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ON WEAK SUBFAMILIES AND THE CLASSIFICATION OF FAMILIES OF VARIETIES

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To the memory of Umberto Gasapina

In the paper we investigate through some examples the class of measurability, in Stoka's integral geometry theory, of weak subfamilies of a family of varieties, when we think of these from an intrinsic point of view. Then we suggest a new classification for families of varieties, in connection with late results.

1. Weak subfamilies and independent families.

Let $\mathscr{F}_q$ be a family of varieties, consisting of p-dimensional varieties V_p, defined in an n-dimensional space X_n by analytic equations

$$(1) \qquad F^{\lambda}(x_1, x_2, \dots, x_n, A_1, A_2, \dots, A_q), \quad \lambda = 1, 2, \dots, n-p.$$

Here $x_1, x_2, \dots, x_n$ are coordinates in X_n, while $A_1, A_2, \dots, A_q$ are essential parameters, that is coordinates in the parameter space Y_q.

If G_r is a group of movements acting on X_n and depending on r parameters, then the invariant densities of $\mathscr{F}_q$ with respect to G_r, if existing, are of the type

$$(2) \qquad |\Phi(A_1, A_2, \dots, A_q)| dA_1 \wedge dA_2 \wedge \dots \wedge dA_q,$$

where Φ is an invariant integral function; this is one of the possible solution of Deltheil's system

(3) $$\sum_{k=1}^{q} \frac{\partial(\xi_k^h(A_1, A_2, \dots, A_q)\Phi)}{\partial A_k} = 0, \quad h = 1, 2, \dots, r,$$

being $\xi_k^h(A_1, A_2, \dots, A_q)$ coefficients of infinitesimal transformations of $\mathscr{F}_q$ with respect to the group H_r, isomorphic to G_r [6], [17], [24], [25].

Definition 1. *A subfamily of the family $\mathscr{F}_q$ is a family $\mathscr{F}_{q_1}$ depending on $q_1 \leq q$ parameters, obtained by replacing $A_1, A_2, \dots, A_q$ with functions*

(4) $$A_k = \varphi_k(B_1, B_2, \dots, B_{q_1}), \quad k = 1, 2, \dots, q,$$

of q_1 new essential parameters. If $q_1 = q$ the subfamily is called weak subfamily [8].

In [8] it was proved the following

Theorem 1. *Let $\mathscr{F}_q$ be a measurable family of varieties, whose density is*

$$d\psi_q = |\Phi(A_1, A_2, \dots, A_q)| dA_1 \wedge dA_2 \wedge \dots \wedge dA_q.$$

Let $\overline{\mathscr{F}}_q$ be a weak subfamily of $\mathscr{F}_q$ defined by

$$A_k = \varphi_k(B_1, B_2, \dots, B_{q_1}), \quad k = 1, 2, \dots, q.$$

Let G_s be a group of invariance with respect to which $\mathscr{F}_q$ is measurable, and $G_h \subseteq G_s$ a group of invariance of $\overline{\mathscr{F}}_q$ ($q \leq h \leq s$); then $\overline{\mathscr{F}}_q$ is measurable with respect to G_h, with density

$$\overline{d\psi}_q = |\Phi(\varphi_1, \varphi_2, \dots, \varphi_q)| \det\left[\frac{\partial \varphi_i}{\partial B_j}\right] dB_1 \wedge dB_2 \wedge \dots \wedge dB_q,$$

$i, j = 1, 2, \dots, q$.

Now let $\mathscr{F}_q(A_1, A_2, \dots, A_q)$ and $\mathscr{F}_t(B_1, B_2, \dots, B_t)$ be two families of varieties, p and s dimensional, placed in X_n and depending on essential parameters $A_1, A_2, \dots, A_q$ and $B_1, B_2, \dots, B_t$ respectively.

Definition 2. *The families $\mathscr{F}_q = \mathscr{F}_q(\underline{A})$ and $\mathscr{F}_t = \mathscr{F}_t(\underline{B})$ are independent if there exists no relation between their parameter sets $\{A_1, A_2, \dots, A_q\}$ and $\{B_1, B_2, \dots, B_t\}$, that is $\{\underline{A}\}$ and $\{\underline{B}\}$ are all essential in the union family $\mathscr{F}_q(\underline{A}) + \mathscr{F}_t(\underline{B})$* [11], [12].

In [11] it has been shown the following

Theorem 2. *Let $\mathcal{F}_m = \mathcal{F}_q + \mathcal{F}_t$ be the family of systems of two independent families $\mathcal{F}_q$ and $\mathcal{F}_t$, $(m = q + t)$, both measurable with respect to the maximal group of invariance G_r of $\mathcal{F}_m$ $(r \geq m)$, and whose densities are $d\psi_q$ and $d\psi_t$ respectively. If the group H_r, isomorphic to G_r, is transitive, then $\mathcal{F}_m$ is measurable, and its density is*

$$d\psi_m = d\psi_q \wedge d\psi_t.$$

Remark. With the sentence *family of systems of two independent families $\mathcal{F}_q$ and $\mathcal{F}_t$* we mean the family in which every variety is a pair $V \cup W$, such that $V \in \mathcal{F}_q$, $W \in \mathcal{F}_t$, $\mathcal{F}_q$ and $\mathcal{F}_t$ are independent, and each group of invariance is a group of invariance of both families. The family $\mathcal{F}_m = \mathcal{F}_q + \mathcal{F}_t$ is also called *independent union of the families $\mathcal{F}_q$ and $\mathcal{F}_t$.*

2. On the measurability of weak subfamilies.

The early classification given by Stoka divided families of varieties, from measurability point of view, in two different classes, to which we can refer as class **A** and class **B**, namely

A. Families of varieties which are measurable according to the first Stoka's condition [22].

B. Families of varieties which are not measurable according to the second Stoka's condition [22].

Remark. A family $\mathcal{F}_q$, whose maximal group of invariance depends on $r < q$ parameters, is of course of class **B**. These are called *trivial non measurable families of varieties.*

We wish to see which is the class of measurability of weak subfamilies of a family of varieties of a given class, when we think of the weak subfamilies from an intrinsic point of view.

1. Let $\mathcal{F}_q$ be of class **A**, and $\overline{\mathcal{F}}_q$ a weak subfamily of $\mathcal{F}_q$. If the maximal group of invariance of $\overline{\mathcal{F}}_q$ depends on $r \geq q$ parameters, then, by Theorem 1, the weak subfamily is again of class **A**.

Example I. Let $\mathscr{F}_9$ be the family of non degenerate quadrics in the projective space P_3, which is of class **A** [23]. The restriction on the weak subfamily of non degenerate ruled quadrics $\overline{\mathscr{F}}_9$ does not change the maximal group of invariance, so $\overline{\mathscr{F}}_9$ is of class **A** [13], [14]. The same result is true for the family of non degenerate quadrics with elliptic points [13], [14].

Example II. There are also examples in which the restrictions on subfamilies change the maximal group of invariance. This occurs, for instance, when we take the family of non degenerate conics of the projective plane P_2, whose maximal group of invariance is the projective group G_8, and restrict it on the families of non degenerate ellipses, or non degenerate hyperbolas. In these cases the maximal group of invariance turns into the affine group G_6; by Theorem 1, as shown in [22], all these families are of class **A**. Other examples are given by taking the weak subfamilies of non degenerate quadrics consisting of non degenerate hyperboloids or non degenerate ellipsoides [7], [19], [20].

2. In case a weak subfamily $\overline{\mathscr{F}}_q$ of $\mathscr{F}_q$ is of class **B**, and it is not a trivial non measurable family, by the considerations developed in [10] we see that $\mathscr{F}_q$ must be of class **B**. However, if $\mathscr{F}_q$ is of class **A**, it is possible to understand the existence of weak subfamilies of class **B** when, in the restriction, the maximal group of invariance turns into a group depending on $r < q$ parameters. This situation may be obtained by building the independent union of two families, in such a way that the group associated to the maximal group of invariance is simply transitive [3], and, in the restrictions on weak subfamilies, it turns into a group whose number of parameters is lowered.

Example III. In the projective space P_3, let $\mathscr{F}_9$ be the family of non degenerate quadrics having elliptic points, and $\mathscr{F}_6$ the family of pairs plane+point, with the point out of the plane. The family $\mathscr{F}_{15} = \mathscr{F}_9 + \mathscr{F}_6$ assumes the projective group G_{15} as maximal group of invariance. The associated group H_{15} is simply transitive [11]. So Theorem 2 ensures that $\mathscr{F}_{15}$ is of class **A** [11], with density given by $d\psi_{15} = d\psi_9 \wedge d\psi_6$, where $d\psi_9$ is the single density of $\mathscr{F}_9$ (example **I** above), and $d\psi_6$ is the single density of $\mathscr{F}_6$ [17]. The independent union $\overline{\mathscr{F}}_{15}$ of the family $\overline{\mathscr{F}}_9$ of non degenerate ellipsoides with the family $\mathscr{F}_6$, is a weak subfamily of $\mathscr{F}_{15}$, and its maximal group of invariance is the affine group G_{12}. Consequently the associated group H_{12} is not transitive, and $\overline{\mathscr{F}}_{15}$ is of class **B**. With the same argument we obtain also that the independent union of the family of non degenerate hyperboloids with elliptic points and $\mathscr{F}_6$ is of class **B**.

3. Now let us take a family of varieties $\mathscr{F}_q$ of class **B**, which is measurable with respect to groups G_a and G_b, $(a \geq q, b \geq q)$, with different measures. Of course, if the maximal group of invariance of a weak subfamily has both G_a and

G_b as subgroups, then the weak subfamily is again of class **B**. On the other side, by Theorem 1, any weak subfamily of $\mathscr{F}_q$, whose maximal group of invariance is a subgroup of G_a or G_b, is of class **A**.

Example IV. A trivial way to obtain weak subfamilies of class **B** of families of class **B**, is to start from trivial non measurable families of varieties, which can be easily built up through independent unions. For instance, in the projective space P_3, let $\mathscr{F}_9'$ be the family of non degenerate hyperboloids with elliptic points. The independent union of $\mathscr{F}_9'$ with the family $\mathscr{F}_6$ of pairs plane+point, the point out of the plane, is of class **B** as pointed out in the previous Example **III**. It may be regarded also as a weak subfamily of the independent union of the family of non degenerate hyperboloids with $\mathscr{F}_6$, which is of class **B**, since it depends on 15 essential parameters while the maximal group of invariance is the affine group of the space.

Example V. Let $\mathscr{F}_4 : x^2 + 2axy + a^2y^2 + 2bx + 2cy + q = 0$ be the family of non degenerate parabolas ($\Delta = -(ab - c)^2 \neq 0$) in the affine plane A_2, which is of class **B** [22]. Let $\overline{\mathscr{F}_4}$ be the weak subfamily of parabolas for which is $q > 0$, V the parabola of equation $y = x^2 + 1$, and G_4 the group

$$G_4 \begin{cases} x = \alpha x' + \beta y' \\ y = \gamma x' + \delta y'. \end{cases} \tag{5}$$

Every $\sigma \in G_4$ keeps $\overline{\mathscr{F}_4}$ invariant, since it does not change q. For any $W \in \overline{\mathscr{F}_4}$ there exists $\tau \in G_4$ such that $\tau(V) = W$, for instance

$$\tau \begin{cases} x = \dfrac{1}{\sqrt{q}}x' + \dfrac{a}{\sqrt{q}}y' \\ y = -\dfrac{2b}{q}x' - \dfrac{2c}{q}y'. \end{cases} \tag{6}$$

Then $\overline{\mathscr{F}_4}$ is the family of varieties G_4-equivalent to V, so it takes the kinematic density of G_4 [10], that is

$$\frac{1}{q(c - ab)^4} da \wedge db \wedge dc \wedge dq. \tag{7}$$

It is easy to show that for any given translation

$$G_2 \begin{cases} x = x' + \mu \\ y = y' + \nu \end{cases}, \tag{8}$$

all parabolas which satisfy

$$b\mu + cv \leq -\frac{(\mu + av)^2 + q}{2}, \tag{9}$$

are changed in parabolas which do not belong to $\overline{\mathscr{F}}_4$, so G_2 does not intersect the maximal group of invariance of the family, which is consequently G_4. Moreover the stabilizer of V in G_4 is $S_V = \{\tau_1, \tau_2\}$, where

$$\tau_1 \begin{cases} x = x' \\ y = y' \end{cases}, \quad \tau_2 \begin{cases} x = -x' \\ y = y' \end{cases}. \tag{10}$$

Then, by [9] (corollary of Theorem 2), $\overline{\mathscr{F}}_4$ is of class **A**, taking the single density (7) on its maximal group of invariance.

4. The above considerations point out that restrictions on weak subfamilies may change the measurability properties of the family. Sometimes, in order to give different forms to a density previously built on a family, a change of parameters is developed. In doing that we have to make sure to obtain again the whole family. On the other side, if it turns in a weak subfamily, we must check whether the class of measurability is kept, otherwise the weak subfamily might be related to a wrong density, as well as regarded as measurable whereas it is not measurable. As an example we take the family of straight-lines in the Euclidean plane E_2 [17], [18], [25].

Example. Let $\mathscr{F}_2 : x\cos\varphi + y\sin\varphi - p = 0$, $(0 \leq \varphi \leq 2\pi,\ p \geq 0)$, be the family of straight lines in E_2. The maximal group of invariance is the group of Euclidean motions

$$G_3 \begin{cases} x' = x\cos\alpha - y\sin\alpha + a \\ y' = x\sin\alpha + y\cos\alpha + b. \end{cases} \tag{11}$$

The family $\mathscr{F}_2$ is of class **A**, being measurable with respect to G_3, with density

$$d\mu_2 = dp \wedge d\varphi. \tag{12}$$

The change of parameters

$$p = \frac{1}{\sqrt{u^2 + v^2}}, \quad \tan\varphi = \frac{v}{u}, \tag{13}$$

allows the family to be written as

$$\mathscr{F}_2 : ux + vy + 1 = 0, \tag{14}$$

which gives again all straight lines of the plane. Indeed, every straight line which does not meet the origin is determined by a finite pair of parameters u, v, while, being impossible to have $u = 0$ and $v = 0$ at the same time, straight lines through the origin can be obtained, for instance, by putting $u \neq 0$, $v = u \tan\varphi$ and by taking the limit as $u \to \infty$. Then the family of straight lines, written in the form (14), assumes the density

$$dv_2 = \frac{1}{(u^2 + v^2)^{\frac{3}{2}}} du \wedge dv, \tag{15}$$

which is determined by replacing (13) in (12).

Now we change parameters as follows

$$p = 1 + \frac{1}{\sqrt{u^2 + v^2}}, \qquad \tan\varphi = \frac{v}{u}. \tag{16}$$

This restricts $\mathscr{F}_2$ on its weak subfamily

$$\overline{\mathscr{F}}_2 : ux + vy - \sqrt{u^2 + v^2} - 1 = 0, \tag{17}$$

consisting of all straight lines whose distance from the origin is

$$1 + \frac{1}{\sqrt{u^2 + v^2}} \geq 1. \tag{18}$$

Of course the density becomes again the dv_2 of (15), which now should be regarded as the invariant density which works on $\overline{\mathscr{F}}_2$, under the group of Euclidean motions. Nevertheless this is not correct. Indeed we can see that under a general translation

$$\begin{cases} x = x' + a \\ y = y' + b, \end{cases} \tag{19}$$

every straight line such that

$$|ua + vb - \sqrt{u^2 + v^2} - 1| < \sqrt{u^2 + v^2}, \tag{20}$$

is transformed in a parallel straight line whose distance from the origin is less then 1, that is in a variety which does not belong to $\overline{\mathscr{F}}_2$. Then, for this family, it is meaningless to give an invariant density with respect to the group of motions. It is straightforward to see that every transformation belonging to the group

$$G_1 \begin{cases} x' = x\cos\alpha - y\sin\alpha \\ y' = x\sin\alpha + y\cos\alpha, \end{cases} \tag{21}$$

keeps $\overline{\mathcal{F}_2}$ invariant, so this is the maximal group of invariance. Consequently (15) must be related to $\overline{\mathcal{F}_2}$ with respect to this group. However G_1 depends on a single parameter, and (15) is now only one among the infinite number of densities that we can build on $\overline{\mathcal{F}_2}$; then this family is trivially not measurable, and so it is of class **B**.
The same argument can be developed in connection with the changes of parameters

$$p = k + \frac{1}{\sqrt{u^2 + v^2}}, \quad \tan\varphi = \frac{v}{u}, \tag{22}$$

where $k > 0$ is a constant. These determine the weak subfamilies of $\mathcal{F}_2$ consisting of straight lines whose distance from the origin is greater or equal k, which are all of class **B**.

Remark. The previous result can be regarded as a new version of the well-known *Bertrand paradox* [2], in the sense that a density, which formally may be related to a family of varieties, is actually an appropriate density only whether it works on a group of invariance of that family.

3. A new classification.

In order to point out the change of measurability properties under the restrictions on weak subfamilies, we referred to the classification of families of varieties given by Stoka. However during the years, other cases were investigated, so, at the present, it is useful to develop a more detailed description. We may introduce the following other classes:

C. Families of varieties which are measurable but the group associated to their maximal group of invariance is only trivially measurable, that is $\Phi = 0$ is the only invariant integral function related to this group.

D. Families of varieties which are not measurable because all transitive groups, associated to groups of invariance, are trivially measurable.

E. Families of varieties which are not measurable because the group associated to their maximal group of invariance takes different non trivial densities.

Remark. When the group associated to the maximal group of invariance G_r of a family of varieties $\mathcal{F}_q$ takes different non trivial densities, then the family is in class **E** only whether $r \geq q$, since, when $r < q$, it is of class **B** being a trivial non measurable family of varieties.

Examples of any class are known. Classes **A** and **B** are the ones of the early classification, and we can find many families of varieties belonging to them in [21], [22], [23], [24], [25]. The only known example of class **C** is developed in [5], while we can find families of varieties of class **D** in [4] and [1]. Class **E** has been pointed out recently [12], and examples of families of varieties which belong to this class can be found in [11], [12], [15], [16].

The same problem of restriction we have dealt with, might be investigated in connection with the above new classes.

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