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QUASI-ABELIAN AND QUASI-SOLVABLE REGULAR SEMIGROUPS

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To the memory of Umberto Gasapina

A regular semigroup is said *quasi-abelian* if every commutator (i.e. every element of the form $[a, b] = a'b'ab$ for $a' \in V(a)$, $b' \in V(b)$) is idempotent. In this note, quasi-abelian regular semigroups are studied and it is proved that they form an e-variety of orthodox semigroups. More, quasi-abelian regular Bruck-Reilly monoids are characterized as extensions of monoids which are (reverse) semidirect products of a group and a semilattice. At last, quasi-abelian congruences are studied and a definition of *quasi-solvability* is given, which generalizes the notion of solvability to the class of regular semigroups: quasi-solvable semigroups of class less than or equal to a given $c \geq 0$ form an e-variety of orthodox semigroups.

The concept of quasi-abelian inverse semigroup is classical in literature (see for example [9], XII): an inverse semigroup is called *quasi-abelian* if every element of the form $a^{-1}b^{-1}ab$ is idempotent. In the present note we extend in a natural way this definition to regular semigroups; quasi-abelian regular semigroups indeed immediately reveal to be orthodox; actually they form an *e-variety* ([4]; a *bivariety* [5]) of orthodox semigroups.

By the definition of quasi-abelian regular semigroups, we can generalize to that class the notion of solvability from Group Theory; solvability was extended in [10] to the class of inverse semigroups, by means of a chain of abelian congruences and it was proved that in groups this definition coincides with the classical one. In [1], F. Albegger proposed, by means of quasi-abelian inverse semigroups, a weaker condition which bypassed some problems that made the original definition not completely satisfactory. He considered so a new (wider) class of inverse semigroups, studying in details many properties; he proved in particular that it is a variety. In the present note, by using quasi-abelian regular semigroups and congruences, we extend to the class of regular semigroups the notion given by Albegger, introducing quasi-solvable regular semigroups, which again result to be orthodox and form an e-variety of orthodox semigroups.

In Section 1 we study quasi-abelian regular semigroups, and in 2 we characterize quasi-abelian semigroups in the class of regular Bruck-Reilly extensions: by means of this characterization, we show that there exist infinitely many non-isomorphic quasi-abelian regular BR-monoids (while do not exist abelian BR-monoids). In Section 3 we describe the least quasi-abelian congruence on a regular semigroup, and in Section 4 we introduce the notion of quasi-solvability for regular semigroups. At last in Section 5 we study the linkage between this definition and the original one in inverse semigroups: we prove that if an inverse semigroup is solvable in the sense of [10] then it is also a quasi-solvable regular semigroup and for many special classes (Groups, Clifford semigroups, E-unitary inverse semigroups, Bruck-Reilly inverse semigroups) the two definitions coincide.

The notations and terminology we use are standard and can be found in [6] or [9].

1. Quasi-abelian regular semigroups.

Definition. Let S be a regular semigroup. We define $S^{(0)} = S$ and for all $i \geq 0$:

$$[S^{(i)}, S^{(i)}] = \{[a, b] = a'b'ab;\ a, b \in S^{(i)} : a' \in V(a) \cap S^{(i)}, b' \in V(b) \cap S^{(i)}\}$$
$$S^{(i+1)} = \langle [S^{(i)}, S^{(i)}] \rangle.$$

The elements $[a, b]$ are called *commutators*.

Usually we shall write S' instead of $S^{(1)}$. Note that the present definition of a commutator is slightly different from the one introduced in [10] ($[a, b] = aba^{-1}b^{-1}$) and reported in [2]: we changed it to remember more closely the classical definition from Group Theory. No confusion may raise since here we are concerned with the subsemigroups of S generated by all the commutators (in [10] we also used single commutators to describe the least commutative congruence on S). In the present situation however the definition of $[a, b]$ is ambiguous: it depends on the inverses $a' \in V(a)$ and $b' \in V(b)$; in the following, when speaking of a commutator $[a, b]$ we always mean *any* of these.

Lemma 1.1. *In a regular semigroup, E and $V(E)$ are included in $[S^{(i)}, S^{(i)}]$ for every $i \geq 0$. If S is orthodox, then $S^{(i)}$ is an orthodox subsemigroup of S, for every $i \geq 0$.*

Proof. For every $e \in E$ and every $e' \in V(e)$ we have $e = eeee = [e, e]$; $e' = e'ee' = e'eee' = [e, e']$.

Suppose that S is orthodox (whence $V(u)V(z) \subseteq V(zu)$ for every $u, z \in S$) and prove that $S^{(i)}$ is regular, whence orthodox, for every $i \geq 0$.
Trivially $S^{(0)}$ is regular; suppose that $S^{(i-1)}$ is regular and let $a = a_1a_2 \ldots a_k \in S^{(i)}$, $a_j = [x, y] = x'y'xy \in [S^{(i-1)}, S^{(i-1)}]$; $x, y \in S^{(i-1)}$, $x' \in V(x) \cap S^{(i-1)} (\neq \emptyset)$, $y' \in V(y) \cap S^{(i-1)} (\neq \emptyset)$. Hence $a_j = y'x'yx \in [S^{(i-1)}, S^{(i-1)}] \cap V(a_j) \subseteq S^{(i)} \cap V(a_j)$ and $a' = a'_k \ldots a'_2a'_1 \in S^{(i)} \cap V(a)$. □

Lemma 1.2. *Let S and T be two regular semigroups and let $T = S\mu$ for some homomorphism μ from S onto T. Then for every integer $i \geq 0$, $S^{(i)}\mu \subseteq T^{(i)}$ and if S is orthodox, then $\forall i \geq 0$, $S^{(i+1)}\mu = T^{(i+1)}$.*

Proof. Trivially, the homomorphism μ maps $S^{(i)}$ into $T^{(i)}$ for every integer $i \geq 0$. Also $S^{(0)} = S$ is regular and $S^0\mu = T^{(0)}$. Now, let S be orthodox and suppose by induction hypotheses that $S^{(i)}\mu = T^{(i)}$ and let $t \in T^{(i+1)}$ be the product of a finite number of commutators $[x_k, y_k] = x'_ky'_kx_ky_k$ with $x_k, y_k \in T^{(i)}$. For any of these, choose a_k and b_k in $S^{(i)}$ so that for some $a'_k \in V(a_k)$ and some $b'_k \in v(b_k)$ one has $a'_k\mu = x'_k$, $b'_k\mu = y'_k$, $a_k\mu = x_k$, $b_k\mu = y_k$ ([6], Example 2.14). S is orthodox, whence $S^{(i)}$ is orthodox and E is included in every $S^{(i)}$ by Lemma 1.1; then $V(a) \subseteq S^{(i)}$ for every $a \in S^{(i)}$, by Proposition VI.1.9 of [6]. Thus t is the image under μ of the product of the commutators $[a_k, b_k] \in S^{(i)}$ and $S^{(i+1)}\mu = T^{(i+1)}$. □

Note that always $S'\mu = T'$ and by Lemma 1.1 if S is orthodox then we have $S^{(i)}\mu = T^{(i)}$ for every integer $i \geq 0$.

Definition. A regular semigroup S is called *quasi-abelian* if every commutator in S is idempotent.

If $S' = E$ then the regular semigroup S is trivially quasi-abelian and orthodox. But in a quasi-abelian regular semigroup S we have $E \subset S' = \langle [a, b] \rangle \subset \langle E \rangle = E$ since by Lemma 1.1 every inverse of an idempotent is a commutator, whence idempotent and this yields that $\langle E \rangle = E$. Thus:

Lemma 1.3. *A regular semigroup is quasi-abelian if and only if $S' = E$. Any quasi-abelian regular semigroup is orthodox.*

Trivially, every abelian regular semigroup is quasi-abelian, as it is an abelian Clifford semigroup (that is a strong semilattice of abelian groups, each of them having a commutator subgroup equal to the identity). The converse is not true; the semigroup $\mathbf{T}(2)$ of partial one-one maps on a set of two elements is not abelian but $\mathbf{T}(2)'$ is a band. In the class of groups these two notions coincide; more generally:

Proposition 1.4. *Any regular subsemigroup T of a quasi-abelian regular semigroup S is quasi-abelian; if T is a subgroup of S then it is abelian. A Clifford semigroup $S = \cup_{j \in J} G_j$ is quasi-abelian if and only if it is abelian.*

Proof. The first part is straightforward. Let $S = \cup_{j \in J} G_j$ be a Clifford semigroup, then it is easy to prove that an element x belongs to S' if and only if $x \in G_j'$ for some $j \in J$, whence S is quasi-abelian if and only if all groups G_j are abelian. □

2. Quasi-abelian regular Bruck-Reilly extensions.

It is well-known that no Bruck-Reilly extension of a monoid is abelian; on the contrary there exist infinitely many non-isomorphic quasi-abelian regular semigroups of this type. To see that, we will characterize quasi-abelian Bruck-Reilly regular semigroups.

For every monoid T with group G of units and every $\alpha \in \mathrm{Hom}(T, G)$, the Bruck-Reilly semigroup $S = BR(T, \alpha)$ is isomorphic to the product $\mathbb{N} \times T \times \mathbb{N}$ ($\mathbb{N}$ denotes the set of natural numbers), with product $(m, a, n)(p, b, q) = (m + p - r, a\alpha^{p-r} b\alpha^{n-r}, n + q - r)$, where $r = \min(n, p)$ and α^0 denotes the identity on T. Such a semigroup is regular if and only if T is regular.

In the following, for every regular monoid T, $(T')_\alpha$ will denote the semigroup of T generated by the elements of the form $(a'\alpha^r)(b'\alpha^s)(a\alpha^t)(b\alpha^u)$, with r, s, t, u non negative integers, $a, b \in T$ and $a' \in V(a)$, $b' \in V(b)$.

Lemma 2.1.

i) $T' \subset (T')_\alpha$.

ii) For every $a \in T$, $a' \in V(a)$, $r, s \in \mathbb{N}$, the product $a'\alpha^r a\alpha^s$ belongs to $(T')_\alpha$.

iii) $[(m, a, n)(p, b, q)] = (n + q - k, a'\alpha^{q-k} b'\alpha^{m-k} a\alpha^{p-k} b\alpha^{n-k}, n + q - k)$, $k = \min(m, n, p, q)$, $a' \in V(a)$, $b' \in V(b)$.

Proof. Part i) is immediate with $r = s = t = u = 0$; part ii) with $b = 1$. The proof of part iii) can be obtained by an easy direct calculation, similar to the one of the first part of Theorem 3.4 of [2]. □

Lemma 2.2. *Let $S = BR(T, \alpha)$ be regular. For $x \in T$, $x \in (T')_\alpha$ if and only if for some non-negative integer m, $(m, x, m) \in S'$.*

Proof. Lemma 2.1, part iii), proves the "if" part.

Now, let $x \in (T')_\alpha$; x is equal to the product of a finite number of elements of the form $w = (a'\alpha^r)(b'\alpha^s)(a\alpha^t)(b\alpha^u)$, with r, s, t, u nonnegative integers, $a, b \in T$ and $a' \in V(a)$, $b' \in V(b)$.

If $0 \in \{r, s, t, u\}$, then ($k = 0$ in Lemma 2.1. iii)) $(r + u, w, r + u) = [(s, a, u), (t, b, r)] \in S'$. If $r, s, t, u > 0$, but $0 \in \{r - 1, s - 1, t - 1, u - 1\}$, then $(r + u - 2, (a'\alpha^{r-1})(b'\alpha^{s-1})(a\alpha^{t-1})(b\alpha^{u-1}), r + u - 2) \in S'$ and at last:

$$\begin{aligned}(r + u - 1, w, r + u - 1) = (r + u - 2, &(a'\alpha^{r-1})(b'\alpha^{s-1})(a\alpha^{t-1})(b\alpha^{u-1}),\\ &r + u - 2)(r + u - 1, 1, r + u + 1) \in S'.\end{aligned}$$

If all the integers $r - 1, s - 1, t - 1, u - 1$ are greater than 0, then consider:

$$(r - 1, (a'\alpha^{r-1})(b'\alpha^{s-1})(a\alpha^{t-1})(b), r - 1) \quad \text{and} \quad (r, (b'\alpha)(b\alpha^u), r).$$

They surely belong to S' as proved above and:

$$(r, w, r) = (r - 1, (a'\alpha^{r-1})(b'\alpha^{s-1})(a\alpha^{t-1})(b), r - 1)(r, (b'\alpha)(b\alpha^u), r) \in S'.$$

To top the proof, we note that if $(m, w, m) \in S'$ for some nonnegative integer m, then, for all $h \geq m$, $(h, w, h) \in S'$, in fact, if $h > m$, then

$$\begin{aligned}(h, w, h) &= (h, 1, m)(m, w, m)(m, 1, h)(m, w', m)(m, w, m)\\ &= [(m, 1, h), (m, w', m)](m, w, m) \in S'. \qquad \square\end{aligned}$$

Theorem 2.3. *Let $S = BR(T, \alpha)$ be a Bruck-Reilly semigroup. The following are equivalent:*

1) S is regular and $(T')_\alpha = E_T$;

2) S is (orthodox) quasi-abelian;

3) the following hold:

i) the group of units G of the monoid T is abelian,

ii) T is unique factorizable as a complex product GE of G and of the band $E = E_T$,

iii) $\alpha|_g = id_G$;

4) T is isomorphic to the (reverse) semidirect product of E by G, that is to the set $G \times E$, endowed with the following operation:

$$(g, e)(h, f) = (gh, h(h, e)f),$$

where G is an abelian group with unity 1_G, E is a band with unity 1_E and $H : G \times E \to E$ is a map which fulfils the following conditions:

– $H(1_G, e) = e$, $H(g, 1_E) = 1_E$

– $H(g, ef) = h(g, e)H(g, f)$

– $H(gh, e) = H(h, H(g, e))$.

Also, $\alpha(g, 1_E) = (g, 1_E)$.

Proof. 1) $\Rightarrow$ 2) as an immediate consequence of Lemmas 2.2 and 1.3.
2) $\Rightarrow$ 3) Let $S = BR(T, \alpha)$ be an orthodox quasi-abelian semigroup. The group of units G of T is abelian by Proposition 1.4. Also, for $a \in T$ and k a nonnegative integer, by Lemmas 2.1 and 2.2, $(a'\alpha^k)a \in (T')_\alpha$ and $(a'\alpha^k)a = e \in E_T$. Choose $k = 1$ to get $a = a\alpha \forall a \in G$ and $T \subseteq GE$, that is $T = GE$; this factorization is unique, since $g, h \in G$, $e, f \in E_T$, $ge = hf$ together imply:

$$g = g\alpha = (ge)\alpha = (hf)\alpha = h\alpha = h \text{ and } e = g^{-1}ge = g^{-1}gf = f.$$

3)$\Rightarrow$ 4) Condition ii) yields, by Theorem 5 of [3], that T is isomorphic to the set $G \times E$ where G is the abelian group of units of T and $E = E_T$, endowed with product:

$$(g, e)(h, f) = (gh, H(h, e)f),$$

where $H(h, e) = h^{-1}eh$ (which immediately fulfils the conditions we asked for). The map $F : G \times E \to g$ of Theorem 5 of [3] is in our case simply the projection onto G.

Also, $\alpha(g, 1_E) = (g, 1_E)$ because of iii).
4) $\Rightarrow$ 1) T is a monoid, since the element $(1_G, 1_E)$ is the unity for T. Also, T is regular: in fact, for every $(g, e) \in T$ the element $(g^{-1}, H(g^{-1}e))$ is an inverse of (g, e), as can be proved with an easy computation.

By Theorem 5 of [3] there exists an isomorphism from G onto $\{(g, 1_E),\ g \in G\}$ and one from E onto $\{(1_G, e),\ e \in E\}$. Thus, we may denote for short $(g, 1_E)$ by g and $(1_G, e)$ by e and the element $(g, e) = (1_G, e)(g, 1_E)$ will be denoted as ge; similarly, 1 denotes the unity of T. In this notation:

$$H(g, e) = g^{-1}eg,$$

since

$$(g^{-1}, 1_E)(1_G, e)(g, 1_E) = (g^{-1}, e)(g, 1_E) = (1_G, H(g, e)).$$

Also, note that we can write $eg = g(g^{-1}eg) = gH(g, e)$ and $ge = (geg^{-1})g = H(g^{-1}, e)g$.

Thus G is (isomorphic to) the group of units of T and the homomorphism $\alpha : T \to G$ is given by $\alpha(ge) = g$, since trivially $\alpha(e) = 1$ and α must be the identity when restricted to G.

As every element ge has an inverse and E_T is a band, then T is orthodox. Every inverse of the element ge has the form $e'g^{-1}$ for $e' \in V(e)$. In fact if $fh \in V(ge)$ with $f \in E, h \in G$, then we have $fh = (fh)(ge)(fh)$ whence (by applying α) $h = hgh$ and $h = g^{-1}$. At last, from $fg^{-1} = (fg^{-1})(ge)(fg^{-1}) = fefg^{-1}$ and $ge = (ge)(fg^{-1})(ge) = gefe$ it follows that $f \in V(e)$.

Now, let w be a generator of $(T')_\alpha$; then for some $g, h \in G$, $e, f \in E$, $r, s, t, u \geq 0$, we have:

$$\begin{aligned}
w &= (ge)'\alpha^r (hf)'\alpha^s (ge)\alpha^t (hf)\alpha^u \\
&= (e'\alpha^r)(g^{-1}\alpha^r)(f'\alpha^s)(h^{-1}\alpha^s)(g\alpha^t)(e\alpha^t)(h\alpha^u)(f\alpha^u) \\
&= (e'\alpha^r)H((g^{-1}\alpha^r)^{-1}, (f'\alpha^s))(g^{-1}\alpha^r)(h^{-1}\alpha^s)(g\alpha^t)(h\alpha^u) \\
&\qquad\qquad H((h\alpha^u), (e\alpha^t))(f\alpha^u) \\
&\qquad \text{(since } G \text{ is abelian every commutator equals the identity)} \\
&= (e'\alpha^r)H((g^{-1}\alpha^r)^{-1}, (f'\alpha^s))H((h\alpha^u), (e\alpha^t))(f\alpha^u) \in E_T. \qquad \square
\end{aligned}$$

3. The least quasi-abelian congruence on a regular semigroup.

If $\mathcal{P}$ is a class of semigroups, a congruence Γ on a semigroup S is said to be a $\mathcal{P}$ congruence if S/Γ is a $\mathcal{P}$ semigroup. The kernel of a congruence Γ on a regular semigroup S is the subset $\ker\Gamma$ of those elements of S which are Γ-congruent to an idempotent element; it is trivially *full*, that is it contains all idempotents of S.

Lemma 3.1. *Let S be a regular semigroup. A congruence Γ on S is quasi-abelian iff S' is included in* $\ker\Gamma$. *In that case* $\ker\Gamma = (S')\Gamma$ *is a full regular subsemigroup of S.*

Proof. Denote by μ the natural homomorphism from S onto S/Γ. First note that if $x \in \ker\Gamma$, then $x\mu \in E_{S/\Gamma} \subset S'\mu$, that is $x\Gamma a$ for some $a \in S'$. By Lemma 1.2, $S'\mu = (S/\Gamma)' = E_{S/\Gamma}$; by Lallements's Lemma, S' is included in $\ker\Gamma$.
Conversely, if $\ker\Gamma$ is the Γ-closure of S', then $(S/\Gamma)' = S'\mu$ is included in $E_{S/\Gamma}$ and S/Γ is quasi-abelian.

The other part follows from Lemma 1.3. $\square$

If U is a subsemigroup of the semigroup S and λ is a congruence on U, recall that *the least congruence on S generated by λ on U* is the least congruence generated by the relation λ_+ such that $\lambda_+|_U = \lambda$ and $\lambda_+|_{S-U} = \mathrm{id}$.

Proposition 3.2. *For any regular semigroup S, there exists the minimal quasi-abelian congruence τ on S: it is the least congruence Γ on S such that S' in included in* $\ker\Gamma$ *and is generated by the least band congruence on S'.*

Proof. As the universal congruence on S is trivially quasi-abelian, it will be enough to prove that for any family $\{\Gamma_j,\ j \in J\}$ of quasi-abelian congruences, the congruence $\tau = \cap\Gamma_j$ is quasi-abelian. This follows immediately from the existence of a monomorphism which maps $(S/\tau)'$ onto a subsemigroup of the direct product of the quotient semigroups $(S/\Gamma_j)'$, all of which are bands.

Let τ be the least quasi-abelian congruence. The restriction of τ to S' is a band congruence and must contain the least band congruence on S'. Consider the congruence on S which is generated by the least band congruence on S'. Its restriction to S' must be a band congruence; since S is regular then $S' \subseteq \ker\tau$, whence it is a quasi-abelian congruence. □

As an example, we want to characterize the least quasi-abelian congruence on a regular Bruck-Reilly semigroup; the following argument will be useful

Lemma 3.3. *Let T be a monoid, G its group of units, Σ any congruence on T. If $g \in G$, then $g\Sigma g^2$ if and only if $g\Sigma 1$. In particular, if Σ is a band congruence then $g\Sigma 1$ $\forall g \in G$.* □

Theorem 3.4. *Let $S = BR(T, \alpha)$ be a regular Bruck-Reilly semigroup; denote by G the group of units of T and by η the least band congruence on the subsemigroup $(T)'_\alpha$*

The least quasi-abelian congruence τ on S can be so defined:

$$(m, a, n)\tau(p, b, q) \text{ if and only if } m = p,\ n = q \text{ and } a\Delta b,$$

where Δ denotes the congruence on T generated by the least band congruence η on $(T')_\alpha$.

The greatest quasi-abelian quotient of S is a quasi-abelian orthodox Bruck-Reilly monoid $U = BR(V, \beta)$, where V is isomorphic to the set $G/(G')_\alpha \times (T')_\alpha/\eta$, endowed with the following operation:

$$(g(G')_\alpha, t\eta)(h(G')_a, u\eta) = (gh(G')_\alpha, (h^{-1}thu)\eta)$$

and

$$\beta(g(G')_\alpha, t\eta) = (g(G')_\alpha, 1\eta).$$

Proof. Let Δ be the least congruence on T generated by η on $(T')_\alpha$. To prove that the relation τ defined as

$$(m, a, n)\tau(p, b, q) \text{ if and only if } m = p,\ n = q \text{ and } a\Delta b$$

is a congruence on S, by [11] Lemma 1.3, we have to show that Δ is α-admissible on T, that is for every $x, y \in T$, $x\Delta y$ yields $x\alpha\Delta y\alpha$. We have

$x\Delta y$ if and only if there exists a sequence $z_0, z_1, \ldots, z_n \in T$ such that $x = z_0$, $y = z_n$ and for every pair $z_i, z_{i+1}, z_i = upv$ and $z_{i+1} = uqv$ for $u, v \in T^1$ and $p\eta_+ q$ (i.e. $p\eta q$ if $p, q \in (T')_\alpha$ and $p = q$ otherwise). Consider the sequence $z_0\alpha, z_1\alpha, \ldots, z_n\alpha \in T$: we have $x\alpha = z_0\alpha$, $y\alpha = z_n\alpha$ and, for every pair $z_i, z_{i+1}, z_i\alpha = u\alpha p\alpha v\alpha$ and $z_{i+1}\alpha = u\alpha q\alpha v\alpha$ for $u\alpha, v\alpha \in T^1$ and $p\alpha\eta_+ q\alpha$: in fact, if $p = q$ then also $p\alpha = q\alpha$; if p and q belong to $(T')_\alpha$, then again $p\alpha\eta q\alpha$, since $p\alpha, q\alpha \in (G')_\alpha$, which consists of a single η-class by Lemma 3.3.

Let Π be any quasi-abelian congruence on S and let $x \in (T')_\alpha$; by Lemma 2.2, for some non-negative integer m, we have $(m, x, m) \in S'$, that is $\Pi(m, x, m) = e$ idempotent in $(S/\Pi)'$.
Thus $\Pi(0, x, 0) = \Pi[(0, 1, m)(m, x, m)(m, 1, 0)] = \Pi(0, 1, m)e\Pi(m, 1, 0)$ is an idempotent element of S/Π, whence $(0, x, 0)\Pi(0, x^2, 0)$ and Π induces a band congruence on $(T')_\alpha$ which contains η. Thus Π contains τ.

Consider $U = BR(W, \beta)$, where W is isomorphic to the set $G/(G')_\alpha \times (T)'_\alpha/\eta$, with the product

$$(g(G')_\alpha, t\eta)(h(G')_\alpha, u\eta) = (gh(G')_\alpha, (h^{-1}thu)\eta)$$

and

$$\beta(gG')_\alpha, t\eta) = (g(G')_\alpha, 1\eta).$$

By Theorem 2.3 U is a regular (orthodox) quasi-abelian semigroup since the above product on W can be defined by means of an appropriate map H : $G/(G')_\alpha \times (T')_\alpha/\eta \to (T')_\alpha/\eta$ which fulfils the required conditions.

Define $H(g(G')_\alpha, t\eta) = (g^{-1}tg)\eta$, which immediately fulfils all the conditions; we have only to prove that it is a map. If $(g(G')_\alpha, t\eta) = (h(G')_\alpha, u\eta)$, then $t\eta u$ and $g^{-1}h\eta h^{-1}g\eta 1$ by Lemma 3.3, since they all belong to $(G')_\alpha$ and η is a band congruence. Whence at last:

$$g^{-1}tg\eta g^{-1}(hh^{-1})t(hh^{-1})g = (g^{-1}h)h^{-1}th(h^{-1}g)\eta h^{-1}th\eta h^{-1}uh.$$

To top the proof of the theorem we will show that the following map Θ is an isomorphism of S/τ onto $U : \Theta((m, a, n)\tau) = (m, (a\alpha(G')_\alpha, (a\alpha)^{-1}a\eta), n)$.

Let $(m, a, n)\tau(p, b, q) : m = p, n = q$ and $a\Delta b$; consider the sequence $z_0, z_1, \ldots, z_n \in T$ such that $a = z_0$, $b = z_n$ and for every pair z_i, z_{i+1} $(0 \le i \le n-1)$ $z_i = upv$ and $z_{i+1} = uqv$ for u, v in T^1 and $p\eta_+ q$. This easily yields for every z_i, z_{i+1} $(0 \le i \le n-1)$:

$$z_i\alpha(G')_\alpha = z_{i+1}\alpha(G')_\alpha \text{ and } (z_i\alpha)^{-1}z_i\eta(z_{i+1}\alpha)^{-1}z_{i+1}.$$

Also

$$\begin{aligned}\Theta((m,a,n)(p,b,q)) &= \Theta(m+p-r, a\alpha^{p-r}b\alpha^{n-r}, n+q-r)\\ &\qquad (r=\min(n,p))\\ &= (m+p-r, (a\alpha\alpha^{p-r}b\alpha\alpha^{n-r}(G')_\alpha,\\ &\qquad (a\alpha\alpha^{p-r}b\alpha\alpha^{n-r})^{-1}a\alpha^{p-r}b\alpha^{n-r}\eta), n+q-r)\end{aligned}$$

and

$$\begin{aligned}\Theta(m,a,n)\Theta(p,b,q) = (m, (a\alpha(G')_\alpha, (a\alpha)^{-1}a\eta), n)\\ (p, (b\alpha(G')_\alpha, (b\alpha)^{-1}b\eta), q)\end{aligned}$$

are equal. In fact

$$\begin{aligned}a\alpha(G')_\alpha &= a\alpha^{p-r+1}(G')_\alpha\,,\\ b\alpha(G')_\alpha &= b\alpha^{n-r+1}(G')_\alpha\,,\\ \beta(g(G')_\alpha, t\eta) &= (g(G')_\alpha, 1\eta);\end{aligned}$$

if $n=p$ then

$$((a\alpha)^{-1}a\eta)((b\alpha)^{-1}b\eta) = (b\alpha)^{-1}(a\alpha)^{-1}ab\eta = ab\eta$$

by Lemma 3.3; otherwise (suppose $n>p=r$):

$$((a\alpha)^{-1}a((b\alpha)^{-1}b)\alpha^{n-r}\eta) = a\eta = (a\alpha b\alpha\alpha^{n-p})^{-1}ab\alpha^{n-p}\eta$$

again by Lemma 3.3.

The homomorphism is onto, since for any element $(m, (g(G')_\alpha, t\eta), n)\in U$ we have:

$$\begin{aligned}\Theta(m, g\alpha t, n) &= (m, (g\alpha^2t\alpha(G')_\alpha, (g\alpha^2t\alpha)^{-1}g\alpha t\eta), n)\\ &= (m, (g(G')_\alpha, t\eta), n),\end{aligned}$$

as $t\alpha\in(G')_\alpha$, whence

$$g(G')_\alpha = g\alpha^2(G')_\alpha = g\alpha^2t\alpha(G')_\alpha,$$

and $(g\alpha^2t\alpha)^{-1}g\alpha\eta 1$.
At last, Θ is injective. In fact, if $\Theta(m,a,n)=\Theta(p,b,q)$, then $m=p$, $n=q$, $a\alpha(b\alpha)^{-1}\in(G')_\alpha$ and $(a\alpha)^{-1}a\eta(b\alpha)^{-1}b$.
Hence $a = a\alpha(a\alpha)^{-1}a\Delta a\alpha(b\alpha)^{-1}b\Delta b$ and thus $(m,a,n)\tau(p,b,q)$. □

The study of regular Bruck-Reilly extensions shows that quasi-abelian congruences are far more general than abelian ones. An abelian congruence on such a semigroup must be a group congruence, since a simple abelian semigroup is a group, but quasi-abelian congruences can be of each type studied in [11]. Consider the semigroup $S = BR(T, \alpha)$, where $T = \{0, a = a^2, 1; 0 < a < 1\}$, $\alpha : T \to 1$; S is quasi-abelian, since T is a band. In [11] it was shown that S admits all three possible types of congruences and all of them are quasi-abelian as their quotients are bands.

4. Quasi-solvable regular semigroups.

Definition. Let S be a regular semigroup, we call S *quasi-solvable of class* c if c is the least integer (greater than 0) such that $S^{(c)} = E$; c is also called the *class of quasi-solvability* of S. S is called *quasi-solvable* if it is quasi-solvable of class c for some integer $c > 0$. In particular a band will be considered quasi-solvable of class 1.

It is well known the linkage between the definition of solvability in Group Theory and commutativity: a group G is solvable if and only if some of its commutator subgroups $G^{(i)}$ equals the identity. From the *quasi-abelian* property, the above definition arises naturally for regular semigroups. When specialized to inverse semigroups, this property is weaker than solvability as defined in [10], even if these two definitions coincide on special classes, in particular for groups.

Lemma 4.1. *Let S be a regular semigroup. For every $c \geq 1$, we have $S^{(c)} = E$ if and only if $[S^{(c-1)}, S^{(c-1)}] = E$. If a regular semigroup S is quasi-solvable, then it is orthodox.*

Proof. Half of the proof is trivial. Now, let $[a, b] \in E$ for every $a, b \in S^{(c-1)}$. By Lemma 1.1, this implies that all inverse elements of an idempotent are idempotent; thus S is orthodox and the product of two idempotents is idempotent, whence all the elements in the subsemigroup $S^{(c)} = \langle [S^{(c-1)}, S^{(c-1)}] \rangle$ are idempotents. $\square$

Denote by τ_i the least quasi-abelian congruence on $S^{(i)}$.

Theorem 4.2. *Let S be a regular semigroup. The following are equivalent:*

i) S is quasi-solvable of class c;
ii) c is the least index such that τ_{c-1} is the identity relation on $S^{(c-1)}$;
iii) there exists a chain of congruence Π_i on S, $0 \leq i \leq c-1$, such that:

$$\Pi_{i+1} \subseteq \Pi_i, \; S^{(i+1)} \subseteq \ker \Pi_i$$

for every $i = 0, \ldots, c-1$ *and* Π_{c-1} *is the identity relation on* S.

Proof. i $\Rightarrow$ ii). If S is quasi-solvable of class c, then $S^{(c)} = (S^{(c-1)})' = E$; hence $S^{(c-1)}$ is quasi-abelian, the least quasi-abelian congruence on $S^{(c-1)}$ is the identity and c must be the least index which fulfils this condition.
ii) $\Rightarrow$ iii) For all i: $0 \leq i \leq c-1$, let Π_i be the congruence on S generated by τ_i on $S^{(i)}$. Thus for every i ($0 \leq i \leq c-1$), $S^{(i+1)}$ is included in $\ker \Pi_i$ since it is included in $\ker \tau_i$ by Lemma 3.1. Also, since $S^{(i+1)} \subseteq S^{(i)}$, then $\Pi_{i+1} \leq \Pi_i$. At last, as τ_{c-1} is the identity on $S^{(c-1)}$, then Π_{c-1}, is the identity on S.
iii) $\Rightarrow$ i) As Π_{c-1} is the identity on S and $S^{(c)} \subseteq \ker \Pi_{c-1}$ then $S^{(c)}$ must coincide with the set E of idempotents of S. $\square$

Lemma 4.3. *Let S be a regular quasi-solvable semigroup and let c be its class of solvability. Then:*

i) If T is a regular subsemigroup of S, then T is quasi-solvable of class k, for some $k \leq c$.

ii) If T is a quotient of S, then T is quasi-solvable of class k, for some $k \leq c$.

Proof. i) We will prove by induction on i that $T^{(i)} \subseteq S^{(i)}$, which is trivial if $i = 0$. If this is true for $i-1$, then for every $a \in S$, $V(a) \cap T^{(i)} \subseteq V(a) \cap S^{(i)}$; thus $[T^{(i)}, T^{(i)}] \subseteq [S^{(i)}, S^{(i)}]$.
ii) S is orthodox, whence (Lemmas 1.1 and 1.2) $S^{(i-1)}$ is orthodox $\forall i > 0$ and $T^{(i)} = S^{(i)}\mu$. Similarly, the following can be proved:

Lemma 4.4. *Let* $\{S_j, j \in J\}$ *be a family of quasi-solvable regular semigroups (with solvability class equal to* c_j *for each* $j \in J$*) and suppose that there exists* $c = \max\{c_j, j \in J\}$. *Then:*

i) the direct product $S = \Pi S_j$, *is quasi-solvable of class c,*

ii) a subdirect product of the semigroups S_j *is quasi-solvable of class k, for some* $k \leq c$.

From Lemmas 4.3 and 4.4 we get (see [4]):

Theorem 4.5. *The class of quasi-solvable regular semigroups of class* $\leq c$ *is an e-variety of orthodox semigroups.*

Proposition 4.6. *Let* Γ *be a congruence on a regular semigroup S such that* S/Γ *is quasi-solvable of class c and* $\ker \Gamma$ *is quasi-solvable of class k. Then S is quasi-solvable of class h, for* $\min(c, k) \leq h \leq k + c$.

Proof. S/Γ is quasi-solvable of class c, thus c is the least integer such that $(S/\Gamma)^{(c)} = E_{S/T}$; whence by Lemma 3.2, $S^{(c)} \subseteq \ker \Gamma$ and $S^{(c+k)} = (S^{(c)})^{(k)} \subseteq (\ker \Gamma)^{(k)} = E$. Thus S is quasi-solvable and its class of quasi-solvability h in less than or equal to $c + k$; by Lemma 4.3 it is impossible that $h < \min(c, k)$. □

When a regular semigroup S is naturally linked to a group G, we expect a precise connection between the quasi-solvability of S and the solvability of G. This is true in most interesting cases. For example (by Lemmas 4.3 and 4.4) any rectangular group $IxMxG$ is quasi-solvability of class c if and only if G is quasi-solvable of class c. Also, by Corollary 7 of [8] it follows immediately that:

Corollary 4.7. *An E-unitary regular semigroup S is quasi-solvable of class c if and only if its greatest group homomorphic image is solvable of class c or $c-1$.*

A similar result holds for bisimple ω-semigroups; more generally:

Corollary 4.8. *A regular Bruck-Reilly semigroup $S = BR(T, \alpha)$ is quasi-solvable of class c if and only if T is quasi-solvable of class c or $c-1$.*

Proof. Since S is regular, T is a regular monoid. The set of those elements x such that for some non-negative integer m, $(m, x, m) \in S'$, is included in $T^{(0)} = T$. Suppose by induction that $\{x : \exists m \geq 0, (m, x, m) \in S^{(i)}\} \subseteq T^{(i-1)}$ and consider the commutator of two elements in $S^{(i)}$

$$\begin{aligned}[(m, x, m), (p, y, p)] &= (m + p - k, x'\alpha^{p-k} y'\alpha^{m-k} x\alpha^{p-k} y\alpha^{m-k}, m + p - k) \\ &= (m + p - k, [x\alpha^{p-k}, y\alpha^{m-k}], m + p - k) \\ &\qquad (\text{with } k = \min(m, p)).\end{aligned}$$

Since $x, y \in T^{(i-1)}$ implies that $x\alpha^{p-k}, y\alpha^{m-k} \in T^{(i-1)}$, then we have:

$$\{x : \exists m \geq 0, (m, x, m) \in S^{(i+1)}\} \subseteq T^{(i)}.$$

Thus, if T is quasi-solvable of class c, then $T^{(c)}$ is a band whence also $S^{(c+1)}$ is. Conversely, by Lemma 4.3, if S is quasi-solvable of class c, then T is quasi-solvable of class at most c. □

5. Quasi-solvability and solvability in inverse semigroups.

In [10] we proposed a definition of solvability for inverse semigroups, by generalizing the corresponding property from Group Theory, that is by means of a chain of commutative quotients.

Definition. ([10]). Let S be an inverse semigroup. Consider the transitive closure $\gamma_{E,S}$ of the following relation on E_S : $e \approx f$ iff $e = abb^{-1}a^{-1}$, $f = baa^{-1}b^{-1}$ for some $a, b \in S$; consider also the following chain of subsemigroups of S:

$$\delta_0(S) = S,$$

$$\delta_i(S) = \{a \in \delta_{i-1}(S) : \text{ for some } e \in E_S, aa^{-1}\gamma_{E,\delta(S)}e \text{ and } ae \in \delta_{i-1}(S)'\}$$

S is called *solvable of solvability class c* if c is the least index i such that $\delta_{i-1}(S)$ is commutative.

This definition was motivated by the following results, which show a strict linkage between the classical notion of solvability in groups and this generalized one:

Proposition 5.1. ([10]). *The following relation is the least commutative congruence on the inverse semigroup S : $\gamma_S = \{(a, b) \in SxS : aa^{-1}\gamma_{E,S}bb^{-1}$ and $ab^{-1} \in \delta_1(S)\}$.*
The semigroup $\delta_1(S)$ is the kernel of γ_S and the relation $\gamma_{E,S}$ is the trace of γ_S. S is solvable of solvability class c if and only if c is the least index i such that $\gamma_{\delta_i(S)}$ is the identity on $\delta_{i-1}(S)$.

Proposition 5.2. ([10], see also [7]). *Let S be a Clifford semigroup. S is solvable of solvability class c if and only if it is a strong semilattice of solvable groups of class $c_i (i \in I)$, such that c is exactly the maximum of the $c_i (i \in I)$.*

In particular, if S is group, our definition exactly coincides with the classical notion of solvability; indeed, for Clifford semigroups, whence also for groups, these also coincide with quasi-solvability. In general we have:

Proposition 5.3. *A solvable inverse semigroup of solvability class c is quasi-solvable of class $k \leq c$. A Clifford semigroup is quasi-solvable of solvability class c if and only if it is solvable of solvability class c.*

Proof. Let S be a solvable inverse semigroup of solvability class c. Then $\delta_{c-1}(S)$ is commutative, hence $\delta_c(S) = E_S$. But we can easily prove by induction that, for every $i \geq 0$, $S^{(i)} \subseteq \delta_i(S)$; in fact, for every $i \geq 0$, $\delta_i(S)' \subseteq \delta_{i+1}(S)$.

By [10] 2.8, in a Clifford semigroup $S = \cup G_j$, the semigroup $\delta_1(S) = S'$ is exactly the strong semilattice of the commutator subgroups G_j' and, again by induction, we can prove that, for every $i \geq 0$, $S^{(i)} = \delta_i(S)$. □

These two notions coincide also in the class of E-unitary inverse semigroups (by [10] 3.12 and Corollary 4.7); a similar result can easily be given for orthogroups.

In general, however, solvability and quasi-solvability are not equivalent properties for inverse semigroups: for example, solvable inverse semigroups do not form a variety (solvability is not inherited by quotients: see [10] for a counterexample). Also they do not coincide in some interesting classes of inverse semigroups:

Proposition 5.4. ([1] 2.2). *Let $S = B(G, 1)$ be a Brandt semigroup. S is quasi-solvable if and only if the group G is solvable. But if $|I| > 1$ then S is not solvable, whatever G is.*

In [10] all symmetric inverse semigroups of injective partial transformations on a set X of elements were proved to be not solvable. On the other hand we have:

Theorem 5.5. *Let n be an integer greater than 1 and let $\mathscr{I}_n$ be the symmetric inverse semigroup on a set X of order n. If $n \leq 4$ then $\mathscr{I}_n$ is quasi-solvable of class $n-1$; if $n > 4$ then $\mathscr{I}_n$ is not quasi-solvable.*

Proof. Consider the permutation group $S_n \subset \mathscr{I}_n$. It is well known that a similar result holds for the solvability of S_n; thus by Lemma 4.3 and Proposition 5.2, $\mathscr{I}_n$ is not quasi-solvable for $n > 4$ and, for $n \leq 4$, the class of quasi-solvability of $\mathscr{I}_n$ is not less than $n-1$.

Suppose $n \leq 4$ and "complete" every element $x \in \mathscr{I}_n$ to an element $y \in S_n$. It will be enough for any $a \in \mathrm{Dom}(x)$, to define $y(a) = x(a)$; if $|Dom(a)| = n$ we have finished, if not then for any $b \notin \mathrm{Dom}(a)$ put $y(b) = c$ with c any element not in $\mathrm{Im}(x)$ (this is certainly possible since $|\mathrm{Dom}(x)| = |\mathrm{Im}(x)| \leq 4$). Note that an element of $\mathscr{I}_n$ is idempotent only if it can be completed to the identity of S_n. This procedure do not uniquely determine y, but is compatible with inverses and with products, so that any commutator of $\mathscr{I}_n$ can be completed to a commutator of S_n; more precisely every element in $(\mathscr{I}_n)^{(c)}$ can completed to an element of $(S_n)^{(c)}$. Whence the result. □

REFERENCES

[1] F. Albegger, *Auflösbare inverse halbgruppen,* Universität Wien, 1991.

[2] C. Bonzini - A. Cherubini - B. Piochi, *The least commutative congruence on simple regular ω-semigroups,* Glasgow Math. J., 32 (1990), pp. 13-23.

[3] F. Catino, *Factorizable semigroups,* Semigroup Forum, 36 (1987), pp. 167-174.

[4] T.E. Hall, *Regular semigroups; amalgamation and the lattice of existence varieties,* Algebra Universalis, 28 (1991), pp. 79-102.

[5] J. Kadourek - M.B. Szendrei, *A new approach in the theory of orthodox semigroups,* Semigroup Forum, 40 (1990), pp. 257-296.

[6] J.M. Howie, *An Introduction to Semigroups theory,* Academic Press, London, 1976.

[7] J.D.P. Meldrum, *Group theoretical results in Clifford semigroups,* Acta Sci. Math., 52 (1988), pp. 3-19.

[8] H. Mitsch, *Subdirect products of E-inversive semigroups,* J. Austral. Math. Soc., 48 (1990), pp. 68-78.

[9] M. Petrich, *Inverse semigroups,* Wiley, New York, 1984.

[10] B. Piochi, *Solvability in inverse semigroups,* Semigroup Forum, 34 (1987), pp. 287-303.

[11] B. Piochi, *Congruences on Bruck-Reilly extension of monoids,* Semigroup Forum, 59 (1995), pp. 179-191.

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