LE MATEMATICHE
Vol. LI (1996) – Supplemento, pp. 189-196

ON EQUATIONAL PROPERTIES OF SUBGROUP LATTICES OF METABELIAN GROUPS

VLADIMIR B. REPNITSKIĬ

To the memory of Umberto Gasapina

The subalgebra lattice of an algebra A is denoted by Sub A. Given a variety $\mathbf{V}$ of algebras, Sub $\mathbf{V}$ denotes the class of all lattices isomorphic to lattices of the form Sub A, $A \in \mathbf{V}$. As was shown by the author (see [1]), for a variety $\mathbf{V}$ of semigroups, Sub $\mathbf{V}$ satisfies a nontrivial identity if and only if $\mathbf{V}$ is periodic, consists of nilpotent extensions of completely simple semigroups, and subgroup lattices of all groups from $\mathbf{V}$ satisfy a nontrivial identity. Thus, the complete description of semigroup varieties $\mathbf{V}$ with a nontrivial identity on Sub $\mathbf{V}$ is reduced to the description of the corresponding (periodic) group varieties. The last varieties must be soluble (see [1], [2]). So the following question arises: *what varieties* $\mathbf{V}$ *of soluble groups have a nontrivial lattice identity on* Sub $\mathbf{V}$? We prove in [3] that any nilpotent variety of groups has such a property. From the main result of the present paper it follows that varieties of metabelian groups have this property as well.

Now we recall some necessary definitions and notation. As usual, the commutator $[a, b]$ of elements a and b of a given group is the element $a^{-1}b^{-1}ab$. A group is called *metabelian* if it satisfies the identity $[[x, y], [z, u]] = 1$. This identity is evidently equivalent both to the identity $[x, y][z, u] = [z, u][x, y]$ and to the identity $[x, y]^{zu} = [x, y]^{uz}$. The subgroup generated by a subset H

of a given group is denoted by $\langle H\rangle$. If s and t are lattice terms, then we shall write briefly $s \leq t$ instead of the lattice identity $s \vee t = t$.

Let us define lattice terms p and q over the alphabet $X = \{x_0, x_1, x_2; y_1, y_2; z_1, z_2, z_3, z_4\}$ by the rule:

$$p_1 = (y_1 \wedge (z_1 \vee z_2)) \vee (y_2 \wedge (z_3 \vee z_4)),$$
$$p = x_0 \wedge ((x_1 \wedge p_1) \vee x_2);$$

$$q_1 = (x_2 \vee y_1 \vee y_2 \vee z_1) \wedge (x_2 \vee y_2 \vee z_2),$$
$$q_2 = (x_2 \vee y_1 \vee y_2 \vee z_3) \wedge (x_2 \vee y_1 \vee z_4),$$
$$q_3 = z_3 \vee q_2 \vee ((x_1 \vee z_3 \vee q_2) \wedge (z_1 \vee q_1)),$$
$$q = x_1 \vee (q_3 \wedge (x_0 \vee x_1)).$$

Theorem. *Let* $\mathbf{V}$ *be a variety of groups. If* $\mathbf{V}$ *is metabelian, then* Sub $\mathbf{V}$ *satisfies the identity* $p \leq q$. *Conversely, if* Sub $\mathbf{V}$ *satisfies the identity* $p \leq q$, *then* $\mathbf{V}$ *satisfies the identity* $[[[x, y], [z, u]], v] = 1$; *in particular, here* $\mathbf{V}$ *is metabelian if its free group of rank five has a trivial center.*

We divide the proof of this theorem into two lemmas.

Lemma 1. *The subgroup lattice of any metabelian group satisfies the identity* $p \leq q$.

Proof. Let G be a metabelian group and $X_0, X_1, X_2, Y_1, Y_2, Z_1, \ldots, Z_4$ arbitrary subgroups of G. We now define the following subgroups of G:

$$P_1 = \langle Y_1 \cap \langle Z_1, Z_2\rangle, Y_2 \cap \langle Z_3, Z_4\rangle\rangle, \tag{1}$$
$$P = X_0 \cap \langle X_1 \cap P_1, X_2\rangle; \tag{2}$$

$$Q_1 = \langle X_2, Y_1, Y_2, Z_1\rangle \cap \langle X_2, Y_2, Z_2\rangle, \tag{3}$$
$$Q_2 = \langle X_2, Y_1, Y_2, Z_3\rangle \cap \langle X_2, Y_1, Z_4\rangle, \tag{4}$$
$$Q_3 = \langle Z_3, Q_2, \langle X_1, Z_3, Q_2\rangle \cap \langle Z_1, Q_1\rangle\rangle, \tag{5}$$
$$Q = \langle X_1, Q_3 \cap \langle X_0, X_1\rangle\rangle. \tag{6}$$

We must prove that $P \subseteq Q$ in Sub G. With that end in view let us take an arbitrary $h \in P$. Then by (2) we have

$$h \in X_0 \tag{7}$$

and

$$h \in \langle X_1 \cap P_1, X_2\rangle. \tag{8}$$

From (8) it follows that, for some $x_1, ..., x_n \in X_1 \cap P_1$ and $y_1, ..., y_n \in X_2$, the equality

$$h = x_1 y_1 \cdots x_n y_n$$

holds. Since $X_1 \cap P_1$ and X_2 are subgroups of G, the easy induction on n gives us existence of elements

$$a, a_1, ..., a_k \in X_1 \cap P_1 \tag{9}$$

and

$$b, b_1, ..., b_k \in X_2 \tag{10}$$

such that

$$h = ab[a_1, b_1]^{\epsilon_1}[a_2, b_2]^{\epsilon_2} \cdots [a_k, b_k]^{\epsilon_k} \tag{11}$$

where $\epsilon_1, \epsilon_2, ..., \epsilon_k \in \{-1, 1\}$.

Let us now consider an arbitrary commutator of the form $[a_i, b_i]$, $1 \leq i \leq k$. From (1) and (9) it follows that there exist elements

$$c_1, ..., c_m \in Y_1 \cap \langle Z_1, Z_2 \rangle \tag{12}$$

and

$$d_1, ..., d_m \in Y_2 \cap \langle Z_3, Z_4 \rangle, \tag{13}$$

for which the equality

$$a_i = c_1 d_1 \cdots c_m d_m$$

is true. From here and from that any group satisfies the identity

$$[xy, z] = [x, z]^y [y, z]$$

we obtain

$$\begin{aligned}[a_i, b_i] &= [c_1 d_1 c_2 d_2 \cdots c_m d_m, b_i] \\ &= [c_1, b_i]^{d_1 c_2 d_2 \cdots c_m d_m} [d_1, b_i]^{c_2 d_2 \cdots c_m d_m} \cdots [c_m, b_i]^{d_m} [d_m, b_i].\end{aligned}$$

Since G is metabelian, this implies that

$$[a_i, b_i] = w_1 w_2 \tag{14}$$

where

$$w_1 = [c_1, b_i]^{c'_1 d'_1} \cdots [c_m, b_i]^{c'_m d'_m} \tag{15}$$

and

$$w_2 = [d_1, b_i]^{d''_1 c''_1} \cdots [d_m, b_i]^{d''_m c''_m} \tag{16}$$

for suitable elements

$$c'_1, \ldots, c'_m, c''_1, \ldots, c''_m \in Y_1 \cap \langle Z_1, Z_2 \rangle \tag{17}$$

and

$$d'_1, \ldots, d'_m, d''_1, \ldots, d''_m \in Y_2 \cap \langle Z_3, Z_4 \rangle. \tag{18}$$

Let us now check that

$$w_1 \in \langle Z_3, Q_2 \rangle \tag{19}$$

and

$$w_2 \in \langle Z_1, Q_1 \rangle. \tag{20}$$

Indeed, by (18) every element d'_j, $1 \leq j \leq m$, can be represent as a product of some elements of the subgroups Z_3 and Z_4. From here and again from that G is metabelian we deduce that, for each j, there exist $f_j \in Z_3$ and $e_j \in Z_4$ with the property

$$[c_j, b_i]^{c'_j d'_j} = [c_j, b_i]^{c'_j e_j f_j}.$$

This is equivalent to that

$$[c_j, b_i]^{c'_j d'_j f_j^{-1}} = [c_j, b_i]^{c'_j e_j}.$$

In addition, using (10), (12), (17) and (18), we have

$$[c_j, b_i]^{c'_j d'_j f_j^{-1}} \in \langle X_2, Y_1, Y_2, Z_3 \rangle$$

and

$$[c_j, b_i]^{c'_j e_j} \in \langle X_2, Y_1, Z_4 \rangle.$$

Now, taking (4) into account, we obtain

$$[c_j, b_i]^{c'_j d'_j f_j^{-1}} = [c_j, b_i]^{c'_j e_j} \in \langle X_2, Y_1, Y_2, Z_3\rangle \cap \langle X_2, Y_1, Z_4\rangle = Q_2.$$

Therefore,

$$[c_j, b_i]^{c'_j d'_j} = ([c_j, b_i]^{c'_j d'_j f_j^{-1}})^{f_j} \in \langle Z_3, Q_2\rangle.$$

The last inclusion holds for all $j \in \{1, 2, ..., m\}$. This and (15) imply (19). The inclusion (20) can be checked similarly with the use of (3), (10), (13), (16), (17) and (18).

Further, as follows from (4), $X_2 \subseteq Q_2$. This inclusion with (9), (10) and (19) implies

$$w_1^{-1}[a_i, b_i] \in \langle X_1, Z_3, Q_2\rangle.$$

Now, taking (14) and (20) into account, we deduce

$$w_2 = w_1^{-1}[a_i, b_i] \in \langle X_1, Z_3, Q_2\rangle \cap \langle Z_1, Q_1\rangle.$$

Hence

$$[a_i, b_i] = w_1 w_2 \in \big\langle Z_3, Q_2, \langle X_1, Z_3, Q_2\rangle \cap \langle Z_1, Q_1\rangle\big\rangle = Q_3 \tag{21}$$

(see (5)).

Now let us note that $b \in X_2 \subseteq Q_2 \subseteq Q_3$ (see (4),(5) and (10)). From here and from that (21) holds for all $i \in \{1, 2, ..., k\}$ we obtain

$$b[a_1, b_1]^{\epsilon_1}[a_2, b_2]^{\epsilon_2} \cdots [a_k, b_k]^{\epsilon_k} \in Q_3.$$

This with (7),(9) and (11) has as a consequence

$$a^{-1}h = b[a_1, b_1]^{\epsilon_1}[a_2, b_2]^{\epsilon_2} \cdots [a_k, b_k]^{\epsilon_k} \in Q_3 \cap \langle X_0, X_1\rangle.$$

Therefore, granting (6), we conclude that

$$h = a(a^{-1}h) \in \big\langle X_1, Q_3 \cap \langle X_0, X_1\rangle\big\rangle = Q.$$

The lemma is proved.

Lemma 2. *If the subgroup lattice of a relatively free group G of rank five satisfies the identity $p \leq q$, then G satisfies the identity $[[[x, y], [z, u]], v] = 1$. In this case G is metabelian whenever G has a trivial center.*

Proof. Let us fix an arbitrary group variety and take in it a free group G of rank five. Let x, y, z, u, v be its free generators and the identity $p \leq q$ hold in $\mathrm{Sub}\, G$. Consider in G the following subgroups:

$$X_0 = \big\langle [[[x, y], [z, u]], v] \big\rangle, \; X_1 = \big\langle [[x, y], [z, u]] \big\rangle, \; X_2 = \langle v \rangle,$$
$$Y_1 = \langle [x, y] \rangle, \; Y_2 = \langle [z, u] \rangle,$$
$$Z_1 = \langle x \rangle, \; Z_2 = \langle y \rangle, \; Z_3 = \langle z \rangle, \; Z_4 = \langle u \rangle.$$

Using them, we construct subgroups P_1, P and Q_1, Q_2, Q_3, Q of G by the above rules (1) – (6). By assumption, the inclusion $P \subseteq Q$ must be hold. We need to prove that then the equality $[[[x, y], [z, u]], v] = 1$ is true in G. In fact, by construction we have

$$[x, y] \in Y_1 \cap \langle Z_1, Z_2 \rangle, \quad [z, u] \in Y_2 \cap \langle Z_3, Z_4 \rangle$$

and hence

$$[[x, y], [z, u]] \in \big\langle Y_1 \cap \langle Z_1, Z_2 \rangle, \langle Y_2 \cap \langle Z_3, Z_4 \rangle \rangle \big\rangle = P_1 .$$

Therefore,

$$[[[x, y], [z, u]], v] \in X_0 \cap \langle X_1 \cap P_1, X_2 \rangle = P. \tag{22}$$

Now we intend to verify sequentially the following equalities:

$$Q_1 = \langle v, [z, u] \rangle , \tag{23}$$
$$Q_2 = \langle v, [x, y] \rangle , \tag{24}$$
$$\langle X_1, Z_3, Q_2 \rangle \cap \langle Z_1, Q_1 \rangle = \langle v \rangle , \tag{25}$$
$$Q_3 = \langle z, v, [x, y] \rangle , \tag{26}$$
$$Q_3 \cap \langle X_0, X_1 \rangle = \{1\}, \tag{27}$$
$$Q = \langle [[x, y], [z, u]] \rangle . \tag{28}$$

Equalities (23) *and* (24). It is easy to see that

$$Q_1 = \langle v, [x, y], [z, u], x \rangle \cap \langle v, [z, u], y \rangle \tag{29}$$

and

$$Q_2 = \langle v, [x, y], [z, u], z \rangle \cap \langle v, [x, y], u \rangle. \tag{30}$$

Let $h \in Q_1$ and consider the endomorphism ϕ of the group G such that $\phi(x) = 1$ and all other free generators are fixed under ϕ. Then we have $\phi(h) = h$, since $h \in Q_1 \subseteq \langle v, [z, u], y\rangle$. On the other hand,

$$\phi(h) \in \phi\langle v, [x, y], [z, u], x\rangle = \langle v, [z, u]\rangle.$$

Hence $h = \phi(h) \in \langle v, [z, u]\rangle$. This means that the inclusion $Q_1 \subseteq \langle v, [z, u]\rangle$ is true. The reverse inclusion evidently follows from (29). Thus the equality (23) is checked. The equality (24) can be checked similarly (here we must act on the right part of (30) by the endomorphism ϕ of G with the property $\phi(z) = 1$ and $\phi(x) = x$, $\phi(y) = y$, $\phi(u) = u$, $\phi(v) = v$).

Equality (25). Using (23) and (24), we obtain

$$(31) \quad \langle X_1, Z_3, Q_2\rangle \cap \langle Z_1, Q_1\rangle = \langle [[x, y], [z, u]], z, v, [x, y]\rangle \cap \langle x, v, [z, u]\rangle.$$

Let an element h belong to the left part of this equality. Consider the endomorphism ϕ_1 of G such that $\phi_1(y) = 1$ and all other free generators are fixed under ϕ_1. Then, since $h \in \langle x, v, [z, u]\rangle$, we have $\phi_1(h) = h$. On the other hand,

$$\phi_1(h) \in \phi_1\langle [[x, y], [z, u]], z, v, [x, y]\rangle = \langle z, v\rangle.$$

Therefore, $h = \phi_1(h) \in \langle z, v\rangle$, i.e. the inclusion

$$\langle X_1, Z_3, Q_2\rangle \cap \langle Z_1, Q_1\rangle \subseteq \langle z, v\rangle \cap \langle x, v, [z, u]\rangle$$

holds. Now let us consider an endomorphism ϕ_2 of G, for which $\phi_2(x) = \phi_2(u) = 1$ and $\phi_2(z) = z$, $\phi_2(v) = v$. Acting by ϕ_2 on the right part of the last inclusion, we obtain by standard arguments that

$$\langle X_1, Z_3, Q_2\rangle \cap \langle Z_1, Q_1\rangle \subseteq \langle v\rangle.$$

Since the reverse inclusion is obviously true (see (31)), the equality (25) is proved.

From (25) we deduce (26):

$$Q_3 = \big\langle Z_3, Q_2, \langle X_1, Z_3, Q_2\rangle \cap \langle Z_1, Q_1\rangle\big\rangle = \langle z, v, [x, y]\rangle .$$

Equality (27). Taking (26) into account, we may wright

$$(32) \quad Q_3 \cap \langle X_0, X_1\rangle = \langle z, v, [x, y]\rangle \cap \langle [[[x, y], [z, u]], v], [[x, y], [z, u]]\rangle.$$

Let ϕ be an endomorphism of G such that $\phi(u) = 1$ and $\phi(x) = x$, $\phi(y) = y$, $\phi(z) = z$, $\phi(v) = v$. Acting by this endomorphism on (32), we obtain (27).

At last, the equality (27) implies

$$Q = \big\langle X_1, Q_3 \cap \langle X_0, X_1\rangle \big\rangle = X_1 = \langle [[x, y], [z, u]] \rangle ,$$

i.e. (28) is true as well.

Now, since $P \subseteq Q$, we obtain from (22) and (28) that

$$[[[x, y], [z, u]], v] \in \langle [[x, y], [z, u]] \rangle .$$

This means that there exists an integer k, for which

$$[[[x, y], [z, u]], v] = [[x, y], [z, u]]^k .$$

Setting here $v = 1$, we deduce $[[x, y], [z, u]]^k = 1$ and, finally, $[[[x, y], [z, u]], v] = 1$, as was to be proved.

Now it remains to note that our theorem directly follows from Lemmas 1 and 2.

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Department of Mathematics and Mechanics,
Ural State University,
Lenina 51,
620083 Ekaterinburg (RUSSIA),
e-mail: vladimir.repnitskii@usu.ru