Γ be an irreducible component of* $\mathrm{Hilb}(X)$ *containing C. Assume $a_i \ge 0$ for every i. Let Z be an integral subvariety of X with* $\dim(Z) = t \le n-2$. *Then for a general curve $D \in \Gamma, \cong \mathbf{P}^1$, we have $D \cap Z = \emptyset$.*

Proof. Since $a_i \ge -1$, for every i, $\mathrm{Hilb}(X)$ is smooth at c and hence Γ is the unique component of $\mathrm{Hilb}(X)$ containing X. Since $a_i \ge 0$ for every i, γ is a covering family of rational curves whose general element is smooth. Let Λ be the open subset of Γ_{reg} parametrizing the smooth rational curves a such that the normal bundle $N_{A/X}$ has splitting type $\{b_i\}_{1 \le i \le n-1}$ with $b_i \ge 0$ for every i, i.e. such that $N_{A/X}$ is spanned. We have $C \in \Lambda$. Fix $D \in \Lambda$ and $P \in D$. Since $h^1(D, N_{D/X}(-P)) = 0$, the family of curves near D and passing through P is smooth of dimension $\chi(N_{D/X}(-P)) = \Sigma_{1 \le i \le n-1} a_i - n + 1$. Hence we see that for every $P \in Z$ the subfamily of Λ formed by the curves containing P has at least codimension $t+1$ in Γ. Hence a general $D \in \Lambda$ does not interest Z.

Theorem 2.3. *Let X be a smooth projective variety and E a semiample vector bundle on X. Set $n :=$* $\dim(X)$, *$r :=$* $\mathrm{rank}(E) \le n-2$ *and assume the existence of $s \in H^0(X,E)$ whose 0-locus Z (as scheme) is $\mathbf{P}^{n-r}$. Assume $n \ge r + 2 + D[E]$. Assume* $\dim(\mathrm{Disamp}(E)) \le n-2$. *Then $X = \mathbf{P}^n$ and E is the direct sum of r line bundles of degree 1.*

Proof. By Lemma 1.4 (part (i) if $n \geq r + 3 + D[E]$, part (j) and the fact $\mathrm{Pic}(\mathbf{P}^{n-r}) \cong \mathbf{Z}$ if $n = r + 2 + D[E]$), the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is an isomorphism. Fix a line $D \subset Z$. Since E is semiample, $N_{Z/X} \mid D$ is semiample and hence we may apply Lemma 2.2 to D. By the adjunction formula X is a Fano manifold and the positive generator L of X has $L \cdot D = 1$. Since $\dim(\mathrm{Disamp}(E)) \leq n - 2$, by Lemma 2.2 there is a deformation D' of D such that $E \mid D'$ is ample. Hence $\deg(E \mid D) \geq r$. By the adjunction formula we have $-K_X \cdot D \geq n + 1$. Hence X is a Fano manifold of index $\geq n + 1$. By a theorem of Kobayashi-Ochiai ([12]) we have $X \cong \mathbf{P^n}$. Furthermore, by the adjunction formula Z is embedded as a linear subspace of X. Fix a plane $\Pi \subset \mathbf{P}^n$ and assume $E \mid \Pi$ ample. Since $\deg(E \mid \Pi) = \mathrm{rank}(E)$, we obtain that $(E \mid \Pi)$ is a uniform vector bundle of splitting typer $(1, \dots, 1)$ and hence $(E \otimes L^*) \mid \Pi$ is a uniform vector bundle of splitting type $(0, \dots, 0)$. This implies that $E \otimes L^* \mid \Pi$ is trivial (see e.g. [17], Chapter I, Theorem 3.2.1). Hence $E \otimes L^*$ is trivial ([5], Corollary 1.7), proving the theorem under the assumption $E \mid \Pi$ ample. For a general plane $\Pi \subset X$ $\mathrm{Disamp}(E) \cap \Pi$ is finite (or empty). $E \mid \Pi$ is semiample and we have $\mathrm{Disamp}(E \mid \Pi) \subseteq \mathrm{Disamp}(E) \cap \Pi$. Since a spanned line bundle which does not contract any curve is ample, if $\mathrm{Disamp}(E \mid \Pi)$ is finite, then it is empty, i.e. $E \mid \Pi$ is ample. Hence the theorem is proved.

Theorem 2.4. *Let X be a smooth projective variety and e a semiample vector bundle on X. Set $n := \dim(X)$, $r := \mathrm{rank}(E) \leq n - 2$ and assume the existence of $s \in H^0(X, E)$ whose 0-locus z (as scheme) is a smooth quadric $\mathbf{Q}$ with $\dim(\mathbf{Q}) = n - r$. Assume $n \geq r + 2 + D[E]$ and $n \geq r + 3$. Assume $\dim(\mathrm{Disamp}(E)) \leq n - 2$. Then (X, E) is one of the following pairs:*

(1) $X = \mathbf{P}^n$, E direct sum of a line bundle of degree 2 and $r - 1$ line bundles of degree 1;

(2) X is a smooth quadric Q and $E \cong \mathbf{O}_Q(1)^{\otimes r}$.

Proof. By Lemma 1.4 (part (i) if $n \geq r + 3 + D[E]$, part (j) and the fact that $\mathrm{Pic}(\mathbf{Q}) \cong \mathbf{Z}$ if $n = r + 2 + D[E]$), the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is an isomorphism. Fix a line $D \subset Z$. since E is semiample $N_{Z/X} \mid D$ is semiample and hence we may apply Lemma 2.2 to D. By the adjunction formula X is a Fano manifold and the positive generator L of X has $L \cdot D = 1$. Since $\dim(\mathrm{Disamp}(E)) \leq n - 2$, by Lemma 2.2 there is a deformation D' of d such that $E \mid D'$ is ample. Hence $\deg(E \mid D) \geq r$. By the adjunction formula we have $-K_X \cdot D \geq n$. Hence X is a Fano manifold of index $x \geq n$. By a theorem of Kobayashi-Ochiai ([12]) X is either $\mathbf{P}^n$ or a smooth quadric Q. Furthermore, by the adjunction formula we see that if $X \cong Q$, then Z is embedded as a linear section of X while if $X \cong \mathbf{P}^n Z$ is embedded as a hypersurface of a linear subspace of X. If $X \cong Q$ and $K \subset X$ is a line such that $E \mid K$ is ample, then by the adjunction formula $E \mid K$ has splitting type $(1, \dots, 1)$. If $X \cong \mathbf{P}^n$ and $K \subset X$ is a line such that $E \mid K$ is ample, then by the adjunction formula $E \mid K$ has splitting type $(2, \dots, 1)$. Since a semiample bundle F on a variety T with $\mathrm{Disamp}(F)$ finite or empty is ample and $\mathrm{Disamp}(E \mid K) \subseteq \mathrm{Disamp}(E) \cap K$, we see that $E \mid K$ is ample for every line K not contained in $\mathrm{Disamp}(E)$. First, we assume $X \cong Q$. We want to find a smooth quadric $A \subset Q$, $\dim(A) = 3$, and such that $E \mid k$ is ample for every line $K \subset A$. A exists and we may take as A general 3-dimensional linear section of Q, unless $\mathrm{Disamp}(E)$ has an irreducible component, M, whose general codimension $n-3$ linear section is a curve containing a line. No such variety M exists in Q. Fix such a quadric A. Since $E \otimes L^* \mid A$ is uniform of splitting type $(0, \dots, 0)$, $E \otimes L^* \mid A$ is trivial (see e.g. [2], Proposition 3). Hence the proof of [5], Corollary 1.7, (see e.g. [2], n. 1, for full details) given that $E \cong L^{\otimes r}$, as wanted. from now on, we assume $X \cong \mathbf{P}^n$. Fix $P \notin \mathrm{Disamp}(E)$. As in the previous case for every line K with $P \in K \subset X$, we have $E \mid K$ ample, and hence $E \mid K$ has splitting type $(2, \dots, 1)$. Since $r \leq n-3$, this condition (a priori weaker than the condition that E is a uniform vector bundle with splitting type $(2, 1, \dots, 1)$) is exactly the condition used in [18] to obtain that E is the direct sum of r line bundles. Hence the theorem is proved also in the case $X \cong \mathbf{P}^n$.

Remark 2.5. As remarked in [13], 2.4, the first part of the proof of Theorem 2.3 and 2.4 applies also if the 0-locus Z of s is assumed to be a Fano manifold of index $\dim(Z) - 1$ and with $\mathrm{Pic}(Z) \cong \mathbf{Z}$.

Theorem 2.6. *Let X be a smooth projective variety and E a semiample vector bundle on X. Set $n := \dim(X)$, $r := \mathrm{rank}(E) \leq n-2$ and assume the existence of $s \in H^0(X, E)$ whose 0-locus Z (as scheme) is a smooth scroll $f : \mathbf{P}(A) \to B$ over a smooth curve B genus > 0. Assume $n \geq r + 3 + D[E]$. Assume $\dim(\mathrm{Disamp}(E)) \leq n-3$. Then X is a $\mathbf{P}^{n-1}$-bundle $\pi : X \to B$ over B and there is $L \in \mathrm{Pic}(X)$ with $\deg(L \mid T) = 1$ for every fiber T of π and a vector bundle J on B with $E \cong \pi^*(J) \otimes L$.*

Proof. By Lemma 1.4 (i) the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is an isomorphic. By Lemma 1.4 (h) we have $\mathrm{Alb}(X) \cong B$ and the Albanese map $f := \alpha_Z : Z = \mathbf{P}(G) \to B$ factors through the Albanese map $\alpha_C : X \to B$. Fix a line $D \subset Z$. Since E is semiample $N_{Z/X} \mid D$ is semiample and hence we may apply Lemma 2.2 to D. Since $\dim(\mathrm{Disamp}(E)) \leq n-2$, by Lemma 2.2 there is a deformation D' of D such that $E \mid D'$ is ample. Hence $\deg(E \mid D) \geq r$. By the adjunction formula we have $-K_X \cdot D \geq n$. Let T be a general fiber of α_X (hence smooth and with $\dim(\mathrm{Disamp}(E) \cap T) < \dim(\mathrm{Disamp}(E))$).

Since α_X is the Albanese fibration, T is connected. Since $n-1 \geq r+2$ and $T \cap Z \cong \mathbf{P}^{n-r-1}$, by Theorem 2.3 $T \cong \mathbf{P}^{n-1}$ and $E \mid T$ is direct sum of r degree 1 line bundles. Furthermore, there is $L \in \mathrm{Pic}(X)$ such that $E \otimes L^*$ is trivial. since $S := \alpha_{X_*}(E \otimes L^*)$ has no torsion, we see that S is a rank r vector bundle on B and the natural map $f : \alpha_X^*(S) \to E \otimes L^*$ has kernel and cokernel supported by finitely many fibers of α_X. Since $\alpha_X^*(S)$ is locally freen $\ker(f) = 0$. Note that L is α_X-ample. Since B is a smooth curve to obtain that $\pi := \alpha_X : X \to B$ is a smooth scroll it is sufficient to check that $\alpha + X$ has no singular fiber. Let M be a fiber of α_X. Since B is a smooth curve and X is smooth. α_X is flat. M has pure dimension $n-1$ and is locally Cohen-Macaulay and the top intersection number $(L \mid M)^{n-1}$ is one ([7], Corollary 10.1). Furthermore, by semicontinuity we have $h^0(M, L \mid M) \geq n$. Hence $(L, L \mid M)$ is a polarized variety with Δ-genera ≤ 0. Since M has only Cohen-Macaulay singularities, we have $(M, L \mid M) \cong (\mathbf{P}^{n-1}, \mathbf{O}(1))$ by a theorem of Goren and Kobayashi- Ochiai (see [6], Theorem 1.1). Now (and only now) we use the strong assumption Codim(Disamp(E)) ≥ 3. Hence for every fiber T of π we have Codim(Disamp($E \mid T$)) ≥ 2 and we may apply 2.3 to $E \mid T$ and obtain that $E \mid T \cong (L \mid T)^{\otimes t}$. Now we will check that $E \cong \pi^*(U)(H)$ with U rank r vector bundle on B and $H \in \mathrm{Pic}(X)$ with $\deg(H \mid T) = 1$ for every fiber T on π. fix any such $H \in \mathrm{Pic}(X)$. It was shown in [15], proof of Theorem A, that the restriction of $E(-H)$ to every fiber of π is trivial. hence $R^1\pi_*(E(-H)) = 0$ and $U := \pi_*(E(-H))$ is a rank r vector bundle on B. It is easy to check that $E \cong \pi^*(u)(H)$.

Remark 2.7. Note that, assuming E ample Theorem 2.6 gives a small improvement of the statement of [15], Theorem A, i.e. a more precise description of the bundle E. This is due only to the last part of the proof of 2.6.

Now we recall the definition of quadric fibration used in [15] and hence used here. A polarized manifold (X, L) (i.e. $L \in \mathrm{Pic}(X)$ is ample) is said to be a quadric fibration over a smooth curve C if there is a surjective morphism $\pi : X \to C$ such that every fiber of π is irreducible and reduced and every smooth fiber T of π is a hyperquadric $Q \subset \mathbf{P}^n$ with $(T, L \mid T) \cong (Q, \mathbf{O}_Q(1))$. Let M be a fiber of π. It was shown in [15], n. 1 that $(L \mid M)^{n-1} = 2$, that $L \mid M$ is very ample with $h^0(M, L \mid M) = n+1$ and that $L \mid M$ embeds M into $\mathbf{P}^n$ as an integral normal quadric. furthermore, setting $G := \pi_*(L)$, G is a rank $n+1$ vector bundles on C, and there is an embedding $i : X \to \mathbf{P}(G)$ and $H \in \mathrm{Pic}(G)$ with $i^*(H) = l$ and $\pi = f \circ i$ (with $f : \mathbf{P}(G) \to C$ the projection).
The proof of the next result is completely different from the proof of the corresponding result with E ample proved in [15], Theorem 5. The drawback

is that for our proof we had to assume $p_a(B) > 0$ and $n \geq r+4$.

Theorem 2.8. *Let X be a smooth projective variety, $H \in \mathrm{Pic}(X)$, H ample and E a semiample vector bundle on X. Assume* $\dim(\mathrm{Disamp}(E)) \leq n-3$. *Set $n := \dim(X)$, $r := \mathrm{rank}(E)$ and assume $n \geq r+4$. Assume the existence of $s \in H^0(X, E)$ whose 0-locus Z (as scheme) is a smooth quadric fibration $Z \to B$ over a smooth curve B of genus > 0 with respect to $H \mid Z$. Assume $n \geq r+3+D[E]$. Then one of the following cases occur:*

(a) there is a surjective morphism $\phi : X \to B$ such that any general fiber T of Φ is a smooth hyperquadric of $\mathbf{P}^{n-r+1}$ with $H \mid T \cong \mathbf{O}_T(1)$ and $E \mid T \cong \mathbf{O}_T(1)^{\otimes r}$;

(b) $\mathrm{Pic}(X) \cong \mathrm{Pic}(Z)$; (X, Z) is a smooth scroll over B; for every fiber $T \cong \mathbf{T}^{n-1}$ of the scroll fibration $\pi : X \to B$ we have $H \mid T \cong \mathbf{O}_T(2) \otimes \mathbf{O}_T(2)^{\otimes(r-1)}$. There are $A \in \mathrm{Pic}(B)$ and a rank $r-1$ vector bundle U on B such that E fits in the following exact sequence:

$$0 \to \pi^*(A) \otimes H^{\otimes 2} \to E \to \pi^*(U) \otimes H \to O. \tag{2}$$

Proof. By Lemma 1.4 (i) the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is an isomorphism. By Lemma 1.4 (h) we have $\mathrm{Alb}(X) \cong B$ and the albanese map $\alpha_Z : Z = \mathbf{P}(G) \to B$ factors through the Albanese map $\alpha_X : X \to B$. By properties of the Albanese fibration a general fiber of α_X is connected. Fix any smooth connected fiber T of α_X. Since $n \geq r+4$ and $\dim(\mathrm{Disamp}(E \mid T)) \leq n-3$, we may apply 2.4 to the pair $(T, E \mid T)$ and obtain that either all such T are smooth quadrics or all such T are projective spaces and in all cases $E \mid T$ is a direct sum of line bundles. If T is a quadric by 2.4 $E(-H)$ is trivial on every fiber of π. set $G := \pi_*(E(-H))$. By a theorem of changing bases the sheaf G is a rank r vector bundle on B and the restriction of the natural morphism $\pi^*(G) \to E(-H)$ to every smooth fiber of π is an isomorphism. Note that for every line D contained in a singular fiber of π and disjoint from $\mathrm{Disamp}(E)$, $E(-H) \mid D$ is trivial and the restriction of b to D is an isomorphism. Hence $E(-H)$ is a vector bundle whose restriction to an open subset T' with $\mathrm{codim}(T \setminus T') \geq 2$ is trivial. Since T is normal, we obtain that $E(-H) \mid T$ is trivial. Since a generically injective map between vector bundles of the same rank drops rank on a divisor, we obtain that b is an isomorphism. Now assume that every smooth fiber of π is a projective space. The last part of the proof of 2.6 shows that π is a smooth scroll. The existence of an exact sequence (2) follows from standard properties of the relative Harder-Narasimhan filtration and the application of a theorem of changing bases made also in the proof of Remark 2.7.

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