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GEOMETRIC GROUP PRESENTATIONS: A COMBINATORIAL APPROACH

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In this paper we obtain combinatorial conditions for the geometricity of group presentations; such a criterion holds both for orientable and for non-orientable manifolds.

1. Introduction

It is always possible to associate a 2-dimensional (canonical) complex to a group presentation (see [14] and [11]).

More precisely, given $\phi = < x_1, \ldots, x_g / R_1, \ldots, R_s >$, the associated complex K_ϕ is *canonical* if it has one 0-cell, g 1-cell, s 2-cells and each 1-cell intersects at least one 2-cell and conversely. In particular, if $g = s$ then ϕ is *balanced.*

By introducing three permutations, deriving from the complex K_ϕ, Neuwirth ([14]) describes an algorithm to check whether a balanced presentation ϕ is *strongly geometric*, i.e. K_ϕ is a spine of a *closed* 3-manifold (see [11] too). More precisely he establishes conditions which assure that K_ϕ is a spine of a closed, *orientable* 3-manifold.

Combinatorial algorithms have been described later by other authors too ([16], [17], [19], [8] and [3]).

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The problem of checking the geometricity of a group presentation has been studied also by Montesinos in [13], where an algorithm is described to establish whether a positive (not necessarily balanced) group presentation ψ is geometric, i.e. there exists a Heegaard diagram of an orientable 3-manifold (possibly with nonempty boundary) M such that ψ is exactly the presentation of $\pi_1(M)$ associated to the diagram (this definition of geometricity is clearly equivalent to the above one). The techniques used in the paper are mainly related with the theory of branched coverings.

In [10], Hog Angeloni investigates about geometricity of (not necessarily balanced, too) group presentations by means of the RR-systems, introduced by Osborne and Stevens ([16], [17], [19])

Moreover in [5] Neuwirth's algorithm is extended to compact orientable 3-manifolds with boundary.

In [8] Grasselli derives Neuwirth's algorithm in terms of coloured graphs by making use of the *bijoin construction* (see [4]) which, starting from a crystallized structure, by means of an associated oriented structure, allows to construct a 4-coloured bipartite graph representing a closed orientable 3-manifold if and only if the oriented structure satisfies some conditions. It is possible, moreover, to characterize the crystallized structures representing spines of the 3-manifold.

In [12], Neuwirth's algorithm is revisited in terms of squashing maps for closed 3-manifolds.

Also in [9] it is possible to find an algorithm to test the geometricity of a group presentation via coloured graphs. Another algorithm to check the geometricity of group presentations by means of coloured graphs, including also the non-orientable case, can be found in [1].

Finally, Skopenkov (in [20]) gives necessary and sufficient conditions for a (finite) 2-dimensional cell complex to be thickenable to a compact 3-manifold. Before stating his theorem, we give some definitions.

Let K be a 2-dimensional (connected) cell complex ([18]). Then we say that the 1-skeleton of K, the graph G, is *locally planar* if the link of each vertex of G is planar; note that if G is a locally planar graph, then supposing d an edge of G and A an end of d, the union of the 2-cells of $F-G$ containing d meets lkA in $S = st_{lkA}(d \cap lkA)$. So, by embedding lkA in a sphere, one induces a cycle of edges of S, hence a cycle of 2-cells containing d and such a cycle is directed iff the sphere is oriented. We say that G is *coherently embeddable* if the union of the links of its vertices can be embedded in a sphere, so that – if d is an edge joining the vertex A and the vertex B – the cycles of 2-cells around d induced by A and by B respectively, are equal, up the orientation. If G is coherently embeddable, we consider the relative (coherent) embedding in a sphere and call G *coherently embedded*. If G is a coherently embedded graph, then there is a handlebody which is a neighbourhood of G and we can assign to any edge the value +1 (resp. -1) if the directed cycles induced by its ends are opposite (resp. equal), i.e. if the edge is the core of an orientable (resp. non-orientable) handle; in this case we say that G is a *labelled graph*.

Now we are ready to state the following:

Theorem 1.1. *(Skopenkov) Let K be a 2-dimensional (connected) cell complex. K is a spine of a compact, connected 3-manifold if and only if the following three conditions hold:*

1. *The 1-skeleton of K is a locally planar graph;*
2. *the 1-skeleton of K is a coherently embedded graph;*
3. *the 1-skeleton of K is a labelled graph and the product of the values of the edges associated to every 2-cell of K is equal to +1.*

In this paper, we obtain algebraic conditions to verify the geometricity of a group presentation $\phi =< x_1,\dots,x_g/R_1,\dots,R_s >$.

Moreover, given a compact, connected 3-manifold M admitting a canonical spine whose group presentation is exactly ϕ, we calculate, by means of different tools, the number of boundary components of M and, in the case of connected boundary, we determine the boundary - surface.

More precisely, in Section 2 we state the geometricity criterion, achieved directly by the presentation ϕ, whereas Section 3 is devoted to the proof and the achievement of further results, via P-graphs ([16], [17], [19], [3]). In Section 3, for example, a criterion is described to recognize geometric presentations corresponding to non-prime 3-manifolds, which extends the criterion stated in [3] for the orientable case.

2. A geometricity criterion

Given a set of permutations $A_1,\dots,A_s$ on a set Y, we will denote by $|A_1,\dots,A_s|$ the number of the orbits of the action on Y of the group generated by $A_1,\dots,A_s$.

Let $\phi =< x_1,\dots,x_g/R_1,\dots,R_s >$, where $s \leq g$, be a group presentation, being $R_i = x_{i_1}^{\varepsilon_{i_1}} \dots x_{i_{n(i)}}^{\varepsilon_{i_{n(i)}}}$, with $\varepsilon_{i_j} \in \{-1,+1\}$, $j = 1,\dots,n(i)$, $i = 1,\dots,s$.

Consider the sets of symbols $X = \{x_h^m / h = 1,\dots,g;\ m = 1,\dots,\alpha_h\}$ and $\bar{X} = \{\bar{x}_h^m / h = 1,\dots,g;\ m = 1,\dots,\alpha_h\}$, where α_h is the total number of the occurrences of x_h in the relators of ϕ .

Now, for each $h \neq k$, $h,k = 1,\dots,g$, for each $m = 1,\dots,\alpha_h$ and for each $l = 1,\dots,\alpha_k$, we define the following transpositions :

$$(\tilde{x}_h^m, \tilde{x}_k^l) = \begin{cases} (x_h^m, x_k^l), & \text{if there is } i = 1, \ldots, s, \text{such that } R_i \text{ contains } x_h^{-1} x_k \\ (x_h^m, \bar{x}_k^l), & \text{if there is } i = 1, \ldots, s, \text{such that } R_i \text{ contains } x_h^{-1} x_k^{-1} \\ (\bar{x}_h^m, x_k^l), & \text{if there is } i = 1, \ldots, s, \text{such that } R_i \text{ contains } x_h x_k \\ (\bar{x}_h^m, \bar{x}_k^l), & \text{if there is } i = 1, \ldots, s, \text{such that } R_i \text{ contains } x_h x_k^{-1} \end{cases}$$

Set $X' = X \cup \bar{X}$, and define on the set X' the following permutations:

(1) $\tilde{A} = \prod (\tilde{x}_h^m, \tilde{x}_k^l), \quad h, k = 1, \ldots, g \ \ (h \neq k); \ \ m = 1, \ldots, \alpha_h; \ \ l = 1, \ldots, \alpha_k.$

(2) $\tilde{B} = \prod (\bar{x}_h^m, x_h^m)$, with $h = 1, \ldots, g$; $m = 1, \ldots, \alpha_h$.

For each $j = 1, \ldots, g$ and for each cyclic permutation σ_j of the symmetric group $\mathscr{S}_{\alpha_j}$, set:

$$\eta_{j,\sigma_j} = (x_j^{\sigma_j(1)} \ldots x_j^{\sigma_j(\alpha_j)}),$$

$$\eta'_{j,\sigma_j} = (\bar{x}_j^{\sigma_j(\alpha_j)} \ldots \bar{x}_j^{\sigma_j(1)}),$$

$$\eta''_{j,\sigma_j} = (\bar{x}_j^{\sigma_j(1)} \ldots \bar{x}_j^{\sigma_j(\alpha_j)}).$$

We are now ready to define:

(3) $\tilde{C} = \prod \eta_{j,\sigma_j} \prod \eta^*_{j,\sigma_j}$,where $j = 1, \ldots, g$ and η^*_{j,σ_j} is either η'_{j,σ_j} or η''_{j,σ_j}.

Denote by $\tilde{\theta}$ the map which associates +1 (resp. -1) to x_j, if and only if $\eta^*_{j,\sigma_j} = \eta'_{j,\sigma_j}$ (resp. if and only if $\eta^*_{j,\sigma_j} = \eta''_{j,\sigma_j}$), $j = 1, \ldots, g$.

If we consider $R_i = \prod_{k=1,\ldots,n} (x_{j_k})^{\varepsilon_{j_k}}$, with $\varepsilon_{j_k} \in \{-1, 1\}, \ \ i = 1, \ldots, s$, then set $\tilde{\theta}(R_i) = \prod_{k=1,\ldots,n} \tilde{\theta}(x_{j_k})$,

Let, finally, $\tilde{\pi}$ be the permutation on X', defined as follows:

$$\tilde{\pi}(x_i^j) = \tilde{B}\tilde{C}(x_i^j)$$

$$\tilde{\pi}(\bar{x}_i^j) = \begin{cases} \tilde{B}\tilde{C}(\bar{x}_i^j), & \text{if } \eta^*_{j,\sigma_j} = \eta'_{j,\sigma_j} \\ \tilde{B}\tilde{C}^{-1}(\bar{x}_i^j), & \text{if } \eta^*_{j,\sigma_j} = \eta''_{j,\sigma_j}. \end{cases}$$

Now, assuming the above notations, we can state the Main Theorem:

Theorem 2.1. *(geometricity criterion) ϕ presents the fundamental group of a compact, connected 3-manifold M if and only if the following conditions hold:*

1. $|\tilde{A}\tilde{C}| - |\tilde{A}| + |\tilde{C}| = 2|\tilde{A},\tilde{C}|$

2. for each relator R of ϕ, $\tilde{\theta}(R)$ = +1.

Theorem 2.2. *M has $|\tilde{A}\tilde{C},\tilde{\pi}|$ boundary components.*

Moreover, if $|\tilde{A}\tilde{C},\tilde{\pi}|$ = 1, then ∂M is an orientable (resp. non-orientable) surface of genus $g-s$ (resp. $2(g-s)$).

Corollary 2.3. *ϕ is a presentation of the fundamental group of a closed, connected 3-manifold M if and only if it satisfies Theorem 2.1 and g = s .*

3. 3-manifold spines associated to group presentations

This section is devoted to the proof of Main Theorem and to find out some geometric properties of the resulting manifold M.

Proof. (Proof of Theorem 2.1) Let $\phi =< x_1,\dots,x_g/R_1,\dots,R_s >$ be a group presentation and let K_ϕ be the 2-dimensional (canonical) complex associated to ϕ, i.e. K_ϕ is the 2-complex – with a unique vertex p, g 1-cells denoted by $x_1,\dots,x_g$ and s 2-cells – whose fundamental group admits ϕ as a presentation.

Consider now the set V obtained as intersection of the boundary of a regular neighborhood H of the 1-skeleton of K_ϕ in K_ϕ and the boundary of a regular neighborhood L of p in K_ϕ. V is a set of points p_i^j , $p_i^{\prime j}$, $i = 1,\dots,g$, lying in a regular neighborhood of x_i, so that p_i^j and $p_i^{\prime j}$ are ends of an arc of $\partial H - L$. By labelling the points in V, we define a permutation C on V, having $V_1,\dots,V_g$, $V_1',\dots,V_g'$ as orbits, where, for each $i = 1,\dots,g$, we set $V_i = \{p_i^j / j = 1,\dots,\alpha(i)\}$ (resp. $V_i' = \{p_i^{\prime j} / j = 1,\dots,\alpha(i)\}$); hence $V = \cup_{i=1}^{g}(V_i \cup V_i')$.

Moreover, observe that a simple curve near the boundary of the 2-cells of K_ϕ meets L in a set Λ of disjoint arcs with (distinct) endpoints in V. We define the permutation A as the involution on V such that, for each $w \in V$, w and $A(w)$ are endpoints of an arc of Λ. Moreover, on H, we consider the set Λ' of the edges e such that e joins p_i^j and $C(p_i^j)$ or $p_i^{\prime j}$ and $C(p_i^{\prime j})$, $i = 1,\dots,g$, $j = 1,\dots,\alpha(i)\}$.

Finally, B is the product of the transpositions $(p_i^j, p_i^{\prime j})$, for each $i = 1,\dots,g$ and $j = 1,\dots,\alpha(i)$.

Now, we can define the graph Γ_ϕ, as follows:

(1) $V(\Gamma_\phi) = V$;

(2) $E(\Gamma_\phi) = \Lambda \cup \Lambda'$.

Note that Γ_ϕ is embedded on the boundary of L; any ordering of the vertices V_i and V_i', $i = 1,\ldots,g$ (hence any choice of C) determines an embedding of this graph into an orientable compact surface F(see [15] or [6]).

Suppose F arbitrarily oriented and let B_i be the cycle in F having the points of V_i as vertices (resp. B_i' the cycle in F having the points of V_i' as vertices) the induced orientation.

Choose C so that the (cyclic) ordering of V_i defined by the orientation of B_i coincides with the (cyclic) ordering induced by C and the (cyclic) ordering of V_i' is opposite or equal to that of V_i.

Let now $\{B_i'/i = 1,\ldots,g\} = \mathscr{B} \cup \mathscr{B}'$, with $\mathscr{B} \cap \mathscr{B}' = \emptyset$ and $B_i' \in \mathscr{B}$ (resp. $B_i' \in \mathscr{B}'$) if and only if V_i' has the opposite ordering of (resp. the same ordering as) V_i.

Obviously, B_i' derives an orientation from the ordering on V_i'; note that the orientation of B_i' is opposite to (resp. the same as) that induced by F if and only if B_i' is in $\mathscr{B}$ (resp. $\mathscr{B}'$).

Since $\#V(\Gamma_\phi)$ equals the number of edges of Λ', $\#E(\Gamma_\phi) = \#\Lambda + \#\Lambda' = \#\Lambda' + |A|$, the number of the regions of $F - K_\phi$ equals $|AC| + |C|$ and the connected components of Γ_ϕ are $|A,C|$, then the Euler characteristic of F is $|AC| - |A| + |C|$; if : $2|A,C| = |AC| - |A| + |C|$, F if the 2-sphere.

So, Γ_ϕ is locally planar; moreover, by definition of the permutation C, Γ_ϕ is coherently embedded. So condition 1. and 2. of Theorem 1.1 are satisfied.

Set now $\theta(x_i)$ = +1 (resp. = -1) if and only if $B_i' \in \mathscr{B}$ (resp. $\mathscr{B}'$).

Then, by Theorem 1.1, K_ϕ is a spine of a compact, connected 3-manifold M if and only if, for each 2-cell $D_j (j = 1,\ldots,s)$ of K_ϕ, the product of the values of $\theta(x_i)$ for all the generators x_i on ∂D_i is equal to +1.

Finally note that, if ϕ is a group presentation and Γ_ϕ is the graph to which ϕ is associated, then the permutations $\tilde{A},\tilde{B},\tilde{C}$ defined by ϕ correspond to the permutations A,B,C which define Γ_ϕ. This completes the proof of Theorem 2.1. □

In order to construct the manifold M, we can perform the join from a inner point of the 3-disc bounded by $\mathbb{S}^2$ on Γ_ϕ, glue to this disc a small thickening T_i for each arc x_i, $i = 1,\ldots,g$. T_i is a 1-handle, orientable (resp. non-orientable) iff $\theta(x_i)$ = +1 (resp. = -1). Therefore, we obtain a handlebody T.

Finally, we glue to T the 2-handles having the 2-cells of K_ϕ as cores.

Introduce, now, a permutation π on V, as follows: for each vertex v, we define $\pi(v)$ as

$$\begin{cases} BC(v) & \text{if, for some } i = 1,\ldots,g,\ v \text{ either } v \in V_i \text{ or } (v \in V_i', \text{and } B_i' \in \mathscr{B}) \\ BC^{-1}(v) & \text{if, for some } i = 1,\ldots,g,\ v\ \in V_i', \text{and } B_i' \in \mathscr{B}'. \end{cases}$$

Lemma 3.1. *The boundary of M has $|AC,\pi|$ connected components; moreover, if $|AC,\pi| = 1$, then ∂M is the orientable (resp. non-orientable) surface of genus $g-s$ (resp. of genus $2g-2s$).*

Proof. For every 2-cell D of K_ϕ, D meets T along a closed, simple curve γ, corresponding to an orbit of the permutation AC.

Moreover, the thickening of D meets ∂M into two 2-discs, say D_1 and D_2. Let D and D' be two (distinct) 2-cells of K_ϕ; then the disks D_i and D'_j, $i,j \in \{1,2\}$ (arising from the thickening of D and D'), belong to the same boundary component of M if and only if there exist an edge of $\Lambda' \cap D_i$ and an edge of $\Lambda' \cap D_j$ lying in the same region of $(T_k - \Gamma_\phi)$, for some k; this region corresponds to an orbit of π.

Now, if $|AC,\pi| = 1$, then ∂M is connected, hence it is a surface S with Euler characteristic $\chi(S) = 2-2g+2s$.

So S is either the orientable surface of genus $g-s$, or the non-orientable surface of genus $2g-2s$.

□

If $|AC,\pi| = 1$, we can easily state a consequence of Lemma 3.1:

Corollary 3.2. *∂M is the 2-sphere if and only if $g = s$.*

For the orientable case, see also [5].

Remark 3.3. The condition $|AC,\pi| = 1$ implies the existence of an ordering $F_1,\dots,F_{|AC|}$ of the regions of $\mathbb{S}^2 - |\Gamma_\phi|$ such that, for each $r = 1,\dots,|AC|-1$ there is an edge of $\partial F_r = \{y, C^\varepsilon(y), AC^\varepsilon(y),\dots,A^{-1}(y)\}$ ($\varepsilon = \pm 1$) with endpoints y_1 and y_2, such that:

(1) $y_2 = C^\varepsilon(y_1)$ with $\varepsilon = -1$ if $y_1, y_2 \in B'_i$, $B'_i \in \mathscr{B}'$ and $\varepsilon = +1$ if either $B(y_1), B(y_2) \in B'_i$ or $y_1, y_2 \in B'_i, B'_i \in \mathscr{B}$;

(2) there is an edge of ∂F_{r+1} with endpoints $B(y_1)$ and $B(y_2) = BC^\varepsilon(y_1)$.

Conversely, the existence of such an ordering implies that $|AC,\pi| = 1$.

Recall that a *Heegaard diagram of genus* g of a closed 3-manifold M (orientable or not) is a triple $(F_g; v = \{v_1,\dots,v_g\}, w = \{w_1,\dots,w_g\})$,where F_g is the surface, orientable of genus g or non-orientable of genus $2g$ (according to M) and v,w are complete systems of meridian curves for F_g.

Remark 3.4. If K_ϕ is a spine of a compact, connected 3-manifold M, then, by the definition of the permutation π, we can define a family of homeomorphisms $\psi_i : B_i \to B'_i$ ($i = 1,\dots,g$) which reverse (resp. preserve) the orientations if $B'_i \in \mathscr{B}$ (resp. $B'_i \in \mathscr{B}'$). By denoting D_i (resp. D'_i) the disc on $\mathbb{S}^2$ bounded by D_i (resp. D'_i), from $\Sigma^2_{2g} = \mathbb{S}^2 - \cup_{i=1}^g (int(D_i) \cup int(D'_i))$, via the identifications induced by

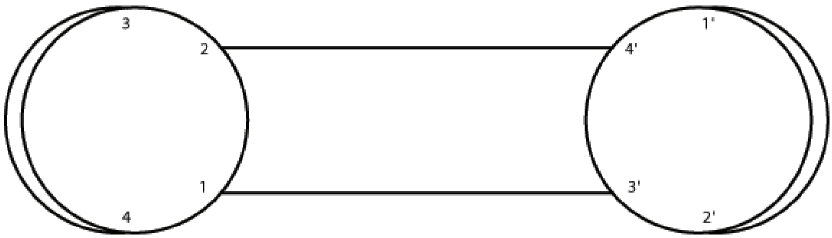

Figure 1

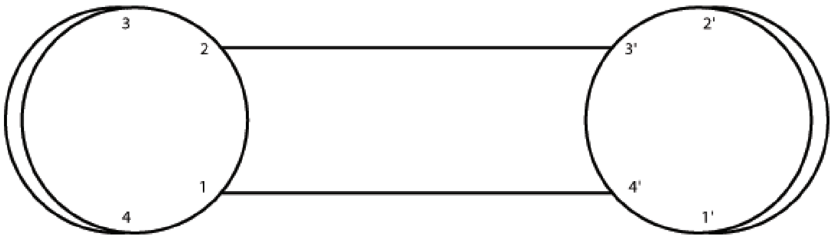

Figure 2

the ψ_i's, we obtain the closed orientable surface of genus g, if $\mathscr{B}' = \emptyset$ (resp. the closed non-orientable surface of genus $2g$, if $\mathscr{B}' \neq \emptyset$); we simply denote such a surface by F_g. The arcs of Λ project on a system w of closed, simple and disjoint curves; furthermore if $v = \{v_1, \ldots, v_g\}$ is the system of meridian curves for F_g corresponding to the B_i's then $(F_g; v, w)$ is a Heegaard diagram of M and ϕ is the associated presentation of the fundamental group.

Moreover, there is a natural way to obtain a representation of M in terms of coloured graphs (for details, see [7]) by starting from $(F_g; v, w)$ [2].

The above Figure 1 (resp. Figure 2) shows Γ_ϕ, for $\phi =< x/xxx^{-1}x^{-1} >$, where:

$$A = (1,3')(2,4')(3,4)(1',2'), (resp. A = (1,4')(2,3')(3,4)(1',2')),$$
$$B = (1,1')(2,2')(3,3')(4,4'),$$
$$C = (1,2,3,4)(4',3',2',1')(resp. = (1,2,3,4)(1',2',3',4')).$$

Note that, in this case, $\mathscr{B}' = \emptyset$ and the resulting 3-manifold is $\mathbb{S}^1 \times \mathbb{S}^2$ (resp. $\mathscr{B}' \neq \emptyset$ and the corresponding 3-manifold is $\mathbb{S}^1 \underset{\sim}{\times} \mathbb{S}^2$.).

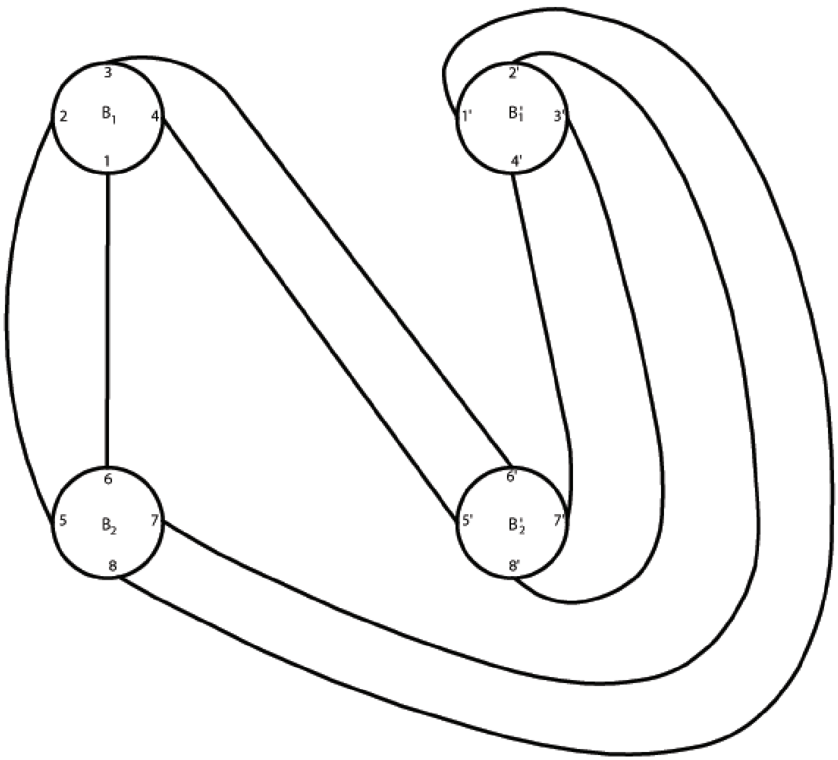

Figure 3

In Figure 3 it is depicted Γ_ϕ, for $\phi =< x, y/yxy^{-1}x^{-1}, yxyx^{-1} >$.
Note that,in this case, the permutations are:

$$A = (1,6)(2,5)(3,6')(4,5')(7,4')(8,1')(2',7')(3',8'),$$
$$B = (1,1')(2,2')(3,3')(4,4')(5,5')(6,6')(7,7')(8,8')$$
$$C = (1,2,3,4)(4',3',2',1')(5,6,7,8)(1',2',3',4').$$

In this case $\mathscr{B}' = \{B_2'\}$; the corresponding 3-manifold is $\mathbb{S}^1 \times \mathbb{RP}^2$.

We are ready to extend the criterion to non prime 3-manifolds, by translating the above results in terms of P-graphs, i.e. we will obtain further results related to the case $|A,C| > 1$, when the graph Γ_ϕ isn't connected.

In the case of a balanced presentation, i.e. if $g = s$, then, following [16], [17] and [3], we can introduce the *P-graph (presentation graph)* P_ϕ associated to ϕ, simply by choosing in Γ_ϕ an internal point Q_i (resp. Q'_i) for each B_i (resp. B'_i) , by deleting B_i (resp. B'_i) and joining Q_i (resp. Q'_i) with the vertices of Γ_ϕ lying on B_i (resp. B'_i) and moreover by choosing the involution B as bijective map from V to V'. Of course, P_ϕ is naturally embedded in $\mathbb{S}^2$.

We say that the P-graph P_ϕ is *faithfully embedded* if the following property holds:

Property (*): for each $y \in V$, we have $BC(y) = C^\varepsilon B(y)$ with:

$\varepsilon = -1$ if either $B(y) \in V'_i$ or $y \in V'_i$ $(B'_i \in \mathscr{B})$;

$\varepsilon = +1$ if either $B(y) \in V'_i$ or $y \in V'_i$ $(B'_i \in \mathscr{B}')$.

In this case, we say that ϕ *fits*.

The Main Theorem can be now restated as follows:

Theorem 3.5. *Provided that* $|AC,\pi| = 1$*, then* K_ϕ *is a spine of a closed, connected 3-manifold M iff* ϕ *fits and the product of the values of the generators* x_i*'s on the boundary of each 2-cell is equal to +1.*

If P_ϕ is a P-graph such that K_ϕ is a spine of a closed, connected 3-manifold and if $|A,C| > 1$, then P_ϕ has more than one connected component, denoted $P^1_\phi, \ldots, P^r_\phi$ (see [3], Proposition 5). Denoting by A_j, B_j, C_j and π_j the restrictions of A, B, C, π, respectively, to the vertices of P^j_ϕ , one can prove:

Theorem 3.6. *With the above notations, if* P^j_ϕ*, j = 1, …,r, is a faithfully embedded P-graph such that* $|A_jC_j, \pi_j| = 1$*, then M is homeomorphic to the connected sum of the closed, connected 3-manifolds represented by* $P^j_\phi, j = 1, \ldots, r$*.*

The proof is perfectly analogous to that of Proposition 5 of [3]; note that the above Theorem 3.6 implies:

Corollary 3.7. *If* $|A,C| > 1$*, then* ϕ *fits if and only if it presents a free product.*

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