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A CHARACTERIZATION OF $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$, $\mathscr{H}$-TRIVIAL QUASI REGULAR SEMIGROUPS

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To the memory of Umberto Gasapina

Introduction.

In [7] we may find a characterization of ρ*-trivial* periodic semigroups, where ρ is a Green relation. In this short note we extend those results to the class of quasi regular semigroups which includes periodic semigroups. We will denote by E the set of the idempotents of a semigroup S, by *Reg* S the set of regular elements of S, by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$, $\mathscr{H}$ the Green relations and by $\mathbb{N}$ the set $\{1, 2, \cdots\}$ of natural numbers.

Definitions and notations not given here can be found in [1], [4], [6].

1. Preliminaries.

Definition 1.1. A semigroup S is *periodic* if $a^m = a^{m+r}$, for every $a \in S$ and for some $m, r \in \mathbb{N}$. If $r = 1$ for every $a \in S$, then S is called an *acyclic* semigroup (see [7]).

Definition 1.2. For any equivalence relation ρ on a semigroup S, S is said ρ-*trivial* if $a\rho b$ implies $a = b$, for every $a, b \in S$ (see [7]).

Theorem 1.3. ([7], Lemma 1.1). *Let S be a periodic semigroup. Then*

i) *S is $\mathscr{R}$-trivial if and only if $(ab)^m = (ab)^m a$, for every $a, b \in S$ and for some $m \in \mathbb{N}$.*
ii) *S is $\mathscr{L}$-trivial if and only if $(ab)^m = b(ab)^m$, for every $a, b \in S$ and for some $m \in \mathbb{N}$.*
iii) *S is $\mathscr{J}$-trivial if and only if $(ab)^m a = (ab)^m = b(ab)^m$, for every $a, b \in S$ and for some $m \in \mathbb{N}$.*
iv) *S is $\mathscr{H}$-trivial if and only if S is acyclic.*

Proposition 1.4. *Let S be a semigroup.*

i) *If $(ab)^m = (ab)^m a$, for every $a, b \in S$ and for some $m \in \mathbb{N}$, then S is acyclic.*
ii) *If $(ab)^m = b(ab)^m$, for every $a, b \in S$ and for some $m \in \mathbb{N}$, then S is acyclic.*

Proof. If we put $a = b$ in i) and ii), we get $a^{2m} = a^{2m+1}$. Then S is acyclic, by Definition 1.1. □

Noting that an acyclic semigroup is periodic, Proposition 1.4 allows to state the following theorem which is equivalent to Theorem 1.3.

Theorem 1.5. *Let S be a semigroup. Then*

i) *S is periodic and $\mathscr{R}$-trivial if and only if $(ab)^m = (ab)^m a$, for every $a, b \in S$ and for some $m \in \mathbb{N}$.*
ii) *S is periodic and $\mathscr{L}$-trivial if and only if $(ab)^m = b(ab)^m$, for every $a, b \in S$ and for some $m \in \mathbb{N}$.*
iii) *S is periodic and $\mathscr{J}$-trivial if and only if $(ab)^m a = (ab)^m = b(ab)^m$, for every $a, b \in S$ and for some $m \in \mathbb{N}$.*
iv) *S is periodic and $\mathscr{H}$-trivial if and only if S is acyclic.*

2. The main results.

Definition 2.1. A semigroup S is *quasi (completely) regular* if a^m is (completely) regular, for every $a \in S$ and for some $m \in \mathbb{N}$ (see [2], Definition 1.4).

Proposition 2.2. *Let S be a semigroup. Then*

i) *S is periodic if and only if $a^m \in E$, for every $a \in S$ and for some $m \in \mathbb{N}$.*
ii) *If S is periodic then S is quasi completely regular.*
iii) *If S is acyclic then S is quasi completely regular.*

Proof. i) If S is a periodic semigroup, then every element of S has a power which is idempotent (see [4], Proposition I.2.7). Conversely, if $a^m \in E$, for every $a \in S$ and for some $m \in \mathbb{N}$, then it is obvious that S is periodic.

ii) Let $a \in S$. By i), there exists $m \in \mathbb{N}$ such that $a^m \in E$. Then, it is obvious that a is quasi completely regular.

iii) It follows from Definition 1.1 and from ii). □

From Propositions 1.4 and 2.2 iii), we obtain immediately

Proposition 2.3. *Let S be a semigroup.*

i) If $(ab)^m = (ab)^m a$, for every $a, b \in S$ and for some $m \in \mathbb{N}$, then S is quasi completely regular.

ii) If $(ab)^m = b(ab)^m$, for every $a, b \in S$ and for some $m \in \mathbb{N}$, then S is quasi completely regular.

Lemma 2.4. *Let S be a semigroup and let $a, b \in S$.*

i) If $(ab)^m \in E$, for some $m \in \mathbb{N}$, then $(ab)^m \mathscr{R} (ab)^m a$.

ii) If $(ab)^m \in E$, for some $m \in \mathbb{N}$, then $(ab)^m \mathscr{L} b(ab)^m$;

iii) If $(ab)^m \in E$, for some $m \in N$, then $(ab)^m \mathscr{J} (ab)^m a \mathscr{J} b(ab)^m$.

Proof. i) If $(ab)^m \in E$ $(a, b \in S$ and $m \in \mathbb{N})$, then $(ab)^m = (ab)^{3m} = (ab)^m a(ba)^{2m-1} b \Rightarrow (ab)^m \in (ab)^m a S^1$. Since $(ab)^m a \in (ab)^m S^1$, we conclude that $(ab)^m \mathscr{R} (ab)^m a$.

ii) The proof is analogous to that given in i).

iii) If $(ab)^m \in E$ $(a, b \in S$ and $m \in \mathbb{N})$, then $(ab)^m \mathscr{R} (ab)^m a$ and $(ab)^m \mathscr{L} b(ab)^m$ by i) and ii). Since $\mathscr{R} \subseteq \mathscr{J}$ and $\mathscr{L} \subseteq \mathscr{J}$, we get $(ab)^m \mathscr{J} (ab)^m a$ and $(ab)^m \mathscr{J} b(ab)^m$. □

Theorem 2.5. *Let S be a semigroup. Then*

i) S is quasi regular and $\mathscr{R}$-trivial if and only if $(ab)^m = (ab)^m a$, for every $a, b \in S$ and for some $m \in \mathbb{N}$.

ii) S is quasi regular and $\mathscr{L}$-trivial if and only if $(ab)^m = b(ab)^m$, for every $a, b \in S$ and for some $m \in \mathbb{N}$.

iii) S is quasi regular and $\mathscr{J}$-trivial if and only if $(ab)^m a = (ab)^m = b(ab)^m$, for every $a, b \in S$ and for some $m \in \mathbb{N}$.

iv) If S is quasi regular and $\mathscr{R}$- or $\mathscr{L}$- or $\mathscr{J}$-trivial, then S is acyclic.

Proof. i) Let S be a quasi regular and $\mathscr{R}$-trivial semigroup. If $(ab)^m \in Reg S$ $(a, b \in S$ and $m \in \mathbb{N})$, then $(ab)^m = (ab)^m x (ab)^m$ $(x \in S) \Rightarrow (ab)^m \mathscr{R} (ab)^m x \in E \Rightarrow (ab)^m = (ab)^m x$, since S is $\mathscr{R}$-trivial $\Rightarrow (ab)^m \in E \Rightarrow (ab)^m \mathscr{R} (ab)^m a$, by Lemma 2.4 i) $\Rightarrow (ab)^m = (ab)^m a$, since S is $\mathscr{R}$-trivial. Conversely, let us suppose that $(ab)^m = (ab)^m a$ (for every $a, b \in S$ and for some $m \in N$). Then S

is acyclic by Proposition 1.4 i), quasi completely regular by Proposition 2.2 iii) and $\mathscr{R}$-trivial by Theorem 1.5 i).

ii) The proof is analogous to that given in i).

iii) Let S be a quasi regular and $\mathscr{J}$-trivial semigroup. If $(ab)^m \in RegS$ $(a, b \in S$ and $m \in \mathbb{N})$, then $(ab)^m = (ab)^m x(ab)^m$ $(x \in S) \Rightarrow (ab)^m \mathscr{R} (ab)^m x \in E \Rightarrow (ab)^m \mathscr{J} x(ab)^m \in E$, since $\mathscr{R} \subseteq \mathscr{J} \Rightarrow (ab)^m = x(ab)^m$, since S is $\mathscr{J}$-trivial $\Rightarrow (ab)^m \in E \Rightarrow (ab)^m \mathscr{J} (ab)^m a \mathscr{J} b(ab)^m$, by Lemma 2.4 iii) $\Rightarrow (ab)^m = (ab)^m a = b(ab)^m$, since S is $\mathscr{J}$-trivial. Conversely, let us suppose that $(ab)^m = (ab)^m a = b(ab)^m$ (for every $a, b \in S$ and for some $m \in \mathbb{N}$). Then S is acyclic by Proposition 1.4, quasi completely regular by Proposition 2.2 and $\mathscr{J}$-trivial by Theorem 1.5 iii).

iv) It follows immediately from i), ii), iii) in this theorem and from Proposition 1.4. □

In [3], Proposition 2.7, we proved that, in a quasi completely regular semigroup, $\mathscr{D} = \mathscr{J}$. So, we have

Proposition 2.6. *Let S be a quasi completely regular semigroup. Then S is $\mathscr{D}$-trivial if and only if S is $\mathscr{J}$-trivial.*

Proposition 2.7. *Let S be a semigroup. Then*

i) S is quasi completely regular and $\mathscr{D}$-trivial if and only if $(ab)^m a = (ab)^m = b(ab)^m$, for every $a, b \in S$ and for some $m \in \mathbb{N}$.

ii) If S is a quasi completely regular and $\mathscr{D}$-trivial then S is acyclic.

Proof. i) If S is a quasi completely regular and $\mathscr{D}$-trivial semigroup then S is $\mathscr{J}$-trivial and $(ab)^m a = (ab)^m = b(ab)^m$ (for every $a, b \in S$ and for some $m \in \mathbb{N}$), by Proposition 2.6 and Theorem 2.5. Conversely, if $(ab)^m a = (ab)^m = b(ab)^m$ (for every $a, b \in S$ and for some $m \in \mathbb{N}$), then S is a quasi completely regular and $\mathscr{J}$-trivial semigroup (see Theorem 2.5 and its proof). We conclude that S is $\mathscr{D}$-trivial, by Proposition 2.6.

ii) It follows immediately from Proposition 2.6 and Theorem 2.5. □

Theorem 2.8. *A semigroup S is quasi completely regular and $\mathscr{H}$-trivial if and only if S is acyclic.*

Proof. Let S be a quasi completely regular semigroup, let $a \in S$, let G be the maximal subgroup of S having $e \in E$ as its identity and let $m \in \mathbb{N}$ be such that $a^m \in G$. We have $a^m \mathscr{H} e$ (see [6], Lemma IV.1.5) and $a^m = e$ since S is $\mathscr{H}$-trivial. By Theorem 1 in [5], $a^n \in G$, for every positive integer $n > m$, and consequently $a^n = e$. If we put $n = m + 1$, we have $a^{m+1} = e$. We conclude that $a^m = a^{m+1}$ and S is an acyclic semigroup. Conversely if S is an acyclic semigroup, then S is quasi completely regular, by Proposition 2.2 iii) and $\mathscr{H}$-trivial by Theorem 1.5 iv). □

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