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# FORMAL THEORY OF INTERNAL CATEGORIES

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*To the memory of Umberto Gasapina*

Categories internal to an elementary topos are regarded as monads in the bicategory of spans of the topos and the main features of the theory are developed: functors, adjointness, module calculus, internal presheaves, internal completeness and cocompleteness, Kan extensions.

## Introduction.

One of the main aspects of category theory regards the fact that many mathematical structures can be "internalized" into suitable categories, sort of "universes" within which they are fully defined. Moreover, the basic constructions of the structures can be performed very much in the same way as in the category *Sets* of sets and many results remain true.

The structure we consider here is category theory itself, internal to a given category $\mathcal{E}$ with finite limits. To develop further the theory however, more properties are needed from $\mathcal{E}$: pullback stable coequalizers to allow composition of modules (*distributors* or *profunctors*) or local cartesian closure (i.e.cartesian closure of the category $\mathcal{E}/X$, for any object $X$)in order to take account of internal completeness of $\mathcal{E}$. In general, it is useful to assume that $\mathcal{E}$ is an elementary topos.

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While the basic elements of internal category theory are well known since some time (see for instance Ch. 2 of Johnstone [5], to which we refer for notions not explained here), the point of view of this paper arose and proved to be useful during the development of locally internal category theory in terms of enriched categories (Betti and Walters [2] and [3]), where it was relevant to consider in the same setting both internal and locally internal categories.

This common setting is provided by the bicategory Span $\mathcal{E}$. Namely, the new aspect of this paper is that internal categories are regarded as monads of the bicategory Span $\mathcal{E}$. The main notions of internal category theory are accordingly introduced: adjointness, presheaves, Kan extensions, completeness and cocompleteness. As a consequence of this language, their basic properties are reproduced purely formally, in the sense that only the bicategorical notions of Span $\mathcal{E}$ are involved into consideration.

The treatment thus allows an essentially new viewpoint that contributes to clarify the whole subject. Moreover, the appropriate "calculus" of modules, which is systematically used, turns out to be a precise and flexible tool for analysing internal structures. With respect to this fact, emphasis is given to particular properties (Kan extensions, tabulation) that can be lifted from Span $\mathcal{E}$ to the bicategory Mod $\mathcal{E}$ of internal categories and modules between them.

## 1. The bicategory Span $\mathcal{E}$.

Let $\mathcal{E}$ be a category with pullbacks. Objects of the bicategory Span $\mathcal{E}$ are the objects of $\mathcal{E}$,arrows $A \longrightarrow\!\!\!\!\!\!+\;\; B$ are spans $(f, g)$ of arrows in $\mathcal{E}$ as in the picture:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$

while 2-cells are arrows $h$ in $\mathcal{E}$ such that the following triangles commute:

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] \arrow[dd, "h"] & \\ A & & B \\ & Y \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Composition is given by pullback and the span $(1_A, 1_A)$ is the identity of $A$.

The category $\mathcal{E}$ can be embedded in the bicategory Span $\mathcal{E}$ by a homomorphism which is the identity on objects and takes an arrow $f$ of $\mathcal{E}$ to the arrow $(1, f)$ of Span $\mathcal{E}$. Such arrows are characterized (up to isomorphism) by the fact of having a right adjoint, namely the span $f^\circ = (f, 1)$. In general in a bicategory, an arrow $\varphi$ with a right adjoint is called a *map* and its right adjoint is denoted by $\varphi^\circ$: the arrows of $\mathcal{E}$ when regarded in Span $\mathcal{E}$ constitute all the maps. With this notation, the span $(f, g)$ can be written as the composite $gf^\circ$.

By the adjointness $f \dashv f^\circ$ one has the natural bijections of 2-cells:

$$\text{(1)} \qquad \begin{array}{c} \sigma : \alpha \longrightarrow f^\circ \beta f \\ \hline \hat{\sigma} : f\alpha \longrightarrow \beta f \\ \hline \check{\sigma} : \alpha f^\circ \longrightarrow f^\circ \beta \\ \hline \bar{\sigma} : f \alpha f^\circ \longrightarrow \beta \end{array}$$

where $\alpha : A -\!\!\!+\!\!\!\to A$ and $\beta : B -\!\!\!+\!\!\!\to B$ are arrows in Span $\mathcal{E}$, and $f : A \longrightarrow B$ is a map.

Observe that the natural bijections (1) provide the correspondences on objects of equivalences between hom-categories:

$$\mathrm{Span}\,\mathcal{E}(A, A) \simeq \mathrm{Span}\,\mathcal{E}(A, B) \simeq \mathrm{Span}\,\mathcal{E}(B, A) \simeq \mathrm{Span}\,\mathcal{E}(B, B)$$

When $\mathcal{E}$ is (finitely) complete and cocomplete, the categories Span $\mathcal{E}(A, B)$ are also (finitely) complete and cocomplete. Moreover, when $\mathcal{E}$ is an elementary topos, the bicategory Span $\mathcal{E}$ admits right Kan extensions and right limits. This means that, for any arrow $\varphi : A -\!\!\!+\!\!\!\to B$, compositions with $\varphi$ on both sides have right adjoints:

$$- \cdot \varphi \dashv \mathrm{Hom}^A(\varphi, -)$$
$$\varphi \cdot - \dashv \mathrm{Hom}_B(\varphi, -)$$

Explicitly, in the case of right Kan extensions, for any arrow $\psi : A -\!\!\!+\!\!\!\to C$ there is an arrow $\mathrm{Hom}^A(\varphi, \psi) : B -\!\!\!+\!\!\!\to C$ $B$ endowed with a 2-cell:

$$\begin{tikzcd}
 & B \arrow[dd, "\mathrm{Hom}^A(\varphi,\psi)"] \\
A \arrow[ur, "\varphi"] \arrow[dr, "\psi"'] \arrow[r, phantom, "\Downarrow"] & \\
 & C
\end{tikzcd}$$

which, by composition, provides a natural bijection of 2-cells:

$$\frac{\gamma\varphi \longrightarrow \psi}{\gamma \longrightarrow \mathrm{Hom}^A(\varphi, \psi)}$$

for any arrow $\gamma : B \longrightarrow\!\!\!\!\!\!\!\!+\ C$.

Analogously, right liftings are characterized by the universal property:

$$\frac{\varphi\gamma \longrightarrow \psi}{\gamma \longrightarrow \mathrm{Hom}_B(\varphi, \psi)}$$

For the existence of right Kan extensions and right liftings, first observe that each is implied by the other one, because by interchanging the two maps of any arrow one has a "symmetry structure" of Span $\mathcal{E}$, namely, an equivalence $(\mathrm{Span}\,\mathcal{E})^{\mathrm{op}} \simeq \mathrm{Span}\,\mathcal{E}$ which is the identity on objects and takes any map $f$ to its right adjoint $f^\circ$. Next, observe that the extension along a composite with $f^\circ$ can be computed as follows:

$$\mathrm{Hom}^A(\varphi f^\circ, \psi) \cong \mathrm{Hom}^A(\varphi, \psi f)$$

and finally that, if $\varphi = gf^\circ$, an explicit formula for the right Kan extension along $\varphi$ is the following (which utilizes the right adjoint $\Pi$ to the pullback functor $(\ )^*$):

$$\mathrm{Hom}^A(gf^\circ, \psi) \cong \mathrm{Hom}^A(g, \psi f) \cong \Pi_{g\times 1}(\psi, f)$$

In particular, observe that the existence of right adjoints to $-\circ\varphi$ and $\varphi\circ -$ ensures that composition with $\varphi$ on both sides preserves colimits.

Another relevant property of Span $\mathcal{E}$ is "tabulation", in the following sense:

**Definition.** *An arrow $\Phi : C \longrightarrow\!\!\!\!\!\!\!\!+\ D$ of a bicategory is said to admit a tabulation if there exist maps $h : A \longrightarrow C$, $k : A \longrightarrow D$ and an invertible 2-cell $\varphi : kh^\circ \cong \Phi$ such that the induced 2-cell $\bar\varphi : k \Longrightarrow \Phi h$ is universal in the sense that for any pair of maps $f : B \longrightarrow C$, $g : B \longrightarrow D$ and any 2-cell $\psi : g \Longrightarrow \Phi f$ there exists a unique map $r : B \longrightarrow A$ such that the following diagram commutes:*

$$\begin{tikzcd} kr \arrow[r, "\bar\varphi r"] \arrow[d, "\cong"'] & \Phi hr \arrow[d, "\cong"] \\ g \arrow[r, "\psi"'] & \Phi f \end{tikzcd}$$

It is not difficult to see that, by definition of its arrows and of its 2-cells, the bicategory Span $\mathcal{E}$ admits the tabulation of any arrow (actually Span $\mathcal{E}$ can be characterized through tabulation, see Carboni, Kasangian and Street [4]).

## 2. Internal categories, functors and natural transformations.

If $\mathbf{A}$ is a *category* internal to $\mathcal{E}$, with domain and codomain arrows $\partial_0$ and $\partial_1$:

$$A_1 \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} A_0$$

and identity:

$$i : A_0 \longrightarrow A_1$$

then it becomes an arrow $\alpha = \partial_1 \partial_0^\circ : A_0 \longrightarrow\!\!\!\!\!\!\!+\;\; A_0$ with a monad structure:

$$\mu : \alpha^2 \Longrightarrow \alpha$$
$$\mathrm{id}_{A_0} : 1 \Longrightarrow \alpha$$

in Span $\mathcal{E}$. Conversely, any monad $A_0 \longrightarrow\!\!\!\!\!\!\!+\;\; A_0$ in Span $\mathcal{E}$ is an internal category whose object of objects is $A_0$. We thus regard internal categories as monads in Span $\mathcal{E}$, and write simply $\mathbf{A} = (A_0, \alpha)$. Observe that an internal category can be also regarded as an one-object category enriched in the bicategory Span $\mathcal{E}$.

With the above description, it is easy to characterize particular internal categories.

**Theorem.** *An internal category* $\mathbf{A} = (A_0, \alpha)$ *is:*

*i)* discrete *if* $\alpha \cong 1_{A_0}$,
*ii)* a monoid *if* $A_0$ *is the terminal object 1,*
*iii)* a poset *if* $f^\circ \alpha f = g^\circ \alpha g$ *implies* $f = g$ *for any pair of parallel maps* $f$ *and* $g$,
*iv)* a groupoid *if* $\alpha$ *is an idempotent monad:* $\alpha^2 \cong \alpha$,
*v)* a group *if it is both a monoid and a groupoid.*

In particular, objects of Span $\mathcal{E}$, when regarded with their trivial monads become *discrete internal categories.*

One can define the *opposite category* $\mathbf{A}^{\mathrm{op}}$ by means of the monad obtained from $\alpha$ by interchanging "domain" and "codomain" (then the definition of $\mathbf{A}^{\mathrm{op}}$ as a category involves the symmetry structure Span $\mathcal{E} \simeq (\mathrm{Span}\, \mathcal{E})^{op}$).

Functors between internal categories are mappings of monads: more precisely, a *functor* $F = (f, \varphi) : (A_0, \alpha) \longrightarrow (B_0, \beta)$ amounts to a map $f : A_0 \longrightarrow B_0$ (which provides the correspondence on objects) and a 2-cell $\varphi : \alpha \Longrightarrow f^\circ \beta f$ (which provides the correspondence on arrows) compatible with compositions and identities in **A** e **B**, in the sense that the following diagrams of 2-cells commute (here, $\eta$ and $\epsilon$ denote unit and counit of the adjunction $f \dashv f^\circ$):

$$\begin{tikzcd}
\alpha^2 \arrow[r, "\psi^2"] \arrow[d, "\mu_A"'] & f^\circ\beta f f^\circ \beta f \arrow[r, "f^\circ\beta\epsilon\beta f"] & f^\circ\beta^2 f \arrow[d, "f^\circ\mu_B f"] \\
\alpha \arrow[rr, "\varphi"'] & & f^\circ\beta f
\end{tikzcd}$$

and:

$$\begin{tikzcd}
1_A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\eta"'] & \alpha \arrow[d, "\varphi"] \\
\alpha \arrow[r, "f^\circ \mathrm{id}_B f"'] & f^\circ\beta f
\end{tikzcd}$$

It is easy to see how composition of internal functors is defined: given $F = (f, \varphi) : (A_0, \alpha) \longrightarrow (B_0, \beta)$ and $G = (g, \psi) : (B_0, \beta) \longrightarrow (C_0, \gamma)$, one takes $gf : A_0 \longrightarrow C_0$ on objects, and the composition:

$$\alpha \xrightarrow{\varphi} f^\circ\beta f \xrightarrow{f^\circ\psi f} f^\circ g^\circ \gamma g f \cong (gf)^\circ \gamma (gf)$$

provides the effect of $GF$ on arrows. Identity functors are trivially defined and one thus has the category Cat $\mathcal{E}$ of internal categories and internal functors.

Particular functors are easily characterized by using the natural bijections (1):

**Theorem.** *The internal functor $F : \mathbf{A} \xrightarrow{F} \mathbf{B}$ is an internal* discrete cofibration *(respectively an internal discrete* fibration*) if and only if the correspondence on arrows $f\alpha \Longrightarrow \beta f$ (respectively $\alpha f^\circ \Longrightarrow f^\circ \beta$) is invertible.*

It is now immediate to prove:

**Theorem.** *Let $\mathbf{A} \xrightarrow{F} \mathbf{B} \xrightarrow{G} \mathbf{C}$ be a diagram of internal functors:*

*i) if $F$ and $G$ are discrete cofibrations, also $GF$ is,*

*ii) if $GF$ and $G$ are discrete cofibrations, then also $F$ is.*

Suppose now that the two functors $(f, \varphi)$ and $(g, \psi)$ are given between the categories $(A_0, \alpha)$ and $(B_0, \beta)$. A *natural transformation* $(f, \varphi) \longrightarrow\!\!\!\!\!+\;(g, \psi)$ is given by a 2-cell $\tau : gf^\circ \Longrightarrow \beta$ in $\mathrm{Span}\,\mathcal{E}$ (the *component* of the natural transformation) such that the following diagram commutes:

$$\begin{tikzcd}
\alpha \arrow[r, "\varphi"] \arrow[d, "\psi"'] & f^\circ \beta f \arrow[r, "\hat{\tau}\beta f"] & g^\circ \beta^2 f \arrow[d, "g^\circ \mu f"] \\
g^\circ \beta g \arrow[r, "g^\circ \beta \check{\tau}"'] & g^\circ \beta^2 f \arrow[r, "g^\circ \mu f"'] & g^\circ \beta f
\end{tikzcd}$$

where the 2-cells $\hat{\tau} : f^\circ \Longrightarrow g^\circ \beta$ and $\check{\tau} : g \Longrightarrow \beta f$ correspond to $\tau$ according to the natural bijections (1), while $\mu$ denotes composition in $(B_0, \beta)$.

Horizontal composition of natural transformations can be defined, so that $\mathrm{Cat}\,\mathcal{E}$ becomes a bicategory. Namely, if $\tau : gf^\circ \Longrightarrow \beta$ and $\sigma : kh^\circ \Longrightarrow \gamma$ are the components of two horizontally composable natural transformations:

$$\begin{tikzcd}
(A_0, \alpha) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\tau\Downarrow"] & (B_0, \beta) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\sigma\Downarrow"] & (C_0, \gamma)
\end{tikzcd}$$

than it is easy to see that the component of $\sigma \circ \tau$ is given by the composite:

$$kgf^\circ h^\circ \xrightarrow{\check{\sigma}\hat{\tau} f^\circ h^\circ} \gamma h \beta f f^\circ h^\circ \xrightarrow{\gamma h \beta \epsilon h^\circ} \gamma h \beta h^\circ \xrightarrow{\gamma \check{\tau}} \gamma^2 \xrightarrow{\mu} \gamma$$

Let $F = (f, \varphi) : \mathbf{C} \longrightarrow \mathbf{D}$ and $G = (g, \psi) : \mathbf{D} \longrightarrow \mathbf{C}$ be functors and let $\gamma$ and $\delta$ be the category structures of $\mathbf{C}$ and $\mathbf{D}$ respectively. Then, *adjointness* is defined as follows: $F \dashv G$ if there is an invertible 2-cell:

$$\delta f \cong g^\circ \gamma$$

which is natural with respect to composition both in $\mathbf{C}$ and in $\mathbf{D}$, in the sense that the following diagrams of 2-cells commute:

(2)
$$\begin{tikzcd}
g^\circ \gamma^2 \cong \delta f \gamma \arrow[r, "\delta f \varphi"] \arrow[d, "g^\circ \mu_{\mathbf{C}}"'] & \delta f f^\circ \delta f \arrow[d, "\delta \epsilon \delta f"] \\
g^\circ \gamma \cong \delta f & \delta^2 f \arrow[l, "\mu_{\mathbf{D}} f"]
\end{tikzcd}$$

and

(2′)
$$\begin{tikzcd}
\delta^2 f \cong \delta g^\circ \gamma \arrow[r, "\psi g^\circ \gamma"] \arrow[d, "\mu_{\mathbf{D}} f"'] & g^\circ \gamma g g^\circ \gamma \arrow[d, "g^\circ \gamma \epsilon \gamma"] \\
\delta f \cong g^\circ \gamma & g^\circ \gamma^2 \arrow[l, "g^\circ \mu_{\mathbf{D}}"]
\end{tikzcd}$$

## 3. Module calculus of internal categories.

In this section we introduce the notion of a **module** of internal categories, together with its appropriate "calculus". With different names, modules are known as *profunctors* or *distributeurs* (Bénabou [1]), and *bimodules* (Lawvere [6]).

Let $\mathbf{A} = (A_0, \alpha)$ and $\mathbf{B} = (B_0, \beta)$ be internal categories. By definition a *module* $\Phi : \mathbf{A} \longrightarrow\!\!\!\!\!\!+\,\, \mathbf{B}$ is given by a span $\varphi : A_0 \longrightarrow\!\!\!\!\!\!+\,\, B_0$ (the *component* of $\Phi$) endowed with a left action of $\mathbf{A}$ and a right action of $\mathbf{B}$, i.e. there are given two 2-cells $\varphi\alpha \Longrightarrow \varphi$ and $\beta\varphi \Longrightarrow \varphi$, which are compatible with composition and unity in $\mathbf{A}$ and in $\mathbf{B}$ and moreover satisfy the *mixed associativity* law expressed by the following commutative diagram of 2-cells in Span $\mathcal{E}$:

$$\begin{tikzcd}
\beta\varphi\alpha \arrow[r] \arrow[d] & \beta\varphi \arrow[d] \\
\varphi\alpha \arrow[r] & \varphi
\end{tikzcd}$$

The notion of a morphism $\Phi \Longrightarrow \Psi$ between modules is obvious: it consists of a 2-cell $\tau : \varphi \Longrightarrow \psi$ between the components, which respects the actions, in the sense that the following diagrams of 2-cells commute:

$$\begin{tikzcd}
\alpha\varphi \arrow[r] \arrow[d, "\alpha\tau"'] & \varphi \arrow[d, "\tau"] \\
\alpha\psi \arrow[r] & \psi
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\varphi\beta \arrow[r] \arrow[d, "\tau\beta"'] & \varphi \arrow[d, "\tau"] \\
\psi\beta \arrow[r] & \psi
\end{tikzcd}
.$$

In this way, one has a category Mod$(\mathbf{A}, \mathbf{B})$ of modules $\mathbf{A} \longrightarrow\!\!\!\!\!\!+\,\, \mathbf{B}$ and their morphisms.

Suppose now that $\mathcal{E}$ has pullback stable coequalizers. Existence and stability of coequalizers allow to define the composite $\Psi \otimes_{\mathbf{B}} \Phi$ in the situation:

$$(A_0, \alpha) \xrightarrow{\Phi} (B_0, \beta) \xrightarrow{\Psi} (C_0, \gamma)$$

Namely, the component $\rho$ of the composite $\Psi \otimes_{\mathbf{B}} \Phi$ is the coequalizer:

$$\psi\beta\varphi \rightrightarrows \psi\varphi \longrightarrow \rho$$

of the two actions of $\beta$ on $\Phi$ and $\Psi$ in the middle.

The universal property of coequalizers ensures, after a calculation, that $\Psi \otimes_{\mathbf{B}} \Phi$ is a module $\mathbf{A} \longrightarrow\!\!\!\!\!\!+\,\, \mathbf{B}$.

Moreover, it is not difficult to prove that composition of modules is associative (up to isomorphisms) and admits as an identity on the left and on the right the hom $\alpha : \mathbf{A} \longrightarrow\!\!\!\!\!+\!\!\!\!\!\!\longrightarrow \mathbf{A}$, regarded as a module with obvious actions. A bicategory $\mathrm{Mod}(\mathcal{E})$ of internal categories and their modules is easily defined.

Any functor $F = (f, \varphi) : (A_0, \alpha) \longrightarrow (B_0, \beta)$ gives rise to an adjoint pair of modules $F_* \dashv F^*$. In the case of enriched categories, this adjointness and its relevance were especially indicated by Lawvere in [6].

In our context, $F_*$ and $F^*$ have components, respectively:

$$f_* = \beta f : A \longrightarrow\!\!\!\!\!+\!\!\!\!\!\!\longrightarrow B$$

$$f^* = f^\circ \beta : B \longrightarrow\!\!\!\!\!+\!\!\!\!\!\!\longrightarrow A$$

The two actions of $F_*$ are given by:

$$\beta f \alpha \xrightarrow{\beta f \varphi} \beta f f^\circ \beta f \xrightarrow{\beta \epsilon f} \beta^2 f \xrightarrow{\mu f} \beta f$$

and:

$$\beta^2 f \xrightarrow{\mu f} \beta f$$

where $\epsilon$ denotes the counit of the adjointness $f \dashv f^\circ$, and $\mu$ denotes composition, both in $\mathbf{A}$ and in $\mathbf{B}$.

Analogously, the actions of $F^*$ are given by:

$$f^\circ \beta^2 \xrightarrow{f^\circ \mu} f^\circ \beta$$

and:

$$\alpha f^\circ \beta \xrightarrow{\varphi f^\circ \beta} f^\circ \beta f f^\circ \beta \xrightarrow{f^\circ \beta \epsilon \beta} f^\circ \beta^2 \xrightarrow{f^\circ \mu} f^\circ \beta$$

Observe that, when $\mathcal{E} = \mathrm{Sets}$, the functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ is an ordinary functor between small categories, and the modules $F_*$ and $F^*$ are given by:

$$F_*(b, a) = \mathrm{Hom}_{\mathbf{B}}(b, F(a))$$

$$F^*(a, b) = \mathrm{Hom}_{\mathbf{B}}(F(a), b)$$

Now, if one goes back to an adjoint pair of functors $F \dashv G$ (Section 2) it is immediate to check that the diagrams (2) and (2′), which express naturality of the isomorphism $\delta f \cong g^\circ \gamma$, in terms of the induced modules ensure exactly that $F_* \cong G^*$.

Another particular feature which is not difficult to prove is that $F_* \dashv F^*$, i.e. that two 2-cells $1_{\mathbf{A}} \Longrightarrow F^* \otimes_{\mathbf{B}} F_*$ and $F_* \otimes_{\mathbf{A}} F^* \Longrightarrow 1_{\mathbf{B}}$ exist, satisfying the triangular identities of an adjunction: in any case, the component of the unit and of the counit of the adjunction is provided by the correspondence on arrows of the functor $F$. The adjunction $F_* \dashv F^*$ gives rise to an adjunction between categories of modules that will be used later: namely, for any internal category $\mathbf{C}$, "composition with $F_*$" is a functor:

$$- \otimes_{\mathbf{A}} F_* : \mathrm{Mod}(\mathbf{C}, \mathbf{A}) \longrightarrow \mathrm{Mod}(\mathbf{C}, \mathbf{B})$$

which admits as a left adjoint "composition with $F^*$":

$$- \otimes_{\mathbf{B}} F^* \dashv - \otimes_{\mathbf{A}} F_*$$

Observe moreover that, for the adjunction $F_* \dashv F^*$, the necessary module compositions always exist, even when $\mathcal{E}$ does not admit pullback stable coequalizers. This depends on the following fact:

**Theorem.** *Given a module $\Phi$ and functors $F$ and $G$ as in the following diagram:*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\Phi", "|" marking] & \mathbf{B} \\ \mathbf{C} \arrow[u, "F"] \arrow[r, dotted] & \mathbf{D} \arrow[u, "G"'] \end{tikzcd}$$

*the composite module $G^*\Phi F_* : \mathbf{C} \longrightarrow\!\!\!\!\!\!\!+\ \mathbf{D}$ exists and has $g^\circ \Phi f$ as a component, where $f$ and $g$ denote respectively the correspondences on objects of $F$ and $G$.*

*Proof.* Just check the universal property of the coequalizers involved. $\square$

By this last result, one has:

$$(GF)_* \cong G_* F_*$$
$$(GF)^* \cong F^* G^*$$

for any pair of composable functors. Hence, if $\mathcal{E}$ admits pullback stable coequalizers, then $\mathrm{Cat}\,\mathcal{E}$ can be embedded in $\mathrm{Mod}\,\mathcal{E}$ by taking the identity on objects (internal categories) and by $F \longmapsto F_*$ on arrows (internal functors). This law preserves composition and identities up to isomorphism, however it reverses 2-cells:

$$\mathrm{Cat}(\mathbf{A}, \mathbf{B}) \longrightarrow \mathrm{Mod}(\mathbf{A}, \mathbf{B})^{\mathrm{op}}$$

and thus it amounts to a homomorphism of bicategories $\mathrm{Cat}\,\mathcal{E} \longrightarrow (\mathrm{Mod}\,\mathcal{E})^{\mathrm{co}}$. It is not difficult to check that this homomorphism is locally fully-faithful.

Analogous considerations relative to the correspondence $F \longmapsto F^*$ lead to a locally full homomorphism:

$$\mathrm{Cat}\,\mathcal{E} \longrightarrow (\mathrm{Mod}\,\mathcal{E})^{\mathrm{op}}$$

## 4. The bicategory Mod $\mathcal{E}$.

We analyze in this section the main properties of the bicategory $\mathrm{Mod}\,\mathcal{E}$, whose (non internal) version is in Bénabou [1].

First, observe that, when $\mathcal{E}$ is locally cartesian closed, right Kan extensions and right lifting can be lifted from $\mathrm{Span}\,\mathcal{E}$ to $\mathrm{Mod}\,\mathcal{E}$. For this, let $\Phi : \mathbf{A} = (A_0, \alpha) \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{B} = (B_0, \beta)$ and $\Psi : \mathbf{A} = (A_0, \alpha) \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{C} = (C_0, \gamma)$ be modules.
Consider the following equalizer in the local category $\mathrm{Span}\,\mathcal{E}(B_0, C_0)$:

$$\epsilon \longrightarrow \mathrm{Hom}^{A_0}(\varphi, \psi) \rightrightarrows \mathrm{Hom}^{A_0}(\varphi, \mathrm{Hom}^{A_0}(\alpha, \psi))$$

where $\varphi$ and $\psi$ denote the components of $\Phi$ and $\Psi$ respectively.

Observe that, in the above diagram, the two parallel arrows express the action of $\Phi$ and $\Psi$ on the category $\mathbf{A}$ and moreover that the equalizer does not depend on the category structure $\beta$ and $\gamma$ of $\mathbf{B}$ and $\mathbf{C}$ respectively. The universal property of equalizers however ensures that $\epsilon$ comes endowed with the action of $\beta$ on the left and $\gamma$ on the right. Thus it provides the component of a module $\mathbf{B} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{C}$. By the universal property of the right Kan extension $\mathrm{Hom}^{A_0}$ in the bicategory $\mathrm{Span}\,\mathcal{E}$ one thus checks that this module is the right Kan extension $\mathrm{Hom}^{\mathbf{A}}(\Phi, \Psi)$ of $\Psi$ through $\Phi$ in the bicategory $\mathrm{Mod}\,\mathcal{E}$.

An analogous formula holds true for right liftings. Precisely, for modules $\Phi : \mathbf{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{B}$ and $\Psi : \mathbf{C} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{B}$, the right lifting $\mathrm{Hom}_{\mathbf{B}}(\Phi, \Psi) : \mathbf{C} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{A}$ has a component given by the equalizer:

$$\cdot \longrightarrow \mathrm{Hom}_{B_0}(\varphi, \psi) \rightrightarrows \mathrm{Hom}_{B_0}(\varphi, \mathrm{Hom}_{B_0}(\beta, \psi))$$

Here is a classical result which allows to "tabulate" any module by means of functors. Its proof results particularly significant with the language of monads in $\mathrm{Span}\,\mathcal{E}$.

**Theorem.** (Bénabou [1]). *For any internal module $\Phi : \mathbf{C} \longrightarrow\!\!\!\!\!\!+\ \mathbf{D}$ there exist functors $H : \mathbf{A} \longrightarrow \mathbf{C}$ and $K : \mathbf{A} \longrightarrow \mathbf{D}$ and an invertible 2-cell $\sigma : \Phi \cong K_* \otimes_{\mathbf{A}} H^*$ in* Mod($\mathbf{C}, \mathbf{D}$). *Moreover, for any pair of functors $F : \mathbf{B} \longrightarrow \mathbf{C}$, $G : \mathbf{B} \longrightarrow \mathbf{D}$ and any 2-cell $\tau : G_* \otimes_{\mathbf{B}} F^* \Longrightarrow \Phi$, there exist a functor $S : \mathbf{B} \longrightarrow \mathbf{A}$, uniquely defined up to isomorphism, endowed with invertible 2-cells $F \cong HS$ and $G \cong KS$ such that the induced $\hat{\tau} : G_* \Longrightarrow \Phi \otimes_{\mathbf{C}} F_*$ factors as:*

$$\begin{tikzcd} K_* \otimes_{\mathbf{A}} S_* \arrow[r, "\hat{\sigma} \otimes_{\mathbf{A}} S_*"] \arrow[d, "\cong"'] & \Phi \otimes_{\mathbf{C}} H_* \otimes_{\mathbf{A}} S_* \arrow[d, "\cong"] \\ G_* \arrow[r, "\hat{\tau}"'] & \Phi \otimes_{\mathbf{C}} F_* \end{tikzcd}$$

*Proof.* Let $\Phi : (C_0, \gamma) \longrightarrow\!\!\!\!\!\!+\ (D_0, \delta)$ be given and denote by:

$$\begin{tikzcd} & A_0 \arrow[dl, "h"'] \arrow[dr, "k"] & \\ C_0 & & D_0 \end{tikzcd}$$

the component of $\Phi$.

By the actions $kh^\circ\gamma \Longrightarrow kh^\circ$ and $\delta kh^\circ \Longrightarrow kh^\circ$ of $\gamma$ and $\delta$, respectively, on $\Phi$, one obtains two 2-cells:

$$h^\circ \gamma h \Longrightarrow k^\circ kh^\circ h$$
$$k^\circ \delta k \Longrightarrow k^\circ kh^\circ h$$

and then the pullback diagram:

$$(3) \qquad \begin{tikzcd} \alpha \arrow[r, "\varphi"] \arrow[d, "\psi"'] & h^\circ \gamma h \arrow[d] \\ k^\circ \delta k \arrow[r] & k^\circ kh^\circ h \end{tikzcd}$$

in the category Span $\mathcal{E}(A_0, A_0)$. It is not difficult to check that $\alpha$ is a monad, thus providing $A_0$ with the structure of an internal category $\mathbf{A}$. For this, only the universal property of the above pullback is used: the identity of the monad

$\alpha$ appears in the following diagram:

$$\begin{tikzcd}
1_A \arrow[r, "\eta_h"] \arrow[d, "\eta_k"'] \arrow[dr] & h^\circ h \arrow[dr] & \\
k^\circ k \arrow[dr] & \alpha \arrow[r] \arrow[d] & h^\circ \gamma h \arrow[d] \\
 & k^\circ \delta k \arrow[r] & k^\circ k h^\circ h
\end{tikzcd}$$

(here the external diagram commutes because the actions on $\Phi$ of the identities both of $\gamma$ and $\delta$ are identities). Analogously, the composition of $\alpha$ arises by the commutativity of the external square in the next diagram, which takes into account associativity of the actions of $\gamma$ and $\delta$ and also the mixed associative law of the actions on $\Phi$ on both sides:

$$\begin{tikzcd}
\alpha^2 \arrow[r, "\varphi^2"] \arrow[d, "\psi^2"'] \arrow[dr] & h^\circ \gamma h h^\circ \gamma h \arrow[r] & h^\circ \gamma^2 h \arrow[d, "\mu_{\mathbf{C}}"] \\
k^\circ \delta k k^\circ \delta k \arrow[d] & \alpha \arrow[r] \arrow[d] & h^\circ \gamma h \arrow[d] \\
k^\circ \delta^2 k \arrow[r, "\mu_{\mathbf{D}}"'] & k^\circ \delta k \arrow[r] & k^\circ k h^\circ h
\end{tikzcd}$$

Again, the universal property of pullbacks shows directly that $h$ and $k$ become the correspondence on objects of functors $H : \mathbf{A} \longrightarrow \mathbf{C}$ and $K : \mathbf{A} \longrightarrow \mathbf{D}$ respectively, whose correspondences on arrows are given by the sides $\varphi$ and $\psi$ of the pullback (3).

To show that $\Phi \cong K_* \otimes_{\mathbf{A}} H^*$, one has to show that the diagonal of the following commutative square:

$$\begin{tikzcd}
\delta k h^\circ \gamma \arrow[r] \arrow[d] & \delta k h^\circ \arrow[d] \\
k h^\circ \gamma \arrow[r] & k h^\circ
\end{tikzcd}$$

is the coequalizer of the compositions which appear as the sides of the following diagram:

$$\begin{array}{ccccc}
\delta k\alpha h^\circ\gamma & \xrightarrow{\delta k\varphi h^\circ\gamma} & \delta kh^\circ\gamma hh^\circ\gamma & \xrightarrow{\delta kh^\circ\gamma\epsilon\gamma} & \delta kh^\circ\gamma^2 \\
{\scriptstyle \delta k\psi h^\circ\gamma}\downarrow & & & & \downarrow{\scriptstyle \delta kh^\circ\mu_{\mathbf{C}}} \\
\delta kk^\circ\delta kh^\circ\gamma & \xrightarrow[\delta\epsilon\delta kh^\circ\gamma]{} & \delta^2kh^\circ\gamma & \xrightarrow[\mu_{\mathbf{D}}kh^\circ\gamma]{} & \delta kh^\circ\gamma
\end{array}$$

This calculation is long, but straightforward.

Consider now a pair of functors $F : \mathbf{B} \longrightarrow \mathbf{C}$ and $G : \mathbf{B} \longrightarrow \mathbf{D}$ and an arrow between modules $\tau : G_* \otimes_{\mathbf{B}} F^* \longrightarrow \Phi$. We describe the components of a functor $S : \mathbf{B} \longrightarrow \mathbf{A}$ which exhibits the pair $(H, K)$ as a tabulation of $\Phi$.

Suppose that $\mathbf{B} = (B_0, \beta)$ and that $f : B_0 \longrightarrow C_0$ and $g : B_0 \longrightarrow D_0$ are the correspondences on objects of, respectively, the functors $F$ and $G$. Then $\hat{\tau} : G_* \Longrightarrow \Phi \otimes_{\mathbf{C}} F^*$ gives rise to a $\bar{\tau} : 1_{\mathbf{B}} \Longrightarrow G^* \otimes_{\mathbf{D}} \Phi \otimes_{\mathbf{C}} F_*$ whose component, by the previous theorem, is a span $\beta \Longrightarrow g^\circ kh^\circ f$. From this, by inserting the identity of $\beta$, one obtains a span:

$$gf^\circ \Longrightarrow kh^\circ$$

Now, by definition of 2-cells in $\mathrm{Span}\,\mathcal{E}$, there is a map $s : B_0 \longrightarrow A_0$ such that:

$$f = hs$$

$$g = ks$$

This is the correspondence on objects of the required functor $S$, whose correspondence on arrows is obtained as follows. Consider the action on arrows of the functors $F$ and $G$:

$$\beta \longrightarrow f^\circ\gamma f \cong s^\circ h^\circ\gamma hs$$

$$\beta \longrightarrow g^\circ\delta g \cong s^\circ k^\circ\gamma ks$$

These two actions induce a unique 2-cell $\beta \Longrightarrow s^\circ\alpha s$ by the universal property of the following pullback (obtained by composing (3) on the left by $s^\circ$ and on the right by $s$):

$$\begin{array}{ccc}
s^\circ\alpha s & \longrightarrow & s^\circ h^\circ\gamma hs \\
\downarrow & & \downarrow \\
s^\circ k^\circ\delta ks & \longrightarrow & s^\circ k^\circ kh^\circ hs
\end{array} \qquad \square$$

We end this section by showing that $\mathrm{Mod}\,\mathcal{E}$ is the "smallest" bicategory containing $\mathrm{Cat}\,\mathcal{E}$ and closed under right Kan extension.

**Theorem.** (Bénabou [1]). *Consider the locally fully-faithful homomorphism:*

$$(\,)^* : \mathrm{Cat}\,\mathcal{E} \longrightarrow (Mod\,\mathcal{E})^{\mathrm{op}}$$

*which is the identity on objects and takes any functor $F$ to the module $F^*$. If $\mathcal{B}$ admits right Kan extensions, any locally fully-faithful homomorphism*

$$G : \mathrm{Cat}\,\mathcal{E} \longrightarrow \mathcal{B}$$

*which is the identity on objects factors uniquely, up to isomorphisms, through $(\,)^*$.*

*Proof.* We just indicate how to define on arrows a homomorphism $G' : (\mathrm{Mod}\,\mathcal{E})^{\mathrm{op}} \longrightarrow \mathcal{B}$ which extends $G$ through $(\,)^*$.

For any $\Phi : \mathbf{C} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{D}$ in $\mathrm{Mod}\,\mathcal{E}$, consider its canonical tabulation $\Phi \cong K_* \otimes_{\mathbf{A}} H^*$, then define $G'(\Phi) = [GK, GH]$, where $[\;,\;]$ denotes right Kan extension in $\mathcal{B}$.

To check that $G'(F) \cong [GK, GH]$ for a functor $F : \mathbf{D} \longrightarrow \mathbf{C}$, when $F^* \cong K_* \otimes_{\mathbf{A}} H^*$ is a tabulation of $F^*$, one has to observe that the module $K_* \otimes_{\mathbf{A}} H^*$ is the left Kan extension $[H, K]$ of $H$ through $K$ in the bicategory of modules. $\square$

## 5. Internal presheaves.

Any object $I$ of $\mathcal{E}$ is regarded as a discrete, internal category.

**Definition.** *An* internal presheaf *of* $\mathbf{C}$ *is a module* $\gamma : \mathbf{C} \longrightarrow\!\!\!\!\!\!\!+\;\; 1$. *A morphism of internal presheaves $\alpha$ and $\beta$ is a morphism $\alpha \Longrightarrow \beta$ of modules.*

Denote by $\mathcal{E}^{\mathbf{C}^{\mathrm{op}}}$ the category of internal presheaves. Then, observing that objects of $\mathcal{E}$ are in (natural) bijection with spans $1 \longrightarrow\!\!\!\!\!\!\!+\;\; 1$, one has that the hom $\mathcal{E}^{\mathbf{C}^{\mathrm{op}}}(\alpha, \beta)$ can be represented by the right Kan extension $[\alpha, \beta] : 1 \longrightarrow\!\!\!\!\!\!\!+\;\; 1$:

$$\mathcal{E}^{\mathbf{C}^{\mathrm{op}}}(\alpha, \beta) \cong \mathrm{Hom}_{\mathcal{E}}(1, [\alpha, \beta])$$

More generally, let $\mathcal{P}\mathbf{C}$ denote the category enriched on $\mathrm{Span}\,\mathcal{E}$ whose objects are modules $\mathbf{C} \longrightarrow\!\!\!\!\!\!\!+\;\; I$ and whose hom between the objects $\gamma : \mathbf{C} \longrightarrow\!\!\!\!\!\!\!+\;\; I$ and $\gamma' : \mathbf{C} \longrightarrow\!\!\!\!\!\!\!+\;\; J$ is the span $I \longrightarrow\!\!\!\!\!\!\!+\;\; J$ given by the right Kan extension $[\gamma; \gamma']$ (observe that identities and composition in $\mathcal{P}\mathbf{C}$ are defined by the universal property of right Kan extensions).

$\mathcal{P}\mathbf{C}$ is the first example of a *locally internal category* in the sense of [2] and [3]. In particular, when $\mathbf{C} = 1$ is the trivial internal category, $\mathcal{P}\mathbf{C}$ reproduces $\mathcal{E}$ as a locally internal category over itself, in a sense that will be more clear later.

Any object $\gamma : \mathbf{C} \longrightarrow\!\!\!\!\!\!+\;\, I$ of $\mathcal{P}\mathbf{C}$ over $I$ gives rise to the internal category $\mathrm{Int}_{\mathbf{C}}(\gamma)$ having $I$ as object of objects and the monad $[\gamma, \gamma]$ as object of arrows. This is the *internal full subcategory* generated by $\gamma$.

Let $\mathbf{B} = (B_0, \beta)$ be an internal category. Define now a $\mathbf{B}$-diagram $\Delta$ of $\mathcal{E}^{\mathbf{C}^{\mathrm{op}}}$ by means of the following data: an object $\delta : \mathbf{C} \longrightarrow\!\!\!\!\!\!+\;\, B_0$ of $\mathcal{P}\mathbf{C}$ over $B_0$, together with an internal functor $\mathbf{B} \longrightarrow \mathrm{Int}_{\mathbf{C}}(\delta)$ which is the identity on objects. It is easy to check that to assign a $\mathbf{B}$-diagram of $\mathcal{E}^{\mathbf{C}^{\mathrm{op}}}$ is equivalent to assign a module $\mathbf{C} \longrightarrow\!\!\!\!\!\!+\;\, \mathbf{B}$ whose component is $\delta$: the action $\beta\delta \Longrightarrow \delta$ corresponds to the action on arrows $\beta \Longrightarrow [\delta, \delta]$ of the given functor. In particular, when $\mathbf{B} = I$ is a discrete category, an $I$-diagram of $\mathcal{E}^{\mathbf{C}^{\mathrm{op}}}$ is exactly an object of $\mathcal{P}\mathbf{C}$ over $I$.

Denoting by $\Delta : \mathbf{B} \longrightarrow \mathcal{E}^{\mathbf{C}^{\mathrm{op}}}$ a $\mathbf{B}$-diagram of $\mathcal{E}^{\mathbf{C}^{\mathrm{op}}}$, one has a bijection:

$$\frac{\Delta : \mathbf{B} \longrightarrow \mathcal{E}^{\mathbf{C}^{\mathrm{op}}}}{\hat{\Delta} : \mathbf{C} \longrightarrow\!\!\!\!\!\!+\;\, \mathbf{B}} \tag{4}$$

Now it is easy to prove that this bijection is *stable under substitution* in the following sense. Suppose $F = (f, \varphi)$ is a functor $\mathbf{A} = (A_0, \alpha) \longrightarrow (B_0, \beta) = \mathbf{B}$: the *substitution* of $\Delta$ along $F$ is the $\mathbf{A}$-diagram $F \cdot \Delta$ whose object over $A_0$ is given by the composition

$$\mathbf{C} \overset{\delta}{\longrightarrow\!\!\!\!\!\!+\;\,} B_0 \overset{f^\circ}{\longrightarrow\!\!\!\!\!\!+\;\,} A_0$$

and whose functor $\mathbf{A} \longrightarrow \mathrm{Int}_{\mathbf{C}}(f^\circ\delta)$ is given by (the identity on objects and) the correspondence on arrows:

$$\alpha \longrightarrow f^\circ\beta f \longrightarrow f^\circ[\delta, \delta]f \cong [f^\circ\delta, f^\circ\delta]$$

By recalling (Theorem of Section 3) that the component of a module of type $\Phi F^*$ is $\varphi f^\circ$, it is easy to check that the bijection (4) is stable under substitutions:

$$\frac{\mathbf{A} \xrightarrow{F} \mathbf{B} \xrightarrow{\Delta} \mathcal{E}^{\mathbf{C}^{\mathrm{op}}}}{\mathbf{C} \overset{\hat{\Delta}}{\longrightarrow\!\!\!\!\!\!+\;\,} \mathbf{B} \overset{F^*}{\longrightarrow\!\!\!\!\!\!+\;\,} \mathbf{A}}$$

As already observed, for $\mathbf{B} = I$ (discrete):

$$\frac{I \longrightarrow \mathcal{E}^{\mathbf{C}^{\mathrm{op}}}}{\mathbf{C} \longrightarrow\!\!\!\!\!\!+\;\, I}$$

hence, in particular for $I = 1$ one has that $\mathcal{P}\mathbf{C}$ can be regarded as the (Span $\mathcal{E}$-enriched) category of families of objects of $\mathcal{E}^{\mathbf{C}^{\mathrm{op}}}$ indexed in $\mathcal{E}$.

**Theorem.** *The tabulation* $\Phi \cong K_* \otimes_{\mathbf{A}} H^*$ *of any object* $\Phi : \mathbf{C} \longrightarrow\!\!\!\!\!\!+\ I$ *of* $\mathcal{P}\mathbf{C}$ *is such that* $H : \mathbf{A} \longrightarrow \mathbf{C}$ *is a discrete fibration. Conversely, any discrete fibration can be constructed in this way for a suitable internal presheaf.*

*Proof.* For the first part it is enough to observe that the diagram:

$$\begin{tikzcd} \alpha \arrow[r, "\varphi"] \arrow[d, "\psi"'] & h^\circ \gamma h \arrow[d] \\ k^\circ \delta k \arrow[r] & k^\circ k h^\circ h \end{tikzcd}$$

which defines the category structure of $\mathbf{A}$ as in the proof of Bénabou's theorem of Section 4 (here we use the same symbols) is a pullback if and only if the corresponding diagram under the adjunction $- \cdot h^\circ \dashv - \cdot h$ is a pullback:

$$\begin{tikzcd} \alpha h^\circ \arrow[r, "\hat{\psi}"] \arrow[d, "\psi h^\circ"'] & h^\circ \gamma \arrow[d] \\ k^\circ \delta k h^\circ \arrow[r] & k^\circ k h^\circ \end{tikzcd}$$

Now, when $\mathbf{D} = I$ is a discrete category, then $\delta$ is the identity of $D_0$ and $\hat{\varphi}$ is an invertible 2-cell.

Conversely, the condition $\hat{\varphi} : \alpha h^\circ \overset{\cong}{\Longrightarrow} h^\circ \gamma$ allows one to show that the span $(h, t) : C \longleftarrow A \longrightarrow 1$ is the component of an internal presheaf $\Phi : \mathbf{C} \longrightarrow\!\!\!\!\!\!+\ 1$ such that its tabulation is given by the pair of functors $H : \mathbf{A} \longrightarrow \mathbf{C}$ and $T : \mathbf{A} \longrightarrow 1$. $\square$

## 6. Internal completeness and cocompleteness.

In this section let $\mathcal{E}$ admit pullback stable coequalizers. If $\mathbf{C}$ is any internal category, consider the (unique) functor $T : \mathbf{C} \longrightarrow 1$. Then, by regarding objects of $\mathcal{E}$ as spans $1 \longrightarrow\!\!\!\!\!\!+\ 1$, composition with $T_*$ becomes a functor:

$$- \otimes_1 T_* : \mathcal{E} \longrightarrow \mathcal{E}^{\mathbf{C}^{\mathrm{op}}}$$

For an $X : 1 \longrightarrow\!\!\!\!\!\!+\ 1$, the object $X \otimes_1 T_*$ of $\mathcal{E}^{\mathbf{C}^{\mathrm{op}}}$ provides exactly a diagram $1 \longrightarrow\!\!\!\!\!\!+\ \mathcal{E}^{\mathbf{C}^{\mathrm{op}}}$ which is easily described when $\mathcal{E} =$ Sets. Namely, in this case, $X \otimes_1 T_*$ is the presheaf $\mathbf{C}^{\mathrm{op}} \longrightarrow$ Sets which is *constant at* $X$.

We know that $- \otimes_1 T_*$ has a left adjoint (Section 3), namely the functor $- \otimes_{\mathbf{C}} T^*$. Hence it is natural to call $- \otimes_1 T_*$ the *internal colimit* functor and to say that $\mathcal{E}$ is *internally cocomplete.* But, when $\mathcal{E}$ is locally cartesian closed, the functor $- \otimes_{\mathbf{C}} T_*$ admits also a right adjoint, namely the functor $\mathrm{Hom}^{\mathbf{C}}(T_*, -)$, hence in this case $\mathcal{E}$ is also *internally complete.*

More generally, consider a functor $F : \mathbf{C} \longrightarrow \mathbf{D}$. One has that the functor $- \otimes_{\mathbf{C}} F^*$ provides the *left Kan extension* along $F$, in the sense of beeing the left adjoint to the functor:

$$- \otimes_{\mathbf{D}} F_* : \mathcal{P}\mathbf{D} \longrightarrow \mathcal{P}\mathbf{C}$$

The right Kan extension along $F$ exists when $\mathcal{E}$ is locally cartesian closed. It needs not exist under weaker assumptions on $\mathcal{E}$. However, when $F$ has a left adjoint $G$, by taking into account (Section 3) that this means exactly that $F_* \cong G^*$, one has that $\mathrm{Hom}^{\mathbf{C}}(F_*, -) \cong - \otimes_{\mathbf{C}} G_*$. Hence:

**Theorem.** *If $\mathcal{E}$ admits pullback stable coequalizers, then there exist left Kan extensions along arbitrary internal functors and right Kan extensions along functors which admit a right adjoint.*

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