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SULLA MISURABILITA' DELLA FAMIGLIA DEI SISTEMI FORMATI DA CONO QUADRICO E RETTA IN A_3

SILVIA IMPOSIMATO

Alla memoria di Umberto Gasapina

The family of system of quadric cone and straight line, in A_3, proves to be measurable.

Il cono quadrico abbia per direttrice, nel piano xy, la conica non degenere di equazione:

$$x^2 + 2axy + by^2 + 2cx + 2dy + e = 0,$$

con

$$\begin{vmatrix} 1 & a & c \\ a & b & d \\ c & d & e \end{vmatrix} \neq 0,$$

e per vertice il punto $V(x_0, y_0, z_0)$.
Quindi, l'equazione del cono sarà:

$$x^2 + by^2 + \left(\frac{x_0^2 + 2ax_0y_0 + by_0^2 + 2cx_0 + 2dy_0 + e}{z_0^2} \right) z^2 +$$

Lavoro eseguito con il contributo del M.U.R.S.T. (40%, 60%).

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$$+ 2axy + \left[-\frac{2}{z_0}(x_0 + ay_0 + c)\right] xz + \left[-\frac{2}{z_0}(ax_0 + by_0 + d)\right] yz + 2cx +$$

$$+ 2dy + \left[-\frac{2}{z_0}(cx_0 + dy_0 + e)\right] z + e = 0 .$$

Le equazioni della retta nello spazio A_3 siano:

$$\begin{cases} x = pz + q \\ y = rz + s. \end{cases}$$

Allora la famiglia dei sistemi formati da cono quadrico e retta in A_3 sarà rappresentata dall'insieme delle tre seguenti equazioni:

$$(1) \quad \begin{aligned} & x^2 + by^2 + \left(\frac{x_0^2 + 2ax_0y_0 + by_0^2 + 2cx_0 + 2dy_0 + e}{z_0^2}\right) z^2 + 2axy + \\ & + \left[-\frac{2}{z_0}(x_0 + ay_0 + c)\right] xz + \left[-\frac{2}{z_0}(ax_0 + by_0 + d)\right] yz + \\ & + 2cx + 2dy + \left[-\frac{2}{z_0}(cx_0 + dy_0 + e)\right] z + e = 0, \\ & x = pz + q \\ & y = rz + s. \end{aligned}$$

I parametri della famiglia sono dodici: $a, b, c, d, e, x_0, y_0, z_0, p, q, r, s$.
Il gruppo massimo di invarianza della famiglia (1) è il gruppo affine nello spazio A_3:

$$(2) \quad \begin{aligned} x &= b_{11}x' + b_{12}x' + b_{13}z' + b_{14}, \\ y &= b_{21}x' + b_{22}x' + b_{23}z' + b_{24}, \\ z &= b_{31}x' + b_{32}x' + b_{33}z' + b_{34}. \end{aligned}$$

I parametri del gruppo (2) sono dodici: $b_{11}, b_{12}, b_{13}, b_{14}, b_{21}, b_{22}, b_{23}, b_{24}, b_{31}, b_{32}, b_{33}, b_{34}$.

Applicando il gruppo (2) alla famiglia (1), si ottiene la nuova varietà:

$$(1')\quad \begin{aligned} & x'^2 + b'y'^2 + \left(\frac{x_0'^2 + 2a'x_0'y_0' + b'y_0'^2 + 2c'x_0' + 2d'y_0' + e'}{z_0'^2} \right) z'^2 + \\ & + 2a'x'y' + \left[-\frac{2}{z_0'}(x_0' + a'y_0' + c') \right] x'z' + \\ & + \left[-\frac{2}{z_0'}(a'x_0' + b'y_0' + d') \right] y'z' + 2c'x' + 2d'y' + \\ & + \left[-\frac{2}{z_0'}(c'x_0' + d'y_0' + e') \right] z' + e' = 0, \\ & x' = p'z' + q', \\ & y' = r'z' + s'. \end{aligned}$$

Poniamo:

$$A = \frac{x_0^2 + 2ax_0y_0 + by_0^2 + 2cx_0 + 2dy_0 + e}{z_0^2},$$

$$B = -\frac{2}{z_0}(x_0 + ay_0 + c),$$

$$D = -\frac{2}{z_0}(ax_0 + by_0 + d),$$

$$E = -\frac{2}{z_0}(cx_0 + dy_0 + e),$$

$$T = b_{11}^2 + bb_{21}^2 + Ab_{31}^2 + 2ab_{11}b_{21} + Bb_{11}b_{31} + Db_{21}b_{31},$$

$$H = b_{11}b_{13} + bb_{21}b_{23} + Ab_{31}b_{33} + ab_{11}b_{23} + ab_{13}b_{21} + \frac{B}{2}b_{11}b_{33} + \frac{B}{2}b_{13}b_{31} + \frac{D}{2}b_{21}b_{33} + \frac{D}{2}b_{23}b_{31},$$

$$L = b_{12}b_{13} + bb_{22}b_{23} + Ab_{32}b_{33} + ab_{12}b_{23} + ab_{13}b_{22} + \frac{B}{2}b_{12}b_{33} + \frac{B}{2}b_{13}b_{32} + \frac{D}{2}b_{22}b_{33} + \frac{D}{2}b_{23}b_{32},$$

$$M = b_{13}b_{14} + bb_{23}b_{24} + Ab_{33}b_{34} + ab_{13}b_{24} + ab_{14}b_{23} + \frac{B}{2}b_{13}b_{34} + \frac{B}{2}b_{14}b_{33} + \frac{D}{2}b_{23}b_{34} + \frac{D}{2}b_{24}b_{33} + cb_{13} + db_{23} + \frac{E}{2}b_{33},$$

$$\begin{aligned}
\Delta\ &= H(a'd'-b'c')+L(a'c'-d')+M(b'-a'^2),\\
\Delta x_0 &= H(b'e'-d'^2)+L(c'd'-a'e')+M(a'd'-b'c'),\\
\Delta y_0 &= H(c'd'-a'e')+L(e'-c'^2)+M(a'c'-d'),\\
\Delta z_0 &= d'^2+c'^2b'+a'^2e'-2a'c'd'-b'e',
\end{aligned}$$

$$\begin{aligned}
T_1 &= b_{11}b_{12}+bb_{21}b_{22}+Ab_{31}b_{32}+ab_{11}b_{22}+ab_{12}b_{21}+\frac{B}{2}b_{11}b_{32}+\\
&\quad+\frac{B}{2}b_{12}b_{31}+\frac{D}{2}b_{21}b_{32}+\frac{D}{2}b_{22}b_{31},\\
T_2 &= b_{12}^2+bb_{12}^2+Ab_{32}^2+2ab_{12}b_{22}+Bb_{12}b_{32}+Db_{22}b_{32},\\
T_3 &= b_{11}b_{14}+bb_{22}b_{24}+Ab_{31}b_{34}+ab_{14}b_{21}+ab_{11}b_{24}+\frac{B}{2}b_{11}b_{34}+\\
&\quad+\frac{B}{2}b_{14}b_{31}+\frac{D}{2}b_{21}b_{34}+\frac{D}{2}b_{24}b_{31}+cb_{11}+db_{21}+\frac{E}{2}b_{31},\\
T_4 &= b_{12}b_{14}+bb_{22}b_{24}+Ab_{32}b_{34}+ab_{12}b_{24}+ab_{14}b_{22}+\frac{B}{2}b_{12}b_{34}+\\
&\quad+\frac{B}{2}b_{14}b_{32}+\frac{D}{2}b_{22}b_{34}+\frac{D}{2}b_{24}b_{32}+cb_{12}+db_{22}+\frac{E}{2}b_{32},\\
T_5 &= b_{14}^2+bb_{24}^2+Ab_{34}^2+2ab_{14}b_{24}+Bb_{14}b_{34}+Db_{34}b_{24}+2cb_{14}+\\
&\quad+2db_{24}+Eb_{34}+e.
\end{aligned}$$

Con queste notazioni possiamo esprimere in forma compatta le equazioni del gruppo isomorfo associato a (2):

$$a'=\frac{T_1}{T},\ b'=\frac{T_2}{T},\ c'=\frac{T_3}{T},\ d'=\frac{T_4}{T},\ e'=\frac{T_5}{T},$$

$$x_0'=\frac{\Delta x_0}{\Delta},\ y_0'=\frac{\Delta y_0}{\Delta},\ z_0'=\frac{\Delta z_0}{\Delta},$$

$$\begin{aligned}
p' &= \frac{(b_{13}-pb_{33})(b_{22}-rb_{32})-(b_{23}-pb_{33})(b_{11}-pb_{32})}{(b_{21}-rb_{31})(b_{12}-pb_{32})-(b_{22}-rb_{32})(b_{11}-pb_{31})},\\
q' &= \frac{(rb_{34}-b_{24}+s)(b_{12}-pb_{32})-(pb_{34}-b_{14}+q)(b_{22}-rb_{32})}{(b_{21}-rb_{31})(b_{12}-pb_{32})-(b_{22}-rb_{32})(b_{11}-pb_{31})}\\
r' &= \frac{(b_{13}-pb_{33})(b_{21}-rb_{31})-(b_{23}-rb_{33})(b_{11}-pb_{31})}{(b_{21}-rb_{31})(b_{12}-pb_{32})-(b_{22}-rb_{32})(b_{11}-pb_{31})},\\
s' &= \frac{(rb_{34}-b_{24}+s)(b_{11}-pb_{31})-(pb_{34}-b_{14}+q)(b_{21}-rb_{31})}{(b_{21}-rb_{31})(b_{12}-pb_{32})-(b_{22}-rb_{32})(b_{11}-pb_{31})}.
\end{aligned}$$

I coefficienti delle trasformazioni infinitesime sono raccolti nella seguente tabella:

$$\xi_1^1 = -a,\ \xi_1^2 = -2b,\ \xi_1^3 = -c,\ \xi_1^4 = -2d,\ \xi_1^5 = -2e,\ \xi_1^6 = -x_0,$$

$$\xi_1^7 = 0,\ \xi_1^8 = -2z_0,\ \xi_1^9 = -p,\ \xi_1^{10} = -q,\ \xi_1^{11} = 0,\ \xi_1^{12} = 0,$$

$$\xi_2^1 = 1,\ \xi_2^2 = 2a,\ \xi_2^3 = 0,\ \xi_2^4 = c,\ \xi_2^5 = 0,\ \xi_2^6 = -y_0,$$

$$\xi_2^7 = 0,\ \xi_2^8 = 0,\ \xi_2^9 = -r,\ \xi_2^{10} = -s,\ \xi_2^{11} = 0,\ \xi_2^{12} = 0,$$

$$\xi_3^1 = 0,\ \xi_3^2 = 0,\ \xi_3^3 = 0,\ \xi_3^4 = 0,\ \xi_3^5 = 0,\ \xi_3^6 = -z_0,$$

$$\xi_3^7 = 0,\ \xi_3^8 = 0,\ \xi_3^9 = 1,\ \xi_3^{10} = 0,\ \xi_3^{11} = 0,\ \xi_3^{12} = 0,$$

$$\xi_4^1 = 0,\ \xi_4^2 = 0,\ \xi_4^3 = 1,\ \xi_4^4 = a,\ \xi_4^5 = 2c,\ \xi_4^6 = -1,$$

$$\xi_4^7 = 0,\ \xi_4^8 = 0,\ \xi_4^9 = 0,\ \xi_4^{10} = -1,\ \xi_4^{11} = 0,\ \xi_4^{12} = 0,$$

$$\xi_5^1 = b - 2a^2,\ \xi_5^2 = -2ab,\ \xi_5^3 = d - 2ac,\ \xi_5^4 = -2ad,$$
$$\xi_5^5 = -2ae,\ \xi_5^6 = 0,$$

$$\xi_5^7 = -x_0,\ \xi_5^8 = -2z_0,\ \xi_5^9 = 0,\ \xi_5^{10} = 0,\ \xi_5^{11} = -p,\ \xi_5^{12} = -q,$$

$$\xi_6^1 = a,\ \xi_6^2 = 2b,\ \xi_6^3 = 0,\ \xi_6^4 = d,\ \xi_6^5 = 0,\ \xi_6^6 = 0,$$

$$\xi_6^7 = -y_0,\ \xi_6^8 = 0,\ \xi_6^9 = 0,\ \xi_6^{10} = 0,\ \xi_6^{11} = -r,\ \xi_6^{12} = -s,$$

$$\xi_7^1 = 0,\ \xi_7^2 = 0,\ \xi_7^3 = 0,\ \xi_7^4 = 0,\ \xi_7^5 = 0,\ \xi_7^6 = 0,$$

$$\xi_7^7 = -z_0,\ \xi_7^8 = 0,\ \xi_7^9 = 0,\ \xi_7^{10} = 0,\ \xi_7^{11} = -1,\ \xi_7^{12} = 0,$$

$$\xi_8^1 = 0,\ \xi_8^2 = 0,\ \xi_8^3 = a,\ \xi_8^4 = b,\ \xi_8^5 = 2d,\ \xi_8^6 = 0,$$

$$\xi_8^7 = -1,\ \xi_8^8 = 0,\ \xi_8^9 = 0,\ \xi_8^{10} = 0,\ \xi_8^{11} = 0,\ \xi_8^{12} = -1,$$

$$\xi_9^1 = \frac{D}{2} - Ba,\ \xi_9^2 = -Bb,\ \xi_9^3 = \frac{E}{2} - Bc,\ \xi_9^4 = -Bd,$$
$$\xi_9^5 = -Be,\ \xi_9^6 = 0,$$

$$\xi_9^7 = 0,\ \xi_9^8 = x_0 + 2ay_0 + c,\ \xi_9^9 = p^2,\ \xi_9^{10} = pq,\ \xi_9^{11} = pr,\ \xi_9^{12} = qr,$$

$$\xi_{10}^1 = \frac{B}{2},\ \xi_{10}^2 = D,\ \xi_{10}^3 = 0,\ \xi_{10}^4 = \frac{E}{2},\ \xi_{10}^5 = 0,\ \xi_{10}^6 = 0,$$

$$\xi_{10}^7 = 0,\ \xi_{10}^8 = y_0,\ \xi_{10}^9 = pr,\ \xi_{10}^{10} = ps,\ \xi_{10}^{11} = r^2,\ \xi_{10}^{12} = rs,$$

$$\xi_{11}^1 = 0,\ \xi_{11}^2 = 0,\ \xi_{11}^3 = 0,\ \xi_{11}^4 = 0,\ \xi_{11}^5 = 0,\ \xi_{11}^6 = 0,$$

$$\xi_{11}^7 = 0,\ \xi_{11}^8 = -z_0,\ \xi_{11}^9 = p,\ \xi_{11}^{10} = 0,\ \xi_{11}^{11} = r,\ \xi_{11}^{12} = 0,$$

$$\xi_{12}^1 = 0,\ \xi_{12}^2 = 0,\ \xi_{12}^3 = \frac{B}{2},\ \xi_{12}^4 = \frac{D}{2},\ \xi_{12}^5 = E,\ \xi_{12}^6 = 0,$$
$$\xi_{12}^7 = 0,\ \xi_{12}^8 = -z_0,\ \xi_{12}^9 = 0,\ \xi_{12}^{10} = p,\ \xi_{12}^{11} = 0,\ \xi_{12}^{12} = r.$$

Il corrispondente sistema di Deltheil è costituito dalle seguenti equazioni differenziali:

$$\begin{cases}
a\dfrac{\partial\Phi}{\partial a} + 2b\dfrac{\partial\Phi}{\partial b} + c\dfrac{\partial\Phi}{\partial c} + 2d\dfrac{\partial\Phi}{\partial d} + 2e\dfrac{\partial\Phi}{\partial e} + x_0\dfrac{\partial\Phi}{\partial x_0} + 2x_0\dfrac{\partial\Phi}{\partial z_0} + \\
\qquad + p\dfrac{\partial\Phi}{\partial p} + q\dfrac{\partial\Phi}{\partial q} = -13\Phi, \\
\dfrac{\partial\Phi}{\partial a} + 2a\dfrac{\partial\Phi}{\partial b} + c\dfrac{\partial\Phi}{\partial d} - y_0\dfrac{\partial\Phi}{\partial x_0} - r\dfrac{\partial\Phi}{\partial p} - s\dfrac{\partial\Phi}{\partial q} = 0, \\
-z_0\dfrac{\partial\Phi}{\partial x_0} + \dfrac{\partial\Phi}{\partial p} = 0, \\
\dfrac{\partial\Phi}{\partial c} + a\dfrac{\partial\Phi}{\partial d} + 2c\dfrac{\partial\Phi}{\partial e} - \dfrac{\partial\Phi}{\partial x_0} - \dfrac{\partial\Phi}{\partial q} = 0, \\
(b^2 - 2a^2)\dfrac{\partial\Phi}{\partial a} - 2ab\dfrac{\partial\Phi}{\partial b} + (d - 2ac)\dfrac{\partial\Phi}{\partial c} - 2ad\dfrac{\partial\Phi}{\partial d} - 2ae\dfrac{\partial\Phi}{\partial e} - \\
\qquad - x_0\dfrac{\partial\Phi}{\partial y_0} - 2z_0\dfrac{\partial\Phi}{\partial z_0} - p\dfrac{\partial\Phi}{\partial r} - q\dfrac{\partial\Phi}{\partial s} = (12a + 2)\Phi, \\
a\dfrac{\partial\Phi}{\partial a} + 2b\dfrac{\partial\Phi}{\partial b} + d\dfrac{\partial\Phi}{\partial d} - y_0\dfrac{\partial\Phi}{\partial y_0} - r\dfrac{\partial\Phi}{\partial r} - s\dfrac{\partial\Phi}{\partial s} = -\Phi, \\
z_0\dfrac{\partial\Phi}{\partial y_0} + \dfrac{\partial\Phi}{\partial r} = 0, \\
a\dfrac{\partial\Phi}{\partial c} + b\dfrac{\partial\Phi}{\partial d} + 2d\dfrac{\partial\Phi}{\partial e} - \dfrac{\partial\Phi}{\partial y_0} - \dfrac{\partial\Phi}{\partial s} = 0, \\
(2ay_0 + ax_0 + 2ac - by_0 - d)\dfrac{\partial\Phi}{\partial a} + 2b(x_0 + ay_0 + c)\dfrac{\partial\Phi}{\partial b} + \\
\qquad + (cx_0 + 2acy_0 + 2c^2 - dy_0 - e)\dfrac{\partial\Phi}{\partial c} + \\
\qquad + 2d(x_0 + ay_0 + c)\dfrac{\partial\Phi}{\partial d} + 2e(x_0 + ay_0 + c)\dfrac{\partial\Phi}{\partial e} + \\
\qquad + z_0(x_0 + 2ay_0 + 2c)\dfrac{\partial\Phi}{\partial z_0} + z_0p^2\dfrac{\partial\Phi}{\partial p} + \\
\qquad + z_0pq\dfrac{\partial\Phi}{\partial q} + z_0pr\dfrac{\partial\Phi}{\partial r} + z_0qr\dfrac{\partial\Phi}{\partial s} = -4(3ay_0 + 2x_0 + 3c + pz_0)\Phi, \\
(x_0 + ay_0 + c)\dfrac{\partial\Phi}{\partial a} + 2(ax_0 + by_0 + d)\dfrac{\partial\Phi}{\partial b} + (cx_0 + dy_0 + e)\dfrac{\partial\Phi}{\partial d} +
\end{cases}$$

$$
\left|\begin{array}{l}
+ z_0 y_0 \frac{\partial \Phi}{\partial z_0} - p r z_0 \frac{\partial \Phi}{\partial p} - p s z_0 \frac{\partial \Phi}{\partial q} - r^2 y_0 \frac{\partial \Phi}{\partial r} + \\
+ r s z_0 \frac{\partial \Phi}{\partial z_0} = (4 r z_0 - 4 y_0)\Phi, \\
-z_0 \frac{\partial \Phi}{\partial z_0} + p \frac{\partial \Phi}{\partial p} + r \frac{\partial \Phi}{\partial r} = -\Phi, \\
(x_0 + a y_0 + c)\frac{\partial \Phi}{\partial c} + (a x_0 + b y_0 + d)\frac{\partial \Phi}{\partial d} + 2(c x_0 + d y_0 + e)\frac{\partial \Phi}{\partial e} + \\
+ z_0 \frac{\partial \Phi}{\partial z_0} - p z_0 \frac{\partial \Phi}{\partial q} - r z_0 \frac{\partial \Phi}{\partial s} = -4\Phi.
\end{array}\right.
$$

Tale sistema, dividendo ogni sua equazione per Φ, riducesi ad un sistema di equazioni algebriche, lineare non omogeneo, crameriano nelle dodici incognite:

$$
\frac{\partial ln\Phi}{\partial a}, \frac{\partial ln\Phi}{\partial b}, \frac{\partial ln\Phi}{\partial c}, \frac{\partial ln\Phi}{\partial d}, \frac{\partial ln\Phi}{\partial e}, \frac{\partial ln\Phi}{\partial x_0},
$$
$$
\frac{\partial ln\Phi}{\partial y_0}, \frac{\partial ln\Phi}{\partial z_0}, \frac{\partial ln\Phi}{\partial p}, \frac{\partial ln\Phi}{\partial q}, \frac{\partial ln\Phi}{\partial r}, \frac{\partial ln\Phi}{\partial s},
$$

il quale ammette un'unica soluzione, cioè un valore per $\frac{\partial ln\Phi}{\partial a}$, uno per $\frac{\partial ln\Phi}{\partial b}$, ..., uno per $\frac{\partial ln\Phi}{\partial s}$.
Ne segue che si potrà determinare l'unica soluzione Φ, a meno di costanti moltiplicative, per il sistema stesso.
Pertanto il gruppo isomorfo a (2) è misurabile. Ciò basta, stante la prima condizione di Stoka, per concludere con il

Teorema. *La famiglia dei sistemi formati da cono quadrico e retta in A_3 è misurabile.*

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