LE MATEMATICHE
Vol. LI (1996) – Supplemento, pp. 115-125

AMPLE VECTOR BUNDLES WITH SECTIONS VANISHING ON SURFACES OF KODAIRA DIMENSION ZERO

ANTONIO LANTERI

To the memory of Umberto Gasapina

Let $\mathcal{E}$ be an ample vector bundle of rank $r \geq 2$ on a compact complex manifold X of dimension $n = r + 2$ having a section whose zero locus is a smooth surface Z. Triplets $(X, \mathcal{E}, Z)$ as above are investigated under the assumption that Z has Kodaira dimension zero. It turns out that either X is a $\mathbb{P}^{n-2}$-bundle over a smooth surface, $\mathcal{E}$ restricts to every fibre as $\mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$ and the $\mathbb{P}$-bundle projection restricted to Z is a birational morphism contracting some exceptional curves, or, up to contracting some (-1)-hyperplanes, X is a Fano manifold with $-K_X = \det \mathcal{E}$, in which case Z is a K3 surface.

Introduction.

It is well known that imposing to a projective manifold X to contain a given manifold Z as an ample divisor gives very strong restrictions. This philosophy, which arose in the context of hyperplane sections long time ago, was made explicit by Sommese ([16], p. 56) by saying that a projective manifold X is as special as any of its ample hypersurfaces. Recently Maeda and the author [4], [5], [6] started to revisit this philosophy in the setting of ample vector

1991 Mathematics Subject Classification: Primary 14J60; Secondary 14C20, 14F05, 14J40.

bundles discovering some analogies which deserve to be further investigated. The appropriate set-up to do this is the following.

(0.1) $\mathscr{E}$ is an ample vector bundle of rank $r \geq 2$ on a complex projective manifold X of dimension n such that there exists a section $s \in \Gamma(\mathscr{E})$ whose zero locus $Z = (s)_0$ (as a scheme) is a submanifold of X of the expected dimension $n - r$.

Note that (0.1) includes the case of ample and spanned (i.e. globally generated) vector bundles, since in this case any general section satisfies the condition above due to the Bertini theorem (e.g. see [12], Theorem 1.10 or [15], Teorema 3.8).

Pairs $(X, \mathscr{E})$ as in (0.1) with $n - r \geq 1$ have been studied in [4] for Z a projective space or a quadric, in [5] for Z a ruled surface, and in [6] for Z a scroll or a quadric fibration over a smooth curve with respect to a polarizing ample line bundle on X. The aim of this paper is to investigate $(X, \mathscr{E})$ when Z is a surface of Kodaira dimension zero. The main result is the following

Theorem. *Let* $(X, \mathscr{E})$ *be as in* (0.1) *and assume that* Z *is a surface of Kodaira dimension* $\kappa(Z) = 0$. *Then* $(X, \mathscr{E})$ *is one of the following:*

(1) $X = \mathbb{P}_S(\mathscr{F})$, *where* $\mathscr{F}$ *is an ample vector bundle of rank* $n - 1$ *over a smooth surface* S *with* $\kappa(S) = 0$ *and* $\mathscr{E} = \pi^*\mathscr{V} \otimes H$, *where* $H = H(\mathscr{F})$ *is the tautological line bundle on* X, $\mathscr{V}$ *is a vector bundle of rank* $n - 2$ *on* S *and* $\pi : X \to S$ *is the bundle projection; moreover* $\pi_{|Z} : Z \to S$ *is a birational morphism, but not an isomorphism;*

(2) *there exist a birational morphism* $\varphi : X \to X'$ *expressing* X *as a projective manifold* X' *blown-up at a finite set* B *and an ample vector bundle* $\mathscr{E}'$ *of rank* $n - 2$ *on* X', *such that* $\mathscr{E} = \varphi^*\mathscr{E}' \otimes [-\varphi^{-1}(B)]$, *having a section whose zero locus* Z' *satisfies* (0.1). *Moreover* X' *is a Fano manifold with* $-K_{X'} = \det \mathscr{E}'$ *and* Z' *is a* K3 *surface dominated by* Z *via the birational morphism* $\varphi_{|Z}$.

Both cases are effective. Note that in case (2) B could contain some infinitely near point; in this case $\varphi^{-1}(B)$ has to be intended as a non reduced divisor.

The idea of the proof is the following. Since Z is smooth and of the expected dimension, its normal bundle can be identified with $\mathscr{E}_Z$ so that $K_Z = (K_X + \det \mathscr{E})_Z$, by adjunction. Hence if Z is not minimal $K_X + \det \mathscr{E}$ cannot be nef. By using [9] we thus see that there is a long but very precise list of possibilities for $(X, \mathscr{E})$; but for the most part of them it turns out that all smooth 2-dimensional zero loci of sections of $\mathscr{E}$ are ruled surfaces, contradicting our assumption on Z. This easily leads to case (1) or, up to a sequence of contractions, one gets a similar situation with Z minimal. On the other hand if

Z is minimal, then the Lefschetz-Sommese theorem combined with elementary properties of Fano manifolds immediately leads to (2) with $X = X'$.

The paper is organized as follows. In Section 1 the case when Z is a minimal surface with $\kappa(Z) = 0$ is discussed from a slightly more general point of view and some examples are given. The case of non-minimal surfaces occurring as zero loci is settled in Section 2. The Theorem is proved in Section 3. In Section 4 I discuss some more details on case (1) and improve a result of [5] as an application of the Theorem.

I would like to thank Hidetoshi Maeda for many helpful discussions and his valuable criticism.

1. Zero loci with numerically trivial canonical bundle.

Throughout all the paper varieties are defined over the complex number field $\mathbb{C}$. We use the standard notation from algebraic geometry. Projective manifold means smooth projective variety. We make no distinction between vector bundles and locally free sheaves. The tensor products of line bundles are denoted additively, while we use multiplicative notation for intersection products in Chow rings. Let X be a projective manifold. The pull-back of a vector bundle $\mathscr{E}$ on X by an embedding $i : Y \hookrightarrow X$ is denoted by $\mathscr{E}_Y$. The canonical bundle of X is denoted by K_X and $\kappa(X)$ stands for the Kodaira dimension of X. X is called a *Fano manifold* if its anticanonical bundle $-K_X$ is ample. A projective manifold X is said to be *regular* if $h^1(\mathscr{O}_X) = 0$. Let $(X, \mathscr{E})$ be a pair as in (0.1) and assume that $n - r \geq 1$. We point out the following fact mentioned in the Introduction. Since Z is smooth of the expected dimension $n - r$, the normal bundle $N_{Z/X}$ of Z in X is

$$N_{Z/X} \cong \mathscr{E}_Z$$

(e.g. [2], Example 6.3.4 or [12], p. 266). Hence, by adjunction, we get

$$K_Z = (K_X)_Z + \det N_{Z/X} = (K_X + \det \mathscr{E})_Z. \tag{1.0.1}$$

Lemma 1.1. *Let $(X, \mathscr{E})$ be a pair as in (0.1) with $n - r \geq 2$ and assume that K_Z is torsion. Then Z is regular, K_Z is trivial and X is a Fano manifold with $-K_X = \det \mathscr{E}$. In particular all extremal rays of X have length $\geq r$.*

Proof. By the Lefschetz-Sommese theorem ([6], Theorem 1.1) the restriction homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is injective (with torsion free cokernel). Thus, since K_Z is torsion, in view of (1.0.1) there exists a positive integer m such that $m(K_X + \det \mathscr{E}) = \mathscr{O}_X$. This implies that $-K_X$ is ample, i.e.

X is Fano. But then Pic(X) has no torsion (e.g see [7], Lemma 1.3), hence $K_X + \det \mathscr{E}$ itself is trivial. Thus K_Z is trivial by (1.0.1). Moreover since Fano manifolds are regular, by the Lefschetz-Sommese theorem again we get $h^1(\mathscr{O}_Z) = h^1(\mathscr{O}_X) = 0$. Finally, for every rational curve $C \subset X$, looking at the normalization $\nu : \mathbb{P}^1 \to C$ we have

$$-K_X C = (\det \mathscr{E})C = \deg(\nu^* \mathscr{E}_C) \geq r,$$

due to the ampleness of $\mathscr{E}$. So the last assertion follows from the definition of the length $l(R)$ of an extremal ray R of X

$$l(R) := \min\{-K_X C \mid C \subset X \text{ rational curve with } [C] \in R\}. \qquad \square$$

1.2. For $n - r = 3$, 1.1 says that Z is a Calabi-Yau threefold. To give an example of this situation consider the Cayley bundle $\mathscr{C}$ on the 5-dimensional smooth hyperquadric $\mathbb{Q}^5$ [14]. $\mathscr{C}(1)$ is a rank-2 vector bundle with $c_1 = 1$. Moreover $\mathscr{C}(2)$ is spanned ([14], Theorem 3.7); hence the rank-2 vector bundle $\mathscr{E} := \mathscr{C}(3)$ is ample and spanned. Thus, by combining the Lefschetz-Sommese theorem with (1.0.1), we conclude that the zero locus of the general section of $\mathscr{E}$ is a Calabi-Yau threefold.

For $n - r = 2$ we get the situation described in case (2) of the Theorem for $(X', \mathscr{E}', Z')$.

Corollary 1.3. *Let* $(X, \mathscr{E})$ *and* Z *be as in* (0.1). *If* Z *is a minimal surface of Kodaira dimension* $\kappa(Z) = 0$, *then* X *is a Fano manifold with* $-K_X = \det \mathscr{E}$ *and* Z *is a* K3 *surface.*

1.4. Note that this situation is effective as shown by the following examples.

i) Let X be a Mukai manifold of dimension n, i.e. $-K_X = (n-2)\mathscr{H}$ with $\mathscr{H} \in \text{Pic}(X)$ ample. Recently Mella [10] proved that the general element of the fundamental system $|\mathscr{H}|$ is smooth; it thus follows (e.g. see [11], Proposition 1) that, when $b_2(X) = 1$, $|\mathscr{H}|$ is base point free. Then $\mathscr{E} := \mathscr{H}^{\oplus(n-2)}$ is ample and spanned and its general section vanishes along a K3 surface. This is true "a fortiori" when $b_2(X) \geq 2$ and $\mathscr{H}$ is very ample; see [11], Example 2, for concrete examples.

ii) A less obvious example is the following. On $\mathbb{Q}^6$ let $\mathscr{E} := \mathscr{S}(2)$, where $\mathscr{S}$ is one of the two spinor bundles. Recall that $\mathscr{S}(1)$ is spanned as a quotient of $\mathscr{O}_{\mathbb{Q}}^{\oplus 8}$ ([13], Theorem (2.8), (ii)); hence $\mathscr{S}(2)$ is ample and spanned and by adjunction it is easy to see that its general section vanishes along a K3 surface.

iii) For more examples see ([8], (2.5) and the list in Theorem 3.1).

Another obvious consequence of 1.1, extending a well known result in the setting of ample divisors ([16], Corollary I-A), is the following

Corollary 1.5. *For $n - r \geq 2$ no abelian variety can occur as the zero locus of a section of an ample vector bundle of rank r on a projective manifold of dimension n.*

2. Non-minimal surfaces as zero loci.

As a first thing we have the following

Lemma 2.1. *Let $(X, \mathcal{E})$ and Z be as in (0.1) with $n - r = 2$ and assume that Z is not ruled. If Z is not minimal, then either*

a) *$X = \mathbb{P}_S(\mathcal{F})$, $\mathcal{F}$ denoting an ample vector bundle of rank $n - 1$ over a smooth surface S and $\mathcal{E} = \pi^*\mathcal{V} \otimes H$, where $H = H(\mathcal{F})$ is the tautological line bundle on X, $\mathcal{V}$ is a vector bundle of rank $n - 2$ on S and $\pi : X \to S$ is the bundle projection, or*

b) *there exists an effective divisor E on X such that*

$$(E, \mathcal{O}_E(E), \mathcal{E}_E) \cong (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}}(-1), \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}).$$

Proof. Since Z contains some exceptional curve K_Z is not nef, hence $K_X + \det \mathcal{E}$ is not nef in view of (1.0.1). Therefore $(X, \mathcal{E})$ fits into one the 13 cases listed in [9], Theorem. However in cases (1) – (7) and (9) – (12) any smooth 2-dimensional zero locus S of a section of $\mathcal{E}$ is a ruled surface. Actually, S is $\mathbb{P}^2$ in case (1), and a quadric in cases (2), (5) and (6); in cases (3), (4) S is a Del Pezzo surface of degree 3, 4 respectively; in cases (7), (9) S is again a Del Pezzo surface. This is obvious in case (9). In case (7) we have $\mathcal{E} = \mathcal{S}(2)$, $\mathcal{S}$ being a spinor bundle on $\mathbb{Q}^4$. Since $\mathcal{E}$ is ample and spanned, its general section vanishes along a smooth surface S and since $c_1(\mathcal{E}) = c_1(\mathcal{S}) + 4 = 3$, by (1.0.1) we get $K_S = (\mathcal{O}_{\mathbb{Q}}(-4) + c_1(\mathcal{E}))_S = (\mathcal{O}_{\mathbb{Q}}(-1))_S$, so that S is Del Pezzo. Finally S is a scroll in case (10) and a conic fibration in cases (11) and (12). As to the surviving cases in the list, (8) gives rise to b), while in case (13) we have that $X = \mathbb{P}_S(\mathcal{F})$ for some vector bundle $\mathcal{F}$ of rank $n-1$ on a smooth surface S and

$$\mathcal{E}_F \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$$

for every fibre F of the bundle projection $\pi : X \to S$. Hence there exists a vector bundle $\mathcal{V}$ of rank $n - 2$ on S such that $\mathcal{E} \otimes [-H] = \pi^*\mathcal{V}$, where

$H = H(\mathscr{F})$ denotes the tautological line bundle of $\mathscr{F}$ on X. In particular this gives

$$\det \mathscr{E} = (n-2)H + \pi^* \det \mathscr{V}. \tag{2.1.1}$$

We also have

$$K_X = -(n-1)H + \pi^*(K_S + \det \mathscr{F}). \tag{2.1.2}$$

Now recall that the line bundle defined by the Wiśniewski relation

$$L := (n-3)K_X + (n-2)\det \mathscr{E}, \tag{2.1.3}$$

is ample and gives (X, L) the structure of a scroll over S (see [9], (2.2.8) and (2.10)). We thus get from (2.1.3), (2.1.2), (2.1.1)

$$L = -(n-1)(n-3)H + (n-3)\pi^*(K_S + \det \mathscr{F}) + (n-2)(\pi^* \det \mathscr{V} + (n-2)H),$$

i.e.

$$L - H = \pi^*((n-3)(K_S + \det \mathscr{F}) + (n-2)\det \mathscr{V}).$$

This shows that up to twisting $\mathscr{F}$ by a suitable line bundle on S we can assume that $\mathscr{F}$ is ample. This gives case a). □

Let $(X, \mathscr{E})$ and Z be as in 2.1, case b). Since

$$(E, \mathscr{O}_E(E)) \cong (\mathbb{P}^{n-1}, \mathscr{O}_{\mathbb{P}}(-1))$$

there exists a birational morphism $\varphi : X \to X'$ onto a projective manifold X' of dimension n contracting E. More information on this case are provided by the following

Lemma 2.2. *Let* $(X, \mathscr{E})$ *and* Z *be as in* (2.1), *case* b) *and let* $\varphi : X \to X'$ *be the contraction of* E. *Then there exist an ample vector bundle* $\mathscr{E}'$ *of rank* $n-2$ *on* X' *and a section* $s' \in \Gamma(\mathscr{E}')$ *such that* $(X', \mathscr{E}')$ *and* $Z' := (s')_0$ *satisfy* (0.1) *and* $\varphi_{|Z} : Z \to Z'$ *is a birational morphism. Moreover* $(X', \mathscr{E}')$ *cannot be as in case* a).

Proof. By [5], Lemma 5.1, we know that there exists an ample vector bundle $\mathscr{E}'$ of rank $n-2$ on X' such that

$$\mathscr{E} = \varphi^* \mathscr{E}' \otimes \mathscr{O}_X(-E). \tag{2.2.1}$$

Now look at the section $s \in \Gamma(\mathcal{E})$ vanishing on Z. Due to our assumption, its restriction to E is an element $s_E \in \Gamma(\mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)})$. Hence $(s_E)_0 = Z \cap E$ is a linear subspace of E of dimension ≥ 1. Of course it cannot be a 2-dimensional linear subspace of E; otherwise we would get $Z = (s_E)_0 \cong \mathbb{P}^2$, contradicting the assumption that Z is not ruled. Hence $(s_E)_0$ is a line inside E. Set $e := (s_E)_0$. Then e is a smooth rational curve inside Z; moreover

$$e^2 = E_Z e = \mathcal{O}_Z(E)e = \mathcal{O}_E(-1)e = -1.$$

Therefore e is a (-1)-curve inside Z, which is contracted by $\varphi_{|Z}$. Hence $Z' := \varphi(Z)$ is a smooth surface inside X', birational to Z. Moreover let $s' \in \Gamma(\mathcal{E}')$ be the section corresponding to s via the isomorphism induced by (2.2.1); then $(s')_0 = Z'$, so that $(X', \mathcal{E}')$ and Z' satisfy (0.1). To conclude the proof assume, by contradiction, that $(X', \mathcal{E}')$ is as in case a) of 2.1 and let $\pi : X' \to S$ be the bundle projection. Let $x = \varphi(E)$, let F be the fibre of π containing x and let $\tilde{F}$ be its proper transform. Note that $\tilde{F} = \mathbb{P}(N_{x/F})$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-3}$ whose fibres are the proper transforms of the lines in F passing through x; moreover $E \cap \tilde{F}$ is a section of this $\mathbb{P}^1$-bundle. So, if f is a fibre of $\tilde{F}$ we have $\mathcal{O}_X(E)f = \mathcal{O}_{\tilde{F}}(E)f = 1$. Recalling that $\mathcal{E}'_F \cong \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}$ we thus get by (2.2.1)

$$\mathcal{E}_f = (\varphi^*\mathcal{E}' \otimes \mathcal{O}_X(-E))_f = \mathcal{O}_f^{\oplus(n-2)},$$

contradicting the ampleness of $\mathcal{E}$. □

3. Proof of the Theorem.

The theorem will follow by combining 2.1 and 2.2 with 1.3.

Proof. If Z is a minimal surface then the assertion follows from 1.3. So assume that Z is not minimal. Then 2.1 applies. If we are in case a) then for every fibre F of $\pi : X \to S$, since $s_F \in \Gamma(\mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)})$, we see that $(s_F)_0 = Z \cap F \neq \emptyset$. This shows that $\pi_{|Z} : Z \to S$ is a surjective morphism. Note that, due to the irreducibility of Z, it can be $\dim(Z \cap F) > 0$ only for a finite number of fibres, and for any such a fibre F_0, $\dim(Z \cap F_0) = 1$. So, $f_0 := Z \cap F_0$ is a a line inside F_0, hence a smooth rational curve inside Z; moreover f_0 is a (-1)-curve since $\pi_{|Z}$ contracts it to a smooth point, due to the smoothness of S. This shows that $\pi_{|Z}$ is a birational morphism. Note that $\mathrm{Pic}(X) \cong \pi^*\mathrm{Pic}(S) \oplus \mathbb{Z}$, X being a $\mathbb{P}$-bundle over S. Since the restriction homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is injective by the Lefschetz-Sommese theorem [6], Theorem 1.1, this gives an injective homomorphism

$$\pi_{|Z}^*\mathrm{Pic}(S) \oplus \mathbb{Z} \to \mathrm{Pic}(Z),$$

which shows that $\pi_{|Z}$ cannot be an isomorphism. So this gives case (1) of the Theorem. Now suppose we are in case b). By applying inductively 2.2 we thus see that there exists a birational morphism $\varphi : X \to X'$ leading to a triple $(X', \mathscr{E}', Z')$ satisfying (0.1), where Z' is a minimal surface with $\kappa(Z') = 0$. Thus by applying 1.3 to $(X', \mathscr{E}', Z')$ we get the situation described in case (2) of the Theorem. □

4. Final remarks.

As a first thing let us give some more details on the situation described in case (1) of the Theorem, without assuming that $\kappa(S) = 0$. So,

4.0. let S be a smooth surface and let $\mathscr{F}$ and $\mathscr{V}$ be an ample vector bundle of rank $n-1$ and a vector bundle of rank $n-2$ on S respectively. Let $X := \mathbb{P}_S(\mathscr{F})$ and

(4.0.1) $$\mathscr{E} := H(\mathscr{F}) \otimes \pi^*\mathscr{V},$$

where $\pi : X \to S$ stands for the bundle projection and $H(\mathscr{F})$ is the tautological line bundle on X. Assume furthermore that $\mathscr{E}$ is ample and that (0.1) is satisfied.

Note that the situation described in 4.0 gives rise to a pair $(X, \mathscr{E})$ as in case (1) of the Theorem, apart from the assumption on $\kappa(Z)$. The same argument as in Section 3 shows that for any section of $\mathscr{E}$ vanishing on a smooth surface Z, $\pi_{|Z} : Z \to S$ is a birational morphism. Here we want to compute the number of the exceptional curves it contracts. We have

Proposition 4.1. *Let things be as in* 4.0 *and let* t *be the number of the exceptional curves contracted by* $\pi_{|Z}$*; then*

$$t = c_2(\mathscr{F}) + c_1(\mathscr{F})c_1(\mathscr{V}) + c_1(\mathscr{V})^2 - c_2(\mathscr{V}).$$

Proof. Set $H = H(\mathscr{F})$. Recalling (2.1.1) and (2.1.2) we have

$$K_X + \det \mathscr{E} = -H + \pi^*(K_S + \det \mathscr{F} + \det \mathscr{V}).$$

So, recalling also (1.0.1), we get

(4.1.1) $$K_Z = (-H + \pi^*(K_S + \det \mathscr{F} + \det \mathscr{V}))_Z.$$

On the other hand since (0.1) is satisfied, Z represents the top Chern class $c_{n-2}(\mathscr{E})$. Hence by (4.0.1) the class of Z in the Chow ring of X is given by

(4.1.2) $$Z = H^{n-2} + H^{n-3}\pi^*c_1(\mathscr{V}) + H^{n-4}\pi^*c_2(\mathscr{V}).$$

Taking into account the Leray-Hirsch relation for $\mathscr{F}$

$$H^{n-1} - H^{n-2}\pi^*c_1(\mathscr{F}) + H^{n-3}\pi^*c_2(\mathscr{F}) = 0,$$

we thus get from (4.1.1), (4.1.2), after some computations,

$$K_Z^2 = -c_2(\mathscr{F}) - c_1(\mathscr{F})c_1(\mathscr{V}) + c_2(\mathscr{V}) - c_1(\mathscr{V})^2 + K_S^2.$$

Then the assertion follows recalling that $t = K_S^2 - K_Z^2$. □

Note that for $\mathscr{V} = \mathscr{O}_S^{\oplus(n-2)}$ 4.1 gives $t = c_2(\mathscr{F})$, a fact which is well known at least in the setting of hyperplane sections or, more generally, when $\mathscr{F}$ is ample and spanned.

4.2. Looking at (4.0.1) the ampleness of $\mathscr{E}$ seems to suggest that $\mathscr{V}$ cannot be too much negative. Though I could not state this in a precise form, there is an obvious inequality involving the Chern classes of $\mathscr{V}$ and $\mathscr{F}$. Actually the same argument as in Section 3 shows that

$$t \geq 1 \tag{4.2.1}$$

regardless any assumption on $\kappa(S)$. So 4.1 gives the following inequality:

$$c_1(\mathscr{V})^2 - c_2(\mathscr{V}) + c_1(\mathscr{V})c_1(\mathscr{F}) \geq 1 - c_2(\mathscr{F}). \tag{4.2.2}$$

Note that (4.2.2) is certainly satisfied when $\mathscr{V}$ is nef, since in this case the Schur polynomial $c_1^2 - c_2$ is non-negative ([1], Theorem 2.5) and of course $c_1(\mathscr{V})c_1(\mathscr{F}) \geq 0$ due to the ampleness of $\mathscr{F}$. On the other hand, in view of several results on ample vector bundles with low top Chern class on surfaces, it seems reasonable to wonder whether (4.2.1) could be sharpened for $\kappa(S) \geq 0$.

We conclude this Section by improving a result obtained in [5]. Referring to what we said at the beginning of the Introduction, a natural way to rephrase the speciality of a submanifold of X is to suppose that it is not of general type. If $\mathscr{E}$ is an ample line bundle on X having a section which vanishes on a smooth hypersurface Z, then the condition that $\kappa(Z) < \dim Z$ implies $\kappa(X) = -\infty$ (see [7], Proposition 1.1 and [3], Proposition 5). It is a natural question to ask whether this implication continues to hold in the setting of ample vector bundles satisfying (0.1). A partial affirmative answer was provided in [5], Section 5, discussing the case $r = n - 2$ with Z a ruled surface. As a consequence of our Theorem we have the following improvement

Corollary 4.3. *Let* $(X, \mathscr{E})$ *and* Z *be as in* (0.1) *with* $n - r = 2$. *If* $\kappa(Z) \leq 0$ *then* $\kappa(X) = -\infty$.

Proof. In view of [5], Corollary 5.2, we can assume that $\kappa(Z) = 0$, hence $(X, \mathscr{E})$ is as in the Theorem. In case (1) having $(K_X)_F = K_F \cong \mathscr{O}_{\mathbb{P}}(1-n)$ for every fibre F of π, we see that $h^0(mK_X) = 0$ for all $m > 0$. In case (2) we have $\kappa(X) = \kappa(X')$ and of course $h^0(mK_{X'}) = 0$ for all $m > 0$, since $-K_{X'}$ is ample. □

REFERENCES

[1] J.P. Demailly - Th. Peternell - M. Schneider, *Compact complex manifolds with numerically effective tangent bundles,* J. Algebraic Geometry, 3 (1994), pp. 295-345.

[2] W. Fulton, *Intersection Theory,* Ergeb. Math. Grenzgeb. (3), vol. 2 Springer-Verlag, Berlin, 1984.

[3] P. Ionescu, *Generalized adjunction and applications,* Math. Proc. Cambridge Philos. Soc., 99 (1986), pp. 456-472.

[4] A. Lanteri - H. Maeda, *Ample vector bundles with sections vanishing on projective spaces or quadrics,* Intern. J. Math., 6 (1995), pp. 587-600.

[5] A. Lanteri - H. Maeda, *Geometrically ruled surfaces as zero loci of ample vector bundles,* Forum Math., 9 (1997), pp. 1-15.

[6] A. Lanteri - H. Maeda, *Ample vector bundle characterizations of projective bundles and quadric fibrations over curves,* Higher Dimensional Complex Varieties (M. Andreatta and Th. Peternell, eds), Proc. Intern. Conf. Trento, June 1994, De Gruyter, Berlin-New York,1996, pp. 247-259.

[7] A. Lanteri - M. Palleschi, *Remarks on recent work by Sommese,* Proc. Conference Algebraic Geometry, Berlin, 1985 (H. Kurke and M. Roczen, eds), Texte Math., vol. 92, Teubner Leipzig, 1986, pp. 228-235.

[8] A. Lanteri - C.S. Sacchi, *On some special Fano manifolds,* Beitr. z. Alg. Geom., 37 (1996), pp. 87-101.

[9] H. Maeda, *Nefness of adjoint bundles for ample vector bundles,* Le Matematiche, 50 (1995), pp. 73-82.

[10] M. Mella, *Existence of good divisors on Mukai varieties,* preprint ep9705012.

[11] S. Mukai, *Biregular classification of Fano 3-folds and Fano manifolds of coindex 3,* Proc. Natl. Acad. Sci. U.S.A., 86, pp. 3000-3002.

[12] S. Mukai, *Polarized K3 surfaces of genus 18 and 20,* Complex Projective Geometry (G. Ellingsrud, C. Peskine et al., eds), London Math. Soc. Lecture Note Ser., vol. 179 Cambridge Univ. Press, Cambridge, 1992, pp. 264-276.

[13] G. Ottaviani, *Spinor bundles on quadrics,* Trans. Amer. Math. Soc., 307 (1988), pp. 301-316.

[14] G. Ottaviani, *On Cayley bundles on the five dimensional quadric,* Boll. Un. Mat. Ital., (7) 4-A (1990), pp. 87-100.

[15] M. Roggero, *Teoremi di Bertini, fasci riflessivi e curve sottocanoniche,* Seminari di Geometria 1987-1988, Università degli Studi di Bologna, Bologna, 1988, pp. 115-125.

[16] A.J. Sommese, *On manifolds that cannot be ample divisors,* Math. Ann., 221 (1976), pp. 55-72.

Dipartimento di Matematica "F. Enriques",
Università di Milano,
Via C. Saldini, 50,
20133 Milano (ITALY),
*e-mail:*lanteri@vmimat.mat.unimi.it