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ON SEMIGROUPS HAVING THE ω-ITERATION PROPERTY

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To the memory of Umberto Gasapina

Recently it has been proved that a finitely generated semigroup having the ω-iteration property is finite. Here we prove that, under certain conditions, the hypothesis that the generators are finitely many can be omitted. As the added conditions are always satisfied by a group we have that a group with the ω-iteration property is finite.

Finiteness conditions for semigroups (i.e. properties satisfied by any finite semigroup and assuring the finiteness of each semigroup having them) are very important in algebra (Burnside problem) and in automata theory (in principle, any finiteness condition for finitely generated semigroups can be translated in a regularity condition for languages).

In general in the studies of this area, it is required that the semigroups are finitely generated, see for istance [1] and [3]. For example, let us recall the result of Restivo and Reutenauer [7]. It has been proved using (as often it is the case) an argument of combinatorics on words and it says: *a finitely generated semigroup is finite if and only if it is periodic and permutable*, where *permutable* means that for some positive integer n the product of n elements of the semigroup can be re-obtained permuting them in at least a non-trivial way.

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Let us introduce now, in a quite informal way, the following definitions: *a semigroup has the property $\mathscr{E}$ (resp. $\mathscr{I}$) if each infinite sequence of elements of S has an ω-factorization of the form $xe\dot{e}ee\cdots e\cdots$ (resp. has two disjoint segments with the same value).* In [4] and [5] respectively, one can find the formal definitions of $\mathscr{E}$ and $\mathscr{I}$ and the proof that they are finiteness conditions for finitely generated semigroups. In a recent paper [2] Justin and Kelarev proved the following deep result: *properties $\mathscr{E}$ and $\mathscr{I}$ are in fact equivalent* for any semigroup.

This result and the others of [2] stress the importance of the study of the conditions (properties) which ensure the finiteness of a finitely generated semigroup satisfying them. In this paper we are concerned with one of such a condition, namely the *ω-iteration property*. This notion, introduced in [6] where it is proved the following Theorem 1, is hereafter recalled.

Definition. *Let S be a semigroup and n be an integer greater than or equal to 1. We say that a sequence $s_1, s_2, \cdots, s_n$ of elements of S has the iteration property if there exist $i \geq j$, $i, j \in \{1, 2, \ldots, n\}$ such that*

$$s_n s_{n-1} \cdots s_2 s_1 = s_n s_{n-1} \cdots s_{i+1} (s_i \cdots s_j)^2 s_{j-1} \cdots s_2 s_1$$

and we say that $s_1, s_2, \cdots, s_n$ is iteration-free in the opposite case.

Definition. *A semigroup S has the ω-iteration property if for each infinite sequence $s_1, s_2, \ldots, s_i, \ldots$ of elements of S there exist $n \geq 1$, depending on the chosen sequence, such that the left factor $s_1, s_2, \ldots, s_n$ has the iteration property.*

Theorem 1. ([6]). *A finitely generated semigroup having the ω-iteration property is finite.*

The proof of this theorem require the Fürstenberg result on uniformly recurrent words and some properties of bi-ideal sequences. In this paper we show that the hypothesis that the generators are finitely many can be replaced by the property of left cancellativity together with the property $\mathscr{F}$ hereafter defined.

Definition. *We say that a semigroup S has the property $\mathscr{F}$ if, for each $a, b \in S$, the equation $xa = b$ has at most finitely many solutions in S.*

Theorem 2. *A left cancellative semigroup having the ω-iteration property and the property $\mathscr{F}$ is finite.*

Proof. Let us consider two cases.

Case 1. All the elements of S are idempotent.

Case 2. At least one element of S is not an idempotent.

Case 1. Suppose by way of contradiction that S is infinite. Let a be a fixed element of S. For each $x \in S$, we have

$$x^2a = xa$$

and from left cancellativity

$$xa = a.$$

So the equation $xa = a$ admits each element of S as a solution, i.e. $xa = a$ has infinitely many solutions. Contradiction.

Case 2. Again by way of contradiction, suppose that S is infinite.

Claim 1. If

$$s_1, s_2, \ldots, s_n$$

is an iteration-free sequence of elements of S then we can choose in S at least one element, say s_{n+1}, such that the sequence

$$s_1, s_2, \ldots, s_n, s_{n+1}$$

is again iteration-free.

Proof of Claim 1. Suppose, by way of contradiction, that there exists an iteration-free sequence $s_1, s_2, \ldots, s_n$ of maximal length, i.e. non extensible preserving the iteration freeness.

So, for each $x \in S$ the sequence $s_1, s_2, \ldots, s_n, x$ has the iteration property and, for a given $x \in S$, one of the following cases holds:

Case 2.1: there exist i and j in $\{1, \ldots, n\}$ such that

$$xs_n \cdots (s_i \cdots s_j)^2 s_{j-1} \cdots s_2 s_1 = xs_n \cdots s_i \cdots s_j s_{j-1} \cdots s_2 s_1;$$

Case 2.2: $(xs_n \cdots s_i \cdots s_2 s_1)^2 = xs_n \cdots s_i \cdots s_2 s_1$;

Case 2.3: $x^2 s_n \cdots s_i \cdots s_2 s_1 = xs_n \cdots s_i \cdots s_2 s_1$;

Case 2.4: $n \geq 2$ and there exists i in $\{2, \ldots, n\}$ such that

$$(xs_n \cdots s_i)^2 s_{i-1} \cdots s_2 s_1 = xs_n \cdots s_i s_{i-1} \cdots s_2 s_1.$$

Case 2.1 is impossible. In fact, cancelling x, we have

$$s_n \cdots (s_i \cdots s_j)^2 s_{j-1} \cdots s_2 s_1 = s_n \cdots s_i \cdots s_j \cdots s_2 s_1$$

and we are in contradiction as, by assumption, $s_1, s_2, \ldots, s_n$ is iteration-free.

Moreover suppose that for some $x \in S$, Case 2.2 occurs and let $a = s_n \cdots s_i \cdots s_2 s_1$. We have

$$(xa)^2 a = xaa$$

and, usising left cancellativity,

$$xa^2 = a$$

and so, as S has property $\mathscr{F}$, only finitely many elements of S can be in Case 2.2.

Consequently for infinitely many $x \in S$, we would be in Case 2.3 or Case 2.4.

Case 2.3. Using left cancellativity we have

$$x s_n \cdots s_i s_{i-1} \cdots s_2 s_1 = s_n \cdots s_i s_{i-1} \cdots s_2 s_1$$

and this means that the equation $xa = a$ (where $a = s_n \cdots s_i s_{i-1} \cdots s_2 s_1$) has infinitely many solutions in contradiction to the property $\mathscr{F}$ of S.

Case 2.4. Using again left cancellativity we have

$$x s_n \cdots s_i s_{i-1} \cdots s_2 s_1 = s_{i-1} \cdots s_2 s_1$$

and this means that the equation $xa = b$ (where $a = s_n \cdots s_i s_{i-1} \cdots s_2 s_1$ and $b = s_{i-1} \cdots s_2 s_1$) has infinite many solutions in contradiction to the property $\mathscr{F}$ of S.

Now, let s_1 be a non idempotent of S. The sequence s_1 is clearly iteration-free. By Claim 1 there exist a sequence s_1, s_2 which is iteration-free, a sequence s_1, s_2, s_3 which is iteration-free and, continuing in this way, we can construct an infinite sequence

$$s_1, s_2, \cdots, s_n, \cdots$$

such that each of its left factor is iteration-free.
But, as S has the ω-iteration property at least one left factor of $s_1 s_2 \cdots s_n \cdots$ must have the iteration property. Contradiction. $\square$

As each group is left cancellative and has the property $\mathscr{F}$ we have the following

Corollary. *A group with the ω-iteration property is finite.*

Remark. Theorem 2 and its corollary in this paper are not directly deducible from Theorem 1 because a left cancellative semigroup having $\mathscr{F}$ is not necessarily finitely generated.

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