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A FAMILY OF PERMUTABLE COMPLETELY REGULAR SEMIGROUPS

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To the memory of Umberto Gasapina

Semigroups, ideal extensions of a right (left)-zero semigroup by a completely simple semigroup with zero adjoint, whose congruences are pairwise permutable are completely determined.

Introduction.

Several authors investigated the characterization of semigroups (in some classes) by the structure of their congruence lattice and the main results in this topic have been collected in two wide surveys ([9], [10]).

Completely regular semigroups, i.e. semigroups which are union of groups, form a class of semigroups for which this classification problem has been solved with respect to the most important lattice types. In particular the characterizations of completely regular semigroups with an M-symmetric, modular, distributive, relatively complemented, complemented, modular and complemented, or Boolean congruence lattice can be found in [9], and completely regular semigroups with semimodular or strongly semimodular lattice of congruences are characterized in [8] (see also [10]).

The characterization of semigroups (in some classes) whose congruences are pairwise permutable (shortly permutable semigroups) is a very similar

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problem, introduced by Hamilton in [6]; results on this subject are also collected in [9] and [10], but the most recent ones (see, for instance [5]). Permutable completely regular semigroups are described in [4] (see also [10]) modulo two quotient semigroups that are in their turn permutable. These quotient semigroups belong to a subfamily of completely regular semigroups, namely they are ideal extension of a right (left)-zero semigroup by a completely simple semigroup with zero adjoint.

This paper is devoted to prove the characterization of permutable semigroups which are ideal extension of a right-zero semigroup by a completely simple semigroup with zero adjoint, completing the results of [4].

Then some examples of these semigroups are given in order to show how to use the characterization theorem in a constructive way.

At last all completely regular semigroups whose congruences are a totally ordered set are described.

1. We recall the following

Definition 1.1. A semigroup is called a *permutable semigroup* if its congruences are pairwise permutable (see [6]).

Statement 1.1. *Let S be an ideal extension of a right-zero semigroup S_0 by a completely simple semigroup $S_1 = M(G; I, \Lambda; P)$ with zero adjoint. If S is permutable, then $|I|, |\Lambda| \leq 2$.*

Proof. It is well known that if S is permutable, then S_1 is permutable (see for instance [4], a). Then the statement follows from ([4], Statement 1.2).

Notation. In the sequel of this paper S will denote the disjoint union of a right-zero semigroup S_0, which is an ideal of S, and of a completely simple semigroup $S_1 = M(G; I, \Lambda; P)$ where $|I| \leq 2$, $|\Lambda| \leq 2$ and P is normalized, this means that at most one entry of P, to be denoted by p, is different from e, the identity of G. We will write $S = A \oplus B$ to indicate that S is the disjoint union of A and B.

Statement 1.2. *$Sa = a$ for each $a \in S_0$.*

Proof. Let $a \in S_0$ then $Sa = Saa \subseteq S_0a = a$.

Statement 1.3. *If S is a permutable semigroup, then $S_0S_1 = S_0$.*

Proof. Suppose by way of contradiction, $S_0S_1 \subset S_0$. Let τ be the relation $\{(x, y) \in S \times S \mid x = y \text{ or } x, y \in S_1 \cup S_0S_1\}$, τ is a congruence on S. In fact let $(x, y) \in \tau$ with $x \neq y$, then $x, y \in S_1 \cup S_0S_1$, hence $cx, cy \in S_1 \cup S_0S_1$ for every $c \in S$, moreover, if $c \in S_1$ then both xc and yc are in $S_1 \cup S_0S_1$ and if $c \in S_0$ then Statement 1.2 implies $xc = yc = c$. Thus (cx, cy), $(xc, yc) \in \tau$. Now denote by ρ the Rees congruence modulo S_0. Let $x \in S_0 \setminus S_0S_1$, $y \in S_1$, $z \in S_0S_1$ then $(x, z) \in \rho$, $(z, y) \in \tau$, hence $(x, y) \in \rho\tau$ and $(x, y) \notin \tau\rho$. Since S is permutable we obtain $S_0S_1 = S_0$.

Statement 1.4. *If $S_0S_1 = S_0$, then for each $a \in S_0$ there exists $\lambda \in \Lambda$ such that $a = a(i, p_{\lambda i}^{-1}, \lambda)$ for every $i \in I$. Moreover for every $\lambda \in \Lambda$ there exist $a \in S_0$ such that $a = a(i, p_{\lambda i}^{-1}, \lambda)$ for every $i \in I$.*

Proof. For each $a \in S_0$ there exist $b \in S_0$ and $x \in S_1$ such that $a = bx$; let $x = (j, g, \lambda)$ $(j \in I, g \in G, \lambda \in \Lambda)$, thus $a(i, p_{\lambda i}^{-1}, \lambda) = b(j, g, \lambda)(i, p_{\lambda i}^{-1}, \lambda) = a$. Now let $\lambda \in \Lambda$ and $b \in S_0$. Put $a = b(i, e, \lambda)$, then $a(i, p_{\lambda i}^{-1}, \lambda) = b(i, e, \lambda)(i, p_{\lambda i}^{-1}, \lambda) = b(i, e, \lambda) = a$.

Statement 1.5. *If S is a permutable semigroup, then $a_1S_1 \cap a_2S_1 \neq \emptyset$ implies $a_1S_1 = a_2S_1$ for every $a_1, a_2 \in S_0$. Moreover there exist $a_1, a_2 \in S_0$ such that either $S_0 = a_1S_1$ or $S_0 = a_1S_1 \oplus a_2S_1$.*

Proof. First we prove that if $a_1S_1 \cap a_2S_1 \neq \emptyset$ with $a_1, a_2 \in S_0$, then $a_1S_1 = a_2S_1$. Suppose $a_1x = a_2y$ with $x, y \in S_1$ and $y = (i, g, \lambda)$ $(i \in I, g \in G, \lambda \in \Lambda)$. Let $a_2 = a_2(i, p_{\mu i}^{-1}, \mu)$ and put $z = (i, p_{\lambda i}^{-1}g^{-1}p_{\mu i}^{-1}, \mu)$. Thus $a_1xz = a_2(i, g, \lambda)(i, p_{\lambda i}^{-1}g^{-1}p_{\mu i}^{-1}, \mu) = a_2(i, p_{\mu i}^{-1}, \mu) = a_2$. Hence $a_2 \in a_1S_1$ and $a_2S_1 \subseteq a_1S_1$. Similarly we can prove that and $a_1S_1 \subseteq a_2S_1$. That being stated, let τ be the relation defined by $\tau = \{(x, y) \in S \times S \mid x, y \in S_1 \text{ or } x, y \in zS_1$ for some $z \in S_0\}$. The relation τ is a congruence on S, in fact if $x, y \in S_1$ then for every $c \in S_1$, $cx, cy, xc, yc \in S_1$ and for every $c \in S_0$, $cx, cy \in cS_1$ and $xc = yc = c \in cS_1$ by Statement 1.3 and analogously if $x, y \in zS_1$. Moreover the semigroup S/τ is a right-zero semigroup T (isomorphic to $S_0/\tau|_{S_0}$) with identity adjoint. Since every homomorphic image of a permutable semigroup is permutable (see [4], c), S/τ is a permutable semigroup, then T is a permutable semigroup, whence $|T| \leq 2$. Thus if $|T| = 1$ then there exists $a_1 \in S_0$ such that $S_0 = a_1S_1$ otherwise, if $|T| = 2$, there exist $a_1, a_2 \in S_0$ such that $S_0 = a_1S_1 \oplus a_2S_1$.

Statement 1.6. *Let $S_0 = a_1S_1$ or $S_0 = a_1S_1 \oplus a_2S_1$ and suppose $|I| = 2$ and $p_{\lambda i} = e$ for every $i \in I$. If $b_h = a_h(i, g, \lambda)$ $(h \in \{1, 2\})$ for some $i \in I$, $g \in G$ then the following relations hold: $b_h = b_h(i, e, \lambda)$ for each $i \in I$, $b_h(i, g, \mu) = b_h(j, g, \mu)$ for every $g \in G$, $\mu \in \Lambda$, $i, j \in I$.*

Proof. Let $b_h = a_h(i, g, \lambda)$, then

$$b_h(i, e, \lambda) = a_h(i, g, \lambda)(i, e, \lambda) = a_h(i, g, \lambda) = b_h$$

and $b_h(j, e, \lambda) = a_h(i, g, \lambda)(j, e, \lambda) = a_h(i, g, \lambda)b_h$, so $b_h(i, e, \lambda) = b_h$ for each $i \in I$. Now $b_h(i, g, \mu) = b_h(j, e, \lambda)(i, g, \mu) = b_h(j, g, \mu)$ for every $g \in G$, $\mu \in \Lambda$, $i, j \in I$.

Notation. We recall that S is the disjoint union of a right-zero semigroup S_0, ideal of S, and of a completely simple semigroup $S_1 = M(G; I, \Lambda; P)$ where $|I| \leq 2$, $|\Lambda| \leq 2$ and P is normalized, so in the sequel we suppose that $p_{\lambda i} = e$ for every $i \in I$. Then taking into account of previous Statement, for every $a_h \in S_0$ with $a_h = a_h(i, e, \lambda)$, we will write $a_h(g, \nu)$ to denote the element $a_h(i, g, \nu)$ $(= a_h(j, g, \nu)$ if $|I| = 2)$ and, denoting by K a subset of G, we will use the notations $a_h(K, \nu)$ and $a_h(K, \Lambda)$ to indicate respectively the sets $\{a_h(i, k, \nu) \mid i \in I, k \in K\}$ and $\{a_h(i, k, \lambda) \mid i \in I, k \in K, \lambda \in \Lambda\}$. Moreover writing $S_0 = a_1 S_1$ or $S_0 = a_1 S_1 \oplus a_2 S_1$ we assume that the elements a_h $(h \in \{1, 2\})$ satisfy the condition $a_h = a_h(e, \lambda)$. This assumption depends on Statements 1.4, 1.5 and 1.6.

Definition 1.2. For every $a_h \in S_0$ with $a_h = a_h(e, \lambda)$ we call *stabilizer* of a_h in G the set $H_{a_h} = \{g \in G \mid a_h = a_h(g, \lambda)\}$.

Statement 1.7. *The stabilizer of a_h in G is a subgroup of G, and if $b_h = a_h(g, \lambda)$ for some $g \in G$ $(h \in \{1, 2\})$ then the stabilizer of b_h is $g^{-1}H_{a_h}g$.*

Proof. Let $a_h = a_h(e, \lambda)$ and let $g, k \in H_{a_h}$, then $a_h = a_h(g, \lambda) = a_h(k, \lambda)$. Thus we have $a_h(gk, \lambda) = a_h(g, \lambda)(j, k, \lambda) = a_h(j, k, \lambda) = a_h$ and $a_h = a_h(e, \lambda) = a_h(g, \lambda)(j, g^{-1}, \lambda) = a_h(j, g^{-1}, \lambda) = a_h(i, g^{-1}, \lambda)$. Now let $b_h = a_h(g, \lambda)$ and $x \in H_{b_h}$. Then $a_h = a_h(e, \lambda) = a_h(g, \lambda)(i, g^{-1}, \lambda) = b_h(i, g^{-1}, \lambda) = b_h(x, \lambda)(i, g^{-1}, \lambda) = a_h(g, \lambda)(x, \lambda)(i, g^{-1}, \lambda) = a_h(gxg^{-1}, \lambda)$ and $gxg^{-1} \in H_{a_h}$.

Statement 1.8. *Let $S_0 = a_1 S_1$ or $S_0 = a_1 S_1 \oplus a_2 S_1$. For every subgroup M of G containing the stabilizer of a_h the relation $\theta_M = \{(x, y) \in S \times S \mid x = y$ or $x, y \in a_h(Mg, \nu)$ for some $g \in G$, $\nu \in \Lambda$ or $x, y \in a_h(Mg, \Lambda)$ and there exists $m \in M$ such that $a_h(mg, \mu) = a_h(mg, \lambda)$ with $\mu \neq \lambda\}$ is a congruence on S. Conversely for each congruence θ on S there exists a subgroup $M = M_{\theta, a_h}$ containing the stabilizer of a_h such that for every pair $(x, y) \in \theta \cap (a_h S_1 \times a_h S_1)$, $x, y \in a_h(Mg, \Lambda)$ for some $g \in G$ and if P has a non trivial entry p then $(a_h(m_1 g, \lambda), a_h(m_2 g, \mu)) \in \theta$ with $\mu \neq \lambda$ implies $p \in g^{-1}Mg$.*

Proof. Let M be a subgroup containing the stabilizer of a_h, the relation θ_M is obviously an equivalence relation. Let $(x, y) \in \theta_M$ and $c \in S$. By definition of θ_M if $x \neq y$ then $x, y \in S_0$, hence by Statement 1.2 $cx = x$, $cy = y$ and, if $c \in S_0$, $xc = yc = c$. If $c = (j, k, \mu) \in S_1$ and $x, y \in a_h(Mg, \nu)$, then $xc, yc \in a_h(Mgp_{\nu j}k, \mu)$; at last suppose that there exist $m, m_1, m_2 \in M$, $g \in G$ such that $a_h(mg, \mu) = a_h(mg, \lambda)$ with $\mu \neq \lambda$ and $x = a_h(m_1 g, \mu)$, $y = a_h(m_2 g, \lambda)$, then $xc = a_h(m_1 g p_{\mu j} k, \mu)$ and $yc = a_h(m_2 g p_{\lambda j} k, \mu)$. Suppose $p_{\mu j} \neq p_{\lambda j}$, hence $p_{\lambda j} = e$ and $p = p_{\mu j} \neq e$. Then $a_h(mg, \mu) = a_h(mg, \lambda)$ implies $a_h(mgpg^{-1}m^{-1}, \lambda) = a_h(mg, \mu)(j, g^{-1}m^{-1}, \lambda) = a_h(mg, \lambda)(j, g^{-1}m^{-1}, \lambda) = a_h(e, \lambda) = a_h$, so that $mgpg^{-1}m^{-1} \in H_{a_h}$, whence $gpg^{-1} \in M$, so $xc = a_h(m_1 gpg^{-1}gk, \mu) = a_h(m_3 gk, \mu)$ for some $m_3 \in M$. Thus $(xc, yc), (cx, cy) \in \theta_M$ and θ_M is a congruence on S.
Conversely let θ be a congruence on S. Put $M = \{g \in G \mid (a_h(g, \lambda), a_h) \in \theta\}$. It is easy to prove that M is a subgroup of G containing the stabilizer of a_h. Let $x, y \in a_h S_1$ with $(x, y) \in \theta$ and suppose $x = a_h(g_1, \nu)$ and $y = a_h(g_2, \mu)$. Let $i \in I$ such that $p_{\lambda i} = e$ for every $\lambda \in \Lambda$. Then $(a_h(g_1, \nu)(i, g_1^{-1}, \lambda), a_h(g_2, \mu)(i, g_1^{-1}\lambda)) \in \theta$ and since $a_h(g_1, \nu)(i, g_1^{-1}, \lambda) = a_h$, and $a_h(g_2, \mu)(i, g_1^{-1}, \lambda) = a_h(g_2 g_1^{-1}, \lambda)$ we obtain $g_2 g_1^{-1} \in M$, whence $x = a_h(g_1, \nu) \in a_h(Mg_1, \nu)$ and $y = a_h(g_2 g_1^{-1} g_1, \mu) \in a_h(Mg_1, \mu)$. Now suppose $(a_h(m_1 g, \lambda), a_h(m_2 g, \mu)) \in \theta$ with $\mu \neq \lambda$ and $p = p_{\mu j} \neq e$ whence $p_{\lambda j} = e$ and $p_{\mu i} = e$ for $i \neq j$; thus $(a_h(m_1 g, \lambda)(j, g^{-1}m_2^{-1}, \lambda), a_h(m_2 g, \mu)(j, g^{-1}m_2^{-1}, \lambda)) \in \theta$, hence $(a_h(m_1 m_2^{-1}, \lambda), a_h(m_2 gpg^{-1}m_2^{-1}, \lambda)) \in \theta$ whence $(a_h(m_1 m_2^{-1}, \lambda)(i, m_2 gpg^{-1}m_2^{-1}, \lambda), a_h) \in \theta$. Thus $m_1 gpg^{-1}m_2^{-1} \in M$, so $p \in g^{-1}Mg$ and the statement is proved.

Remark 1.1. It is easy to prove that if $b_h = a_h(g, \lambda)$ for some $g \in G$, $(h \in \{1, 2\})$, then M_{θ, b_h} is conjugate to M_{θ, a_h}.

Remark 1.2. Let M be a subgroup of G containing the stabilizer of a_h, if P has a non trivial entry p, then by the previous proof the relation $\varphi_M = \{(x, y) \in S \times S \mid x = y$ or $xy \in a_h(Mg, \nu)$ for some $g \in G$, $\nu \in \Lambda$ or $x, y \in a_h(Mg, \Lambda)$ and $p \in g^{-1}Mg\}$ is a congruence on S.

Statement 1.9. *Let S be a permutable semigroup and let $S_0 = a_1 S_1$ or $S_0 = a_1 S_1 \oplus a_2 S_1$. Then every pair of subgroups of G containing the stabilizer of a_h is permutable.*

Proof. Let M, N be two subgroups of G both containing the stabilizer of a_h and let θ_M and θ_N be the congruences built according Statement 1.8. Let $x = a_h(m, \lambda)$, then for each $y \in a_h(MNm, \lambda)$, $(x, y) \in \theta_N \theta_M$, hence, by permutability of S, $(x, y) \in \theta_M\theta_N$, but if $(x, z) \in \theta_M$ there exists $\nu \in \Lambda$ such that $z \in a_h(M, \nu)$ for some $\nu \in \Lambda$, and if $(z, y) \in \theta_N$ then $y \in a_h(NM, \eta)$ for some $\eta \in \Lambda$, whence $MNM \subseteq NM$ so $MN \subseteq NM$. Analogously we can prove the opposite inclusion.

Statement 1.10. *Let S be a permutable semigroup and let $S_0 = a_1S_1$ or $S_0 = a_1S_1 \oplus a_2S_1$. If $\Lambda = \{\lambda, \mu\}$ then there exist an integer h ($h \in \{1, 2\}$), and an element $g \in G$ such that $a_h(g, \lambda) = a_h(g, \mu)$.*

Proof. Suppose by way of contradiction that for every a_h and for every $g \in G$ it is $a_h(g, \lambda) \neq a_h(g, \mu)$. If there are $g_1, g_2 \in G$ such that $a_h(g_1, \lambda) = a_h(g_2, \mu)$, then, choosing $i \in I$ such that $p_{\mu i} = p_{\lambda i} = e$, we have $a_h = a_h(g_1, \lambda)(i, g_1^{-1}, \lambda) = a_h(g_2, \mu)(i, g_1^{-1}, \lambda) = a_h(g_2g_1^{-1}, \lambda)$, whence $g_2g_1^{-1}$ belongs to the stabilizer of a_h, hence $a_h(g_2, \lambda) = a_h(g_1, \lambda) = a_h(g_2, \mu)$. Then, for every $g_1, g_2 \in G$, $a_h(g_1, \lambda) \neq a_h(g_2, \mu)$. So $\psi = \{(x, y) \in S \times S \mid x \in u(I, G, \nu),\ y \in v(I, G, \nu)$ for some $\nu \in \Lambda,\ u, v \in S\}$ is an equivalence relation on S. Moreover ψ is a congruence on S, in fact let $x = u(g, \nu)$, $y = v(k, \nu)$ for some $\nu \in \Lambda$, $g, k \in G$, $u, v \in S$ and let $c \in S$, then $cx = cu(g, \nu)$ and $cy = cv(k, \nu)$, and if $c \in S_0$ then $xc = yc = c$, otherwise if $c = (j, g', \eta) \in S_1$ $xc = u(g, \nu)c = u(gp_{\nu j}g', \eta)$, $yc = v(k, \nu)c = v(kp_{\nu j}g', \eta)$, whence $(cx, cy) \in \psi$ and $(xc, yc) \in \psi$. The relation ψ is not permutable with the Rees congruence ρ modulo S_0. In fact let $x = u(i, g_1, \lambda)$ with $u \in S_1$, $z = a_h(i, g_1, \lambda)$, $y = a_h(i, g_1, \mu)$, then $(x, z) \in \psi$ and $(z, y) \in \rho$ whence $(x, y) \in \psi\rho$, but $(x, y) \notin \rho\psi$. Then there are $g \in G$ and $a_h \in S_0$ such that $a_h(g, \lambda) = a_h(g, \mu)$.

Statement 1.11. *Let S be a permutable semigroup and let $S_0 = a_1S_1$ or $S_0 = a_1S_1 \oplus a_2S_1$ and let $\Lambda = \{\lambda, \mu\}$. Let M_{a_h} be a proper subgroup of G containing the stabilizer H_{a_h} of a_h and the entries of P, then for every $m \in M_{a_h}$ and $n \in G \setminus M_{a_h}$ there is an element g belonging either to $M_{a_h}n$ or to $\langle H_{a_h}, n\rangle m$ such that $a_h(g, \lambda) = a_h(g, \mu)$.*

Proof. By way of contradiction suppose that there are $m \in M_{a_h}$ and $n \in G \setminus M_{a_h}$ such that $a_h(g, \lambda) \neq a_h(g, \mu)$ for every $g \in M_{a_h}n \cup \langle H_{a_h}, n\rangle m$. If there are g_1, g_2 both of them belonging either to $M_{a_h}n$ or to $\langle H_{a_h}, n\rangle m$ such that $a_h(g_1, \lambda) = a_h(g_2, \mu)$, then by the same argument used in the previous proof we deduce $a_h(g, \lambda) = a_h(g, \mu)$ for some $g \in M_{a_h}n \cup \langle H_{a_h}, n\rangle m$, so we can assume $a_h(g_1, \lambda) \neq a_h(g_2, \mu)$ and $\tau = \{(x, y) \in S \times S \mid x = y$ or $x, y \in a_h(M_{a_h}, \Lambda)$ or $x, y \in a_h(M_{a_h}g, \nu)$, or $x, y \in a_h(M_{a_h}g, \Lambda)$ and there exists $m \in M_{a_h}$ such that $a_h(mg, \mu) = a_h(mg, \lambda)$ with $\mu \neq \lambda\}$ is an equivalence relation on S. Moreover τ is a congruence. In fact suppose $(x, y) \in \tau$ with $x \neq y$, with the same arguments used in Statement 1.8 we prove that $(cx, cy) \in \tau$ and that if $c \in S_0$ or if $x, y \notin a_h(M_{a_h}, \Lambda)$ then $(xc, yc) \in \tau$. Then suppose $c = (i, g, \mu) \in S_1$ and $x = a_h(m, \nu)$, $y = a_h(n, \lambda)$ for some $m, n \in M_{a_h}$, $\nu, \lambda \in \Lambda$, hence $xc = a_h(mp_{\nu i}g, \mu)$, $yc = a_h(np_{\lambda i}g, \mu)$, and since the entries of P belong to M_{a_h}, then $(xc, yc) \in \tau$. Now put $N_{a_h} = \langle H_{a_h}, n\rangle$ and consider the congruences $\theta_{M_{a_h}}$ and $\theta_{N_{a_h}}$ defined in Statement 1.8; then $(a_h(m, \lambda), a_h(e, \mu)) \in \tau$, $(a_h(e, \mu), a_h(n, \mu)) \in \theta_{N_{a_h}}$, hence $(a_h(m, \lambda), a_h(n, \mu)) \in \tau\theta_{N_{a_h}}$. But the τ-class containing $a_h(n, \mu)$ is $a_h(M_{a_h}n, \mu)$ and the $\theta_{N_{a_h}}$-class containing $a_h(m, \lambda)$ is $a_h(N_{a_h}m, \lambda)$, hence τ and $\theta_{N_{a_h}}$ are not permutable. Then there is $g \in M_{a_h}n \cup \langle H_{a_h}, n\rangle m$ such that $a_h(g, \lambda) = a_h(g, \mu)$.

Statement 1.12. *Let S be a permutable semigroup and let $S_0 = a_1S_1 \oplus a_2S_1$. Then the stabilizers of a_1 and a_2 generate G.*

Proof. Suppose by way of contradiction that $\langle H_{a_1}, H_{a_2}\rangle = K \subset G$. The relation $\sigma = \{(x, y) \in S \times S \mid x = y \text{ or } x, y \in a_1(G, \Lambda)\}$ is obviously an equivalence relation. Let $c \in S$ and $(x, y) \in \sigma$ with $x \neq y$, then $x = a_1(g, \lambda)$, $y = a_1(m, \nu)$ for some $g, m \in G$, $\lambda, \nu \in \Lambda$. Then $cx = x$, $cy = y$ and if $c \in S_0$ then $xc = yc = c$, otherwise if $c = (j, g', \eta) \in S_1$ then $xc = a_1(g, \lambda)(j, g', \eta) \in a_1(G, \Lambda)$ and so $yc = a_1(m, \nu)(j, g', \eta) \in a_1(G, \Lambda)$, then $(cx, cy), (xc, yc) \in \sigma$. The relation $\psi_K = \{(x, y) \in S \times S \mid x = y$ or $x, y \in a_1(Kg, \nu) \cup a_2(Kg, \nu)$ for some $g \in G$, $\nu \in \Lambda$ or $x, y \in a_1(Kg, \Lambda) \cup a_2(Kg, \Lambda)$ and there exist $k \in K$, $h \in \{1, 2\}$ such that $a_h(kg, \mu) = a_h(kg, \lambda)$ with $\mu \neq \lambda\}$ is a congruence on S. In fact let $(x, y) \in \psi_K$ with $x \neq y$, if $x, y \in a_hS_1$ the same argument used to prove that θ_K is a congruence can be applied. So let $x \in a_1S_1$ and $y \in a_2S_1$. Obviously for every $c \in S$ it is $cx = x$ and $cy = y$ hence $(cx, cy) \in \psi_K$. If $c \in S_0$ it is $xc = yc = c$ hence $(xc, yc) \in \psi_K$, then suppose $c = (j, m, \eta)$, $x = a_1(kg, \lambda)$, $y = a_2(kg, \nu)$, $y = a_2(kg, \nu)$, whence $xc = a_1(kgp_{\lambda j}m, \eta)$, $yc = a_2(kgp_{\nu j}m, \eta)$, so if $p_{\lambda j} = p_{\nu j}$ then $(xc, yc) \in \psi_K$, otherwise $\lambda \neq \nu$ and there exist $k \in K$, $h \in \{1, 2\}$ such that $a_h(kg, \nu) = a_h(kg, \lambda)$ that implies that both $p_{\lambda j}$ and $p_{\nu j}$ are in $g^{-1}Kg$, hence $xc \in a_1(Kgm, \eta)$, $yc \in a_2(Kgm, \eta)$, and again $(xc, yc) \in \psi_K$. Moreover the congruences σ and ψ_K are not permutable. In fact, being $a_2 = a_2(e, \lambda)$, $(a_2, a_1(e, \lambda)) \in \psi_K$ and $(a_1(e, \lambda), a_1(g, \lambda)) \in \sigma$ for every $g \in G$. Hence $(a_2, a_1(g, \lambda)) \in \psi_K\sigma$ for every $g \in G$. Since $(a_2, x) \in \sigma$ if and only if $a_2 = x$ and $(a_2, y) \in \psi_K$ implies $y \in a_1(K, \Lambda) \cup a_2(K, \Lambda)$ then for each $g \in G \setminus K$ $(a_2, a_1(g, \lambda)) \in \psi_K\sigma$ but $(a_2, a_1(g, \lambda)) \notin \sigma\psi_K$. Then $K = G$.

Theorem 1.1. *Let S be a completely regular semigroup which is the ideal extension of a right-zero semigroup S_0 by a semigroup $S_1 = M(G; I, \Lambda; P)$ (where P is normalized) with zero adjoint. Then S is permutable if and only if the following hold:*

(1) $|I| \leq 2$, $|\Lambda| \leq 2$

and denoting by e the identity of G and supposing $\lambda \in \Lambda$ *such that* $p_{\lambda i} = e$ *for every* $i \in I$ *either*

(2) $S_0 = a_1 S_1$ *with* $a_1 \in S_0$ *and* $a_1 = a_1(e, \lambda)$.
(3) $MN = NM$ *for every subgroups* M, N *of* G *containing the stabilizer of* a_1.
(4) *If* $|\Lambda| = 2$, *putting* $\Lambda = \{\lambda, \mu\}$, *there exists* $g \in G$ *such that* $a_1(g, \lambda) = a_1(g, \mu)$. *Moreover for every proper subgroup* M *of* G *containing each entry of* P *and a conjugate* $g^{-1} H_{a_1} g$ *of the stabilizer* H_{a_1} *of* a_1, *and for every* $m \in M$, $n \in G \setminus M$ *there exists* k *belonging either to* Mn *or to* $\langle g^{-1} H_{a_1} g, n \rangle m$ *such that* $a_1(gk, \lambda) = a_1(gk, \mu)$
or
(2′) $S_0 = a_1 S_1 \oplus a_2 S_1$ *with* $a_1, a_2 \in S_0$ *and* $a_1 = a_1(e, \lambda), a_2 = a_2(e, \lambda)$.
(3′) $MN = NM$ *for every subgroups* M, N *of* G *containing either the stabilizer of* a_1 *or the stabilizer of* a_2.
(4′) *If* $|\Lambda| = 2$, *putting* $\Lambda = \{\lambda, \mu\}$, *there exist* $g \in G$ *and some* $h \in \{1, 2\}$ *such that* $a_h(g, \lambda) = a_h(g, \mu)$. *Moreover for every* $h \in \{1, 2\}$ *and for every proper subgroup* M *of* G *containing each entry of* P *and a conjugate* $g^{-1} H_{a_h} g$ *of the stabilizer* H_{a_h} *of* a_h, *and for every* $m \in M$, $n \in G \setminus M$ *there exists* k *belonging either to* Mn *or to* $< g^{-1} H_{a_h} g, n > m$ *such that* $a_h(gk, \lambda) = a_h(gk, \mu)$.
(5′) *Each pair of subgroups* H, K *respectively conjugated to the stabilizers of* a_1 *and* a_2 *generate* G.

Proof. The only if part easily follows from previous Statements. To prove the converse we start showing that for every congruence ρ on S, $\rho \cap (S_0 \times S_1) \neq \emptyset$ implies $\rho = \omega$, where ω denotes, as usual, the universal congruence on S. Let $s_0 \in S_0$ and let $s_1 = (i, g, \eta) \in S_1$ with $(s_0, s_1) \in \rho$, then, being $s_0 = (I, G, \eta)s_0$, $\{s_0\} \times \{(I, G, \eta)(i, g, \lambda)\} \subseteq \rho$ whence $\{s_0\} \times (I, G, \lambda) \subseteq \rho$. Suppose $s_0 = a_1(k, \nu)$ and choose i such that $p_{\nu i} = e$, then $\{s_0\} \times (I, G, \lambda) \subseteq \rho$ implies $\{s_0(i, k^{-1}, \lambda)\} \times \{(I, G, \lambda)(i, k^{-1}, \lambda)\} = \{a_1\} \times (I, G, \lambda) \subseteq \rho$ and $\{a_1\} \times \{a_1(G, \lambda)\} \subseteq \rho$ hence $\{a_1\} \times \{a_1(G, \lambda) \cup (I, G, \lambda)\} \subseteq \rho$. Moreover if $S_0 = a_1 S_1 \oplus a_2 S_1$, $\{a_1\} \times (I, G, \lambda) \in \rho$ implies $\{a_2 a_1\} \times \{a_2(G, \lambda)\} \subseteq \rho$ hence $\{a_1\} \times \{a_2(G, \lambda)\} \subseteq \rho$ then $\{a_1\} \times \{a_1(G, \lambda) \cup a_2(G, \lambda) \cup (I, G, \lambda)\} \subseteq \rho$. Now by (4) or (4’) there exist $h \in \{1, 2\}$ and $g \in G$ such that $a_h(g, \lambda) = a_h(g, \mu)$, whence $\{a_h(G, \lambda)\} \times \{a_h(G, \mu)\} \subseteq \rho$. Then, being $\{a_h(G, \lambda)\} \times (I, G, \lambda) \subseteq \rho$, it is also $\{a_h(G, \lambda)\} \times (I, G, \mu) \subseteq \rho$, whence $\rho = \omega$.

Now let ρ, τ be two non universal congruences on S, to prove that they are permutable it is enough to show that their restrictions to S_0 and to S_1 are permutable. Since S_1 is a permutable semigroup and the restrictions to S_1 of congruences on S are congruences on S_1 we have only to prove that $\rho|_{S_0}$ and $\tau|_{S_0}$ are permutable. Let $b \in S_0$ and suppose $b = a_1(g_1, \nu)$. Put

$M_{a_1} = \{g \in G \mid (a_1(g, \lambda), a_1) \in \rho\}$. From Statement 1.8, if $b\rho \subseteq a_1 S_1$ then either

(a) $b\rho = \{b\}$, or
(b) $b\rho = \{a_1(M_{a_1} g_1, \nu)\}$, or
(c) $b\rho = \{a_1(M_{a_1} g_1, \Lambda)\}$, or
(d) $b\rho = S_0$.

If there exists $a_2(g_2, \mu)$ such that $(b, a_2(g_2, \mu)) \in \rho$, then we easily obtain $(a_1, a_2(g_2 g_1^{-1} H_{a_1}, \lambda)) \in \rho$, hence $(b, a_2(g_2 g_1^{-1} H_{a_1} g_1, \nu)) \in \rho$ so $(a_2(g_2, \mu), a_2(g_2 g_1^{-1} H_{a_1} g_1, \nu) \in \rho$, whence $(a_2, a_2(g_2 g_1^{-1} H_{a_1} g_1 g_2^{-1}, \lambda)) \in \rho$. Thus $g_2 g_1^{-1} H_{a_1} g_1 g_2^{-1}$ is contained in H_{a_2} and $H_{a_2} = G$ by (5′). Analogously we obtain $H_{a_1} = G$ whence $b\rho = S_0$. So in any case for every non universal congruence ρ the ρ-classes of b are of the types (a), (b), (c), (d).

If for every $b \in S_0$ both the ρ and the τ-classes are of types (a), (b), (d) (or (a), (c), (d)) then $\rho|_{S_0}$ and $\tau|_{S_0}$ are permutable by (3) and (3′). In fact putting $N_{a_1} = \{g \in G \mid (a_1(g, \lambda), a_1) \in \tau\}$ we obtain $b\rho\tau = \{a_1(N_{a_1} M_{a_1} g_1, \nu)\}$ and $b\tau\rho = \{a_1(M_{a_1} N_{a_1} g_1, \nu)\}$ (respectively $b\rho\tau = \{a_1(N_{a_1} M_{a_1} g_1, \Lambda)\}$ and $b\tau\rho = \{a_1(M_{a_1} N_{a_1} g_1, \Lambda)\}$). Otherwise suppose that there exists $b = a_1(g_1, \nu) \in S_0$ such that $b\rho = \{a_1(M_{a_1} g_1, \nu)\}$ and $b\tau = \{a_1(N_{a_1} g_1, \Lambda)\}$ with $|\Lambda| = 2$. Obviously for each $m \in M_{a_1}$, $n \in N_{a_1}$, $\mu \in \Lambda$, $(a_1(m g_1, \nu), a_1(n g_1, \mu)) \in \rho\tau$. If $m \in N_{a_1}$, it is straightforward that $(a_1(m g_1, \nu), a_1(n g, \mu)) \in \tau\rho$. Suppose $g_1^{-1} m g_1 \notin g_1^{-1} N_{a_1} g_1$; by definition N_{a_1} contains the stabilizer of a_1 so $g_1^{-1} N_{a_1} g_1$ is a proper subgroup containing $g_1^{-1} H_{a_1} g_1$ and we prove that $g_1^{-1} N_{a_1} g_1$ contains all the entries of P. This is obvious if all the entries of P are e, then suppose $p_{\nu j} \neq e$, $(a_1(g_1, \lambda), a_1(g_1, \nu)) \in \tau$ implies $(a_1(g_1, \lambda)(j, g_1^{-1}, \lambda), a_1(g_1, \nu)(j, g_1^{-1}, \lambda)) \in \tau$ whence $(a_1, a_1(g_1 p_{\nu j} g_1^{-1}, \nu)) \in \tau$ that implies $g_1 p_{\nu j} g_1^{-1} \in N_{a_1}$ that is $p_{\nu j} \in g_1^{-1} N_{a_1} g_1$. Thus there is an element g either in $g_1^{-1} N_{a_1} g_1 g_1^{-1} m g_1$ or in $g_1^{-1} M_{a_1} g_1 g_1^{-1} n g_1$ such that $a_1(g_1 g, \lambda) = a_1(g_1 g, \nu)$. If $g = g_1^{-1} n'' m g_1$ then $a_1(n'' m g_1, \lambda) = a_1(n'' m g_1, \nu)$ whence $a_1(m g_1, \nu)\tau = \{a_1(N_{a_1} m g_1, \Lambda)\}$; so $(a_1(m g_1, \nu), a_1(n' m g_1, \lambda)) \in \tau$ for every $n' \in N_{a_1}$, $(a_1(n' m g_1, \lambda), a_1(m' n' m g_1, \lambda)) \in \rho$ for every $m' \in M_{a_1}$, hence by (3) and (3′) $(a_1(m g_1, \nu), a_1(n g_1, \lambda)) \in \tau\rho$. If $g = g_1^{-1} m'' n g_1$ then $a_1(m'' n g_1, \lambda) = a_1(m'' n g_1, \nu)$ whence $a_1(n g_1, \nu)\rho = \{a_1(M_{a_1} n g_1, \Lambda)\}$; so $(a_1(m' n g_1, \nu), a_1(n g_1, \lambda)) \in \rho$ for every $m' \in M_{a_1}$, $(a_1(n' m' n g_1, \nu), a_1(m' n g_1, \nu)) \in \tau$ for every $n' \in M_{a_1}$ hence by (3) and (3′) $(a_1(m g_1, \nu), a_1(n g_1, \nu)) \in \tau\rho$. Then $\rho|_{S_0}$ and $\tau|_{S_0}$ are permutable and the statement is proved.

Remark 1.3. Theorem 1.1 is easily proved to be equivalent to Theorem 1.2 in [4]. Moreover in Theorem 1.1 conditions (2) and (2′) can be rewritten in the following way

(2) S_1 acts transitively on the right on S_0.
(2′) S_1 acts on the right on S_0 generating two orbits.

Remark 1.4. If S_0 is a left-zero semigroup, we can easily reformulate Theorem 1.1, simply considering S_1 acting on the left on S_0.

2. In this section we give some examples of permutable semigroups which are the disjoint union of a right-zero semigroup S_0 which is an ideal of S and of a completely simple semigroup $S_1 = M(G; I, \Lambda; P)$ (i.e. semigroups which are ideal extension of S_0 by S_1 with zero adjoint).

First we consider the case $|\Lambda| = 1$, i.e. $S_1 = I \times G$. Theorem 1.1 becomes:

Corollary 2.1. *Let S be the ideal extension of a right-zero semigroup S_0 by a left group $S_1 = I \times G$ with zero adjoint. Then S is permutable if and only if the following hold:*

(1) $|I| \leq 2$;
(2) *S_1 acts on the right on S_0 generating at most two orbits;*
(3) *$MN = NM$ for every subgroups M, N of G containing the stabilizer of an element of S_0.*
(4) *If $a_1, a_2 \in S_0$ belong to different orbits then their stabilizers generate G.*

If S_1 acts transitively on S_0, let H be a subgroup of G such that every pair of subgroups containing H are permutable (at least $H = G$ satisfies this condition). Let S_0 be a right-zero semigroup disjoint by S_1 with $|S_0| = [G : H]$, where $[G : H]$ denotes, as usual, the cardinality of the set of right cosets of H in G. Fix an one to one map between S_0 and the set of right cosets of H in G and denote by a_{Hg} the element of S_0 corresponding to the coset Hg. In the set $S = S_1 \oplus S_0$ introduce a product in this way: if x, y are both in the same S_i then they are composed by the product of S_i, if $x = (j, g) \in S_1$ and $y = a_{Hk} \in S_0$ then put $xy = y$ and $yx = a_{Hkg}$. The groupoid so generated is a permutable semigroup that is the ideal extension of a right-zero-semigroup S_0 by a left group S_1 with zero adjoint.

If S_1 does not act transitively on S_0, let H, K be subgroups of G such that every pair of subgroups both containing either H or K are permutable, and such that $\langle H, K\rangle = G$ (for instance two maximal subgroups of G, if any, or at least $H = K = G$ satisfy these conditions). Let S_0 be a right-zero semigroup disjoint from S_1 with $S_0 = S_0' \oplus S_0''$ with $|S_0'| = [G : H]$ and $|S_0''| = [G : K]$. Fix an one to one map between S_0' (S_0'') and the set of right cosets of $H(K)$ in G and denote by a_{Hg} (a_{Kg}) the element of S_0 corresponding to the coset Hg (Kg).

In the set $S = S_1 \oplus S_0$ introduce a product in this way: if x, y are both in the same S_i then they are composed by the product of S_i, if $x = (j, g) \in S_1$ and $y = a_{Hk} \in S_0$ ($y = a_{Kk} \in S_0$) then put $xy = y$ and $yx = a_{Hkg}$ ($yx = a_{Kkg}$). The groupoid so generated is a permutable semigroup that is the ideal extension of a right-zero-semigroup S_0 by a left goup S_1 with zero adjoint. Conversely each permutable semigroup that is the ideal extension of a right-zero-semigroup S_0 by a left group S_1 with zero adjoint can be constructed in one of the previous ways.

Then consider the case $|G| = 1$, i.e. $S_1 = I \times \Lambda$ where we suppose $|\Lambda| \geq 2$. Theorem 1.1 becomes:

Corollary 2.2. *Let S be a completely regular semigroup which is the ideal extension of a right-zero semigroup S_0 by a semigroup $S_1 = I \times \Lambda$ with zero adjoint and suppose $|\Lambda| \geq 2$. Then S is permutable if and only if the following hold:*

(1) $|I| \leq 2, |\Lambda| = 2$

(2) *S_1 acts on the right on S_0 generating at most two orbits*

(3) *Putting $\Lambda = \{\lambda, \mu\}$, there exists $a_1 \in S_0$ such that $a_1(i, \lambda) = a_1(i, \mu)$ for every $i \in I$.*

Hence all permutable semigroup we are concerning with are:

a) $S_1 = \{(i, \lambda), (i, \mu)\}$, $S_0 = \{a\}$, both right-zero semigroups with $a(i, \lambda) = (i, \lambda)a = a(i, \mu) = (i, \mu)a = a$ (i.e. a right-zero semigroup of order 2 with zero adjoint)

b) $S_1 = \{(i, \lambda), (i, \mu)\}$, $S_0 = \{a, b\}$, both right-zero semigroups with $a(i, \lambda) = (i, \lambda)a = a(i, \mu) = (i, \mu)a = a$ and $b(i, \lambda) = (i, \lambda)b = b(i, \mu) = (i, \mu)b = b$

c) $S_1 = \{(i, \lambda), (i, \mu)\}$, $S_0 = \{a, b, c\}$, both right-zero semigroups with $a(i, \lambda) = (i, \lambda)a = a(i, \mu) = (i, \mu)a = a$ and $b(i, \lambda) = (i, \lambda)b = (i, \mu)b = b$, $b(i, \mu) = c$, $c(i, \mu) = (i, \mu)c = (i, \lambda)c = c$, $c(i, \lambda) = b$

d) $S_1 = \{(i, \lambda), (i, \mu), (j, \lambda), (j, \mu)\}$ rectangular band, $S_0 = \{a\}$, with $a(i, \lambda) = (i, \lambda)a = a(i, \mu) = (i, \mu)a = a(j, \lambda) = (j, \lambda)a = a(j, \mu) = (j, \mu)a = a$ (i.e. a rectangular band of order 4 with zero adjoint)

e) $S_1 = \{(i, \lambda), (i, \mu), (j, \lambda), (j, \mu)\}$ rectangular band, $S_0 = \{a, b\}$ right-zero semigroup with $a(i, \lambda) = (i, \lambda)a = a(i, \mu) = (i, \mu)a = a(j, \lambda) = (j, \lambda)a = a(j, \mu) = (j, \mu)a = a$ and $b(i, \lambda) = (i, \lambda)b = b(i, \mu) = (i, \mu)b = b(j, \lambda) = (j, \lambda)b = b(j, \mu) = (j, \mu)b = b$

f) $S_1 = \{(i, \lambda)(i, \mu), (j, \lambda), (j, \mu)\}$ rectangular band, $S_0 = \{a, b, c\}$ right-zero semigroup with $a(i, \lambda) = (i, \lambda)a = a(i, \mu) = (i, \mu)a = a(j, \lambda) =$

$(j,\lambda)a = a(j,\mu) = (j,\mu)a = a$ and $b(i,\lambda) = (i,\lambda)b = (i,\mu)b = b(j,\lambda) = (j,\lambda)b = (j,\mu)b = b$, $b(i,\mu) = b(j,\mu) = c$, $c(i,\mu) = (i,\mu)c = (i,\lambda)c = c(j,\mu) = (j,\mu)c = (j,\lambda)c = c$, $c(i,\lambda) = c(j,\lambda) = b$.

Such semigroups were yet described in [3], n. 3.

Now we consider the case $S_1 = M(G; I, \Lambda; P)$ where $|I| \leq 2$, $|\Lambda| \leq 2$ and P is normalized (the case $S_1 = I \times G \times \Lambda$ is hereby included and it is obtained when all the entries of P are the identity e of G). For each given G there are examples of permutable semigroups S which are disjoint union of a right-zero semigroup, ideal of S, and of S_1 whether with S_1 acting transitively on S_0 or not. In fact S_1 with zero adjoint is an example of a permutable semigroup where S_1 acts transitively on S_0, moreover if we take $S_0 = \{a_1, a_2\}$ with $a_h(i, g, \nu) = a_h$, for every $h \in \{1, 2\}$, $i \in I$, $g \in G$, $\nu \in \Lambda$ or $S_0 = \{a_1, a_2, a_3\}$ with $a_1(i, g, \nu) = a_1$, for every $i \in I$, $g \in G$, $\nu \in \Lambda$, $a_2(i, g, \lambda) = a_2$, $a_2(i, g, \mu) = a_3$, $a_3(i, g, \mu) = a_3$, $a_3(i, g, \lambda) = a_2$, for every $i \in I$, $g \in G$, we again get permutable semigroups S where S_1 does not act transitively on S_0 and the stabilizers of a_h for $h \in \{1, 2\}$ are G. Conversely each permutable semigroup which is the disjoint union of a right-zero semigroup S_0, ideal of S, and of a completely simple semigroup $S_1 = M(G; I, \Lambda; P)$, satisfying the condition $a_h(g, \lambda) = a_h$ for every a_h such that $a_h(e, \lambda) = a_h$, is one of the three above semigroups.

In general to construct a permutable semigroup which is the disjoint union of a right-zero semigroup, ideal of S, and of a given semigroup $S_1 = M(G; I, \Lambda; P)$, we have to select one or two subgroups of G satisfying either condition (3) or condition (3′) and (5′) according with S_1 acts transitively on S_0 or not, then we have to construct an S_0 such that conditions (2) and (4) or (2′) and (4′) are satisfied, in such construction the selected subgroup are conjugates to the stabilizers of the elements of S_0. Obviously conditions (4) and (4′) may be fulfilled in different ways depending on the choice of the subgroups, and we have not a general tool to use. We notice that when these subgroups are either maximal or normal, then conditions (4) and (4′) become simpler.

As an example we construct all the permutable semigroups which are disjoint union of a right-zero semigroup, ideal of S, and of a semigroup $S_1 = M(S_3; I, \Lambda; P)$, where S_3 denotes as usual the substitution group on three elements. Obviously we can assume $|I| \leq 2$, $|\Lambda| \leq 2$ and we can suppose that $\lambda \in \Lambda$ and $i \in I$ satisfy the condition: $p_{\lambda i} = e$ for each $i \in I$, $p_{\mu i} = e$ for each $\mu \in \Lambda$. In the sequel we denote by $T_n(h)$ a right-zero semigroup of order n, whose elements are indexed by h, different h are used to indicate disjoint zero-semigroups in which S_0 can be decomposed. In order to make the construction, first we consider the following semigroups $S = S_0 \oplus S_1$ where S_1 acts on the right on S_0 and $S_1 a = a$ for every $a \in S_0$

1) $S_1 = M(S_3; I, \Lambda; P)$, $S_0 = T_1(h) = \{a_h\}$ with the right action of S_1 on S_0 determined by $a_h(S_3, \Lambda) = a_h$;

2) $S_1 = M(S_3; I, \Lambda; P)$ with $|\Lambda| = 2$, $S_0 = T_2(h) = \{a_h, b_h\}$ with the right action of S_1 on S_0 determined by $a_h(S_3, \lambda) = a_h$, $a_h(S_3, \mu) = b_h$;

3) $S_1 = M(S_3; I, \Lambda; P)$ such that all the entries of P are in the alternating subgroup A_3, $S_0 = T_2(h) = \{a_h, b_h\}$ with the right action of S_1 on S_0 determined by $a_h(A_3, \Lambda) = a_h$, $a_h(A_3 g, \Lambda) = b_h$ for every $g \in S_3 \setminus A_3$;

4) $S_1 = M(S_3; I, \Lambda; P)$ with $|\Lambda| = 2$ such that all the entries of P are in A_3, $S_0 = T_3(h) = \{a_h, b_h, c_h\}$ with the right action of S_1 on S_0 determined by $a_h(A_3, \Lambda) = a_h$, $a_h(A_3 g, \lambda) = b_h$, $a_h(A_3 g, \mu) = c_h$ for every $g \in S_3 \setminus A_3$;

5) $S_1 = M(S_3; I, \Lambda; P)$ with $|\Lambda| = 2$ and p of order 2 a possible entry of P, $S_0 = T_4(h) = \{a_h, b_h, c_h, d_h\}$ with the right action of S_1 on S_0 determined by $a_h(A_3, \lambda) = a_h$, $a_h(A_3 p, \lambda) = b_h$, $a_h(A_3, \mu) = c_h$, $a_h(A_3 p, \mu) = d_h$;

6) $S_1 = M(S_3; I, \Lambda; P)$ such that all the entries of P are e (i.e. $S_1 = I \times S_3 \times \Lambda$), $S_0 = T_3(h) = \{a_h, b_h, c_h\}$ with the right action of S_1 on S_0 determined by $a_h(H, \Lambda) = a_h$, $a_h(Hg, \Lambda) = b_h$, $a_h(Hg^2, \Lambda) = c_h$ for a subgroup H of G of order 2 and $g \in S_3$ of order 3;

7) $S_1 = M(S_3; I, \Lambda; P)$ with $|\Lambda| = 2$ all the entries of P equal to e (i.e. $S_1 = I \times S_3 \times \Lambda$), $S_0 = T_4(h) = \{a_h, b_h, c_h, d_h\}$ with the right action of S_1 on S_0 determined by $a_h(H, \Lambda) = a_h$, $a_h(Hg, \Lambda) = b_h$, $a_h(Hg^2, \lambda) = c_h$, $a_h(Hg^2, \mu) = d_h$ for a subgroup H of G of order 2 and $g \in S_3$ of order 3;

8) $S_1 = M(S_3; I, \Lambda; P)$ with $|\Lambda| = 2$ such that all the entries of P are in a subgroup H of order 2 of G, $S_0 = T_5(h) = \{a_h, b_h, c_h, d_h, e_h\}$ with the right action of S_1 on S_0 determined by $a_h(H, \Lambda) = a_h$, $a_h(Hg, \lambda) = b_h$, $a_h(Hg^2, \lambda) = c_h$, $a_h(Hg, \mu) = d_h$, $a_h(Hg^2, \mu) = e_h$ for $g \in S_3$ of order 3;

9) $S_1 = M(S_3; I, \Lambda; P)$ with $|\Lambda| = 2$ and p of order 3 a possible entry of P, $S_0 = T_6(h) = \{a_h, b_h, c_h, d_h, e_h, f_h\}$ with the right action of S_1 on S_0 determined by $a_h(H, \lambda) = a_h$, $a_h(Hp, \lambda) = b_h$, $a_h(Hp^2, \lambda) = c_h$, $a_h(H, \mu) = d_h$, $a_h(Hp, \mu) = e_h$, $a_h(Hp^2, \mu) = f_h$, for a subgroup H of G of order 2.

Then, taking $h = 1$, the semigroups of types 1), 3), 4), 6), 7), 8) are permutable semigroups in fact $S_0 = a_1 S_1$ with $a_1 \in S_0$ and $a_1 = a_1(e, \lambda)$. Moreover in case 1) the stabilizer of a_1 is S_3, in cases 3) and 4) it is the alternating subgroup A_3 and in cases 6), 7) and 8) is a subgroup H of order 2, hence $MN = NM$ for every subgroups M, N of G containing the stabilizer of a_1. Moreover if $|\Lambda| = 2$ then there is $g \in G$ such that $a_1(g, \lambda) = a_1(g, \mu)$.

Finally if P has an entry different from e then this entry belongs to the stabilizer of a_1, and, by the maximality of these stabilizers, condition (4) of Theorem 1.1 is satisfied. Conversely if S is a permutable semigroups that is the disjoint union of a right-zero semigroup S_0, ideal of S, and of $S_1 = M(S_3; I, \Lambda; P)$ where S_1 acts transitively on S_0, then S is isomorphic to a semigroup of 1), 3), 4), 6), 7), 8). In fact conditions (2) and (3) of Theorem 1.1 imply $S_0 = a_1 S_1$ with the stabilizer of a_1 either equal to S_3, or to a A_3, or to a subgroup of order 2. Moreover condition (4) implies that the possible entry of P different from e is in a subgroup conjugate to the stabilizer of a_1. Since the stabilizer of $a_1(g, \lambda)$ is the conjugate of the stabilizer of a_1 by g, we can assume without loss of generality that this entry belongs to the stabilizer of a_1 (in the opposite case we consider $S_0 = (a_1(g, \lambda))S_1)$ hence we can assume that $a_1(g, \lambda) = a_1(g, \mu)$ for every g belonging to the stabilizer of a_1. Finally if k does not belong to the stabilizer of a_1 we can put $a_1(k, \lambda) = a_1(k, \mu)$ only if the possible entry of P different from e belongs to the conjugate of the stabilizer of a_1 by k, and since in all cases the stabilizer of a_1 is maximal, we can freely choose $a_1(k, \lambda) = a_1(k, \mu)$ or $a_1(k, \lambda) \neq a_1(k, \mu)$ for all the k such that kpk^{-1} belongs to the stabilizer of a_1 for all entries of P. We stress that semigroups of types 2), 5), 9) with $h = 1$ satisfy conditions (1), (2) and (3) of Theorem 1.1 but there is no g in S_3 such that $a_1(g, \lambda) = a_1(g, \mu)$, so condition (4) is not fulfilled.

Now suppose that S_1 does not act transitively on S_0.
Then we consider the following semigroups:

a) $S_1 = M(S_3; I, \Lambda; P)$, $S_0 = T_1(1) \oplus T_1(2)$ right-zero semigroup with the actions of S_1 on $T_1(1)$ and on $T_1(2)$ defined as in 1);

b) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$, $S_0 = T_1(1) \oplus T_2(2)$ right-zero semigroup with the action of S_1 on $T_1(1)$ defined as in 1) and the action of S_1 on $T_2(2)$ defined as in 2);

c) $S_1 = M(S_3; I, \Lambda; P)$ and all entries of P are in the alternating subgroup A_3, $S_0 = T_1(1) \oplus T_2(2)$ right-zero semigroup with the action of S_1 on $T_1(1)$ defined as in 1) and the action of S_1 on $T_2(2)$ defined as in 3);

d) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all entries of P are in the alternating subgroup A_3, $S_0 = T_2(1) \oplus T_2(2)$ right-zero semigroup with the action of S_1 on $T_2(1)$ defined as in 2) and the action of S_1 on $T_2(2)$ defined as in 3);

e) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all entries of P are in the alternating subgroup A_3, $S_0 = T_1(1) \oplus T_3(2)$ right-zero semigroup with the action of S_1 on $T_1(1)$ defined as in 1) and the action of S_1 on $T_3(2)$ defined as in 4);

f) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all entries of P are in the alternating subgroup A_3, $S_0 = T_2(1) \oplus T_3(2)$ right-zero semigroup with the action of S_1 on $T_2(1)$ defined as in 2) and the action of S_1 on $T_3(2)$ defined as in 4);

g) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ $S_0 = T_1(1) \oplus T_4(2)$ right-zero semigroup with the action of S_1 on $T_1(1)$ defined as in 1) and the action of S_1 on $T_4(2)$ defined as in 5);

h) $S_1 = M(S_3; I, \Lambda; P)$, and all the entries of P are equal to e, $S_0 = T_1(1) \oplus T_3(2)$ right-zero semigroup with the action of S_1 on $T_1(1)$ defined as in 1) and the action of S_1 on $T_3(2)$ defined as in 6);

i) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all the entries of P are equal to e, $S_0 = T_2(1) \oplus T_3(2)$ right-zero semigroup with the action of S_1 on $T_2(1)$ defined as in 2) and the action of S_1 on $T_3(2)$ defined as in 6);

j) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all the entries of P are equal to e, $S_0 = T_1(1) \oplus T_4(2)$ right-zero semigroup with the action of S_1 on $T_1(1)$ defined as in 1) and the action of S_1 on $T_4(2)$ defined as in 7);

k) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all the entries of P are equal to e, $S_0 = T_2(1) \oplus T_4(2)$ right-zero semigroup with the action of S_1 on $T_2(1)$ defined as in 2) and the action of S_1 on $T_4(2)$ defined as in 7);

l) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all the entries of P are in subgroup H of order 2 of S_3, $S_0 = T_1(1) \oplus T_5(2)$ right-zero semigroup with the action of S_1 on $T_1(1)$ defined as in 1) and the action of S_1 on $T_5(2)$ defined as in 8);

m) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all the entries of P are in subgroup H of order 2 of S_3, $S_0 = T_2(1) \oplus T_5(2)$ right-zero semigroup with the action of S_1 on $T_2(1)$ defined as in 2) and the action of S_1 on $T_5(2)$ defined as in 8);

n) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ $S_0 = T_1(1) \oplus T_6(2)$ right-zero semigroup with the action of S_1 on $T_1(1)$ defined as in 1) and the action of S_1 on $T_6(2)$ defined as in 9);

o) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all entries of P are equal to e, $S_0 = T_2(1) \oplus T_3(2)$ right-zero semigroup with the action of S_1 on $T_2(1)$ defined as in 3) and the action of S_1 on $T_3(2)$ defined as in 6);

p) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all entries of P are equal to e, $S_0 = T_3(1) \oplus T_3(2)$ right-zero semigroup with the action of S_1 on $T_3(1)$ defined as in 4) and the action of S_1 on $T_3(2)$ defined as in 6);

q) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all entries of P are equal to e, $S_0 = T_2(1) \oplus T_4(2)$ right-zero semigroup with the action of S_1 on $T_2(1)$ defined as in 3) and the action of S_1 on $T_4(2)$ defined as in 7);

r) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all entries of P are equal to e, $S_0 = T_3(1) \oplus T_4(2)$ right-zero semigroup with the action of S_1 on $T_3(1)$ defined as in 4) and the action of S_1 on $T_4(2)$ defined as in 7);

s) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all entries of P are equal to e, $S_0 = T_2(1) \oplus T_5(2)$ right-zero semigroup with the action of S_1 on $T_2(1)$ defined as in 3) and the action of S_1 on $T_5(2)$ defined as in 8);

t) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all entries of P are equal to e, $S_0 = T_3(1) \oplus T_5(2)$ right-zero semigroup with the action of S_1 on $T_3(1)$ defined as in 4) and the action of S_1 on $T_5(2)$ defined as in 8);

u) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all the entries of P are in A_3, $S_0 = T_2(1) \oplus T_6(2)$ right-zero semigroup with the action of S_1 on $T_2(1)$ defined as in 3) and the action of S_1 on $T_6(2)$ defined as in 9);

v) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all the entries of P are in A_3, $S_0 = T_3(1) \oplus T_6(2)$ right-zero semigroup with the action of S_1 on $T_3(1)$ defined as in 4) and the action of S_1 on $T_6(2)$ defined as in 9);

w) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all the entries of P are e, $S_0 = T_4(1) \oplus T_3(2)$ right-zero semigroup with the action of S_1 on $T_4(1)$ defined as in 5) and the action of S_1 on $T_3(2)$ defined as in 6);

y) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all the entries of P are e, $S_0 = T_4(1) \oplus T_4(2)$ right-zero semigroup with the action of S_1 on $T_4(1)$ defined as in 5) and the action of S_1 on $T_3(2)$ defined as in 7);

z) $S_1 = M(S_3; I, \Lambda; P)$, $|\Lambda| = 2$ and all the entries of P are in a subgroup H of order 2, $S_0 = T_4(1) \oplus T_5(2)$ right-zero semigroup with the action of S_1 on $T_4(1)$ defined as in 5) and the action of S_1 on $T_5(2)$ defined as in 8).

All the subgroups of types a)-z) are permutable semigroups, in fact $S_0 = a_1 S_1 \oplus a_2 S_1$ with $a_1, a_2 \in S_0$ and $a_1 = a_1(e, \lambda)$, $a_2 = a_2(e, \lambda)$. Moreover the stabilizers of a_1 and a_2 are either S_3, or A_3, or a subgroup H of order 2, hence $MN = NM$ for every subgroups M, N of G containing the stabilizer of some a_h. Moreover the stabilizers of a_1 and of a_2 in all cases satisfies (5′) of Theorem 1.1. Moreover if $|\Lambda| = 2$ then there are $g \in G$ and $a_h \in S_0$ such that $a_h(g, \lambda) = a_h(g, \mu)$ and by the maximality of the stabilizers of a_h conditions (4′) of Theorem 1.1 is satisfied. Conversely if S is a permutable semigroup that is the disjoint union of a right zero semigroup S_0, ideal of S, and of $S_1 = M(S_3; I, \Lambda; P)$ where S_1 does not act transitively on S_0, then S is isomorphic to a semigroup of types a)–z). In fact conditions (2′) and (3′) of Theorem 1.1 imply $S_0 = a_1 S_1 \oplus a_2 S_1$ with the stabilizers of a_1, a_2 either equal to S_3, or to A_3 or to a subgroup of order 2 and condition (5′) implies that the stabilizers are neither both equal to A_3, nor both of order 2. Moreover it is easy to notice that $S_1 \oplus a_1 S_1$ and $S_1 \oplus a_2 S_1$ are subsemigroups of S and at least one of them is a permutable semigroup that is the disjoint union of a right-zero semigroup $a_1 S_1$ or $a_2 S_1$, ideal of S, and of S_1 where S_1 acts transitively on the right-zero semigroup. Then condition (4′) gives the classification taking into account the previous remarks on condition (4) on permutable semigroups with a unique orbit.

3. The classification of completely regular semigroups whose congruences form a totally ordered set (shortly completely regular Δ-semigroups) can be easily deduced by some results in [1] and [2], and completely regular Δ-semigroups form a subclass of the permutable semigroups here described, so we state the following theorem for sake of completeness.

Theorem 3.1. *A completely regular semigroup S is a Δ-semigroup if and only if either*

(1) *S is a group whose normal semigroup form a chain with respect to the inclusion (shortly Δ-group); or*

(2) *S is a right (left)-zero semigroup of order 2; or*

(3) *S is a right (left)-zero semigroup of order 2 with identity adjoint; or*

(4) *S is the ideal extension of a right (left)-zero semigroup S_0 by a group S_1 with zero adjoint such that:*

- *S_1 is a Δ-group*
- *S_1 transitively acts on the right (left) on S_0*
- *each normal subgroup of S_1 transitively acts on the right (left) on S_0*
- *denoting by H the stabilizer of an element of S_0 the subgroups of S_1 containing H form a chain with respect to the inclusion.*

Proof. If S is a completely regular Δ-semigroup, then S is either a Δ-group or a right (left) zero semigroup of order 2 or the ideal extension of a right (left) zero semigroup by a rectangular band of order 2 with zero adjoint, or the ideal extension of a right (left) zero semigroup by a Δ-group with zero adjoint by statements h), g) and i) in [2]. Moreover if S is the ideal extension of a right (left) zero semigroup by a rectangular band of order 2 with zero adjoint Theorems 3.1 and 3.2 in [1] state that S is a Δ-semigroup if and only if it is a right (left)-zero semigroup of order 2 with identity adjoint. If S is the ideal extension of a right (left) zero semigroup by a Δ-group with zero adjoint, then it is a semigroup called in [2] of type β and by Theorem 3.2 of [2] S is a Δ-semigroup if and only if it satisfies conditions given in point (4) of the statement.

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