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SUBSEMIGROUPS OF INVERSE SEMIGROUPS

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To the memory of Umberto Gasapina

Historical introduction.

Inverse semigroups form the most important class of semigroups besides groups. This article is devoted to a description of those semigroups that can be isomorphically embedded in inverse semigroups. The problem of describing these semigroups was raised by Wagner (whose name is sometimes transliterated as Vagner) in 1952, when he discovered inverse semigroups (called by him "generalized groups"; see [23]) ([1]).

Apparently, this problem is similar to the problem of describing semigroups embeddadle in groups. This second problem had a dramatic history. The so-called van der Waerden problem asks whether all rings without zero divisors are embeddable in division rings. Obviously, the semigroups embeddable in groups are cancellative (on both sides). A semigroup analog of the van der Waerden problem is whether all cancellative semigroups are embeddable in groups. For years it had not been known whether the cancellativity condition was sufficient for embeddability. In 1935 Suschkewitsch published a proof of the sufficiency of this condition (see [20]), but, in 1937, Malcev (see [4]) found yet another necessary condition for embeddability that did not follow

([1]) In 1954 inverse semigroups were rediscovered by Preston (see [7]), who gave them their present name. G. Tallini (see [21]) suggested calling them "gruppidi".

from cancellativity. In 1939 Malcev found an infinite system of quasi identities that described the embeddable semigroups (see [5]) and, in 1940, proved that his set of quasi identities was essentially infinite (that is, not equivalent to any of its *finite* subsets; see [6]). While these brilliant results were correct, proofs of certain auxiliary lemmas raised objections, and their amplified proofs were the subject of at least one doctoral dissertation. The best exposition of Malcev's results can be found in [1], where they are presented, as the authors put it, "with considerable amplifications". Other approaches to that problem were suggested by Joachim Lambek, Vlastimil Pták and Dov Tamari.

It was only natural that when Wagner and his students tried to determine the semigroups embeddable in inverse semigroups they turned to the method of Malcev.

If $\mathbf{S} = (S; \cdot)$ is a semigroup (S is a set of its elements and $\cdot$ an associative multiplication on S), let G_S denote the group freely generated by $\mathbf{S}$. Let the Cayley multiplication table of $\mathbf{S}$ be a set of defining relations for G_S (that is, if $st = u$ for $s, t, u \in S$, we use $st = u$ as a defining relation). As a result, we obtain a group $G(\mathbf{S})$ freely generated by $\mathbf{S}$ and the injective mapping $S \to G_S$ becomes a homomorphism η of $\mathbf{S}$ into $G(\mathbf{S})$. In more modern terms, we can say that $\eta : \mathbf{S} \to G(\mathbf{S})$ is a solution to the universal problem of embedding $\mathbf{S}$ into a group. Clearly, $\mathbf{S}$ is embeddable in a group if and only if η is such an embedding (that is, η is injective). That means that if $\eta(a) = \eta(b)$ for some $a, b \in S$, then $a = b$. Now, $\eta(a) = \eta(b)$ means that there exists a chain of words $w_0, w_1, \ldots, w_n$ in G_S such that w_0 coincides with a, w_n coincides with b, and every two consecutive words w_i and w_{i+1} are obtained by a single application of a defining relation of $G(\mathbf{S})$ (that is, a relation of the form $st = u$ or group relations of the form $ss^{-1} = \Lambda$ and $s^{-1}s = \Lambda$, where $s, t, u \in S$ and Λ is the empty word). The problem was to describe those one-lettered words s and t that were equivalent under these defining relations. Malcev showed that, for each chain $s, \ldots, t$ there existed another chain $s, \ldots, t$ that had a certain "normal" form. Each of his embeddability conditions had the same basic form: "given a normal chain (desribed by certain equalities of products in $\mathbf{S}$), the first and last elements of that chain are equal in $\mathbf{S}$. Thus, each of his conditions had a form: if a certain finite system of equalities between products of elements of S holds, then a certain final equality holds" (that is, each of the conditions was a quasi identity or, in another terminology, a universal Horn formula). Roughly speaking, Malcev's "normal form" of a chain of equivalent words corresponded to a distribution of parentheses in a word made by elements of a groupoid (that is, of a set endowed with a binary multiplication). People in the formal language theory might say that Malcev used a certain connection between groups and Dyck languages.

If we try to apply that method to inverse semigroups, we can construct an inverse semigroup $IG(\mathbf{S})$ freely generated by the set S and subject to the definiting relations of the form $st = u$ and also the relations characterizing inverse semigroups (given any word w, it is equivalent to the word $ww^{-1}w$, and also idempotent words commute). Thus we arrive at a homomorphism $\gamma : \mathbf{S} \to IG(\mathbf{S})$, and $\mathbf{S}$ is embeddable in an inverse semigroup if and only if γ is injective. The difficulty is in "normalizing" chains of transformations beetween equivalent one-lettered words. No such normal form has been found. It would be interesting to attempt finding such a form now, when our knowledge of the structure of free inverse semigroups in somewhat deeper than it was in 1952-1956 and it becomes clearer that the structure of free inverse semigroups is connected with Dyck languages too. Finding a "Malcev-like" solution for the problem of embedding semigroups in inverse semigroups would be a good test of our real understanding of the structure of free inverse semigroups that, in some surprising aspects (see [16]), is *very* different from what we are used to in the case of groups.

The history of the solution of an analogous problem for semigroups embeddable in inverse semigroups was different. I became a university freshman in 1955. First, Wagner's students, and a year later Wagner himself introduced me to these problems. It was clear that a semigroup embeddable in an inverse semigroup has commuting idempotents. It was not clear if commuting idempotents would suffice for embeddability. By 1956 V.L. Izrailevich, one of Wagner's students, found another condition necessary for embeddability and proved that this condition did not follow from the commutativity of idempotents (this has never been published). He found that condition using the argument analogous to that used in [5]. Naturally, that focused everyone's attention on finding a "Malcev-like" solution of the problem. I started working on this particular embedding problem in 1956 and soon realized that a different approach might be more promising. Here is that approach, in a nutshell. As proved by Wagner and then by Preston, every inverse semigroup $\mathbf{S}$ is isomorphic to an inverse semigroup of one-to-one partial transformations of a certain set A. Thus, every semigroup embeddable in $\mathbf{S}$ is isomorphic to a semigroup of one-to-one partial transformations of A. Conversely, if a semigroup $\mathbf{S}$ is isomorphic to a semigroup of one-to-one partial transformations of any set A, then, since *all* one-to-one partial transformations of A form an inverse semigroup $\mathcal{I}(A)$, $\mathbf{S}$ is embeddable in that inverse semigroup. It follows that *a semigroup is embeddable in an inverse semigroup if and only if that semigroup is isomorphic to a semigroup of one-to-one partial transformations of a set.* Thus, all I had to do was to see which semigroups admitted faithful representations by one-to-one partial transformations.

To do that I had to develop a theory of *homomorphic* representations of semigroups by partial transformations and see which of those representations were those by one-to-one partial transformations. "Gluing together" sufficiently many homomorphic representations, one could hope to construct an isomorphic representation (if it existed). I found that representations of semigroups by one-to-one partial transformations were connected with a special type of subsets of semigroups. These subsets were a semigroup analog of cosets of subgroups in a group. By 1957 I had a theory of such subsets developed. To have a faithful representation by one-to-one partial transformations, a semigroup had to have sufficiently many subsets of this type (they had to "separate points", using a topological terminology.) Reading some papers of Robert Croisot and Gabriel Thierrin available at my university library, I found there references to a previous paper by Dubreil (see [2]). It looked as if this paper considered the same or similar subsets, but it was not available at my University library, and I spent three years hunting – without success – for it at all libraires of the USSR, where I lived at the time, and discovered by accident in 1960 that the only copy of the publication was kept . . . in my University library, catalogued as a book (and not as a periodical or a semi-periodical) under a wrong name! Although the times became more vegetarian than earlier, writing to Dubreil was out of question. The role of these special subsets in semigroup theory is considered in [18] and [19].

Anyway, I solved that problem in 1959 and submitted a paper to one of the Soviet journals. The paper was returned to me because it was too long. I made it much shorter omitting many results and making the remaining text more compact (which influenced readability of the paper) and resubmitted it in the beginning of 1960. It appeared in 1961 (see [11]). Some of the remaining parts of this paper made separate short papers. Elementary axioms for semigroups embeddable in inverse semigroups appeared (without proofs) in a short note [10] (altough it was submitted after [11], it appeared earlier, in 1960). The theory of representations of inverse semigroups by partial one-to-one transformations appeared in 1962 (see [12]). Transitive representations of semigroups by partial transformations appeared only in 1963 (see [13]), two years after E.J. Tully Jr. published similar results. I did not publish at that time results on a semigroup generalization of the Jacobson radical connected with transitive representations of semigroups, and I gave a talk on that subject at a big conference only in 1967 (see [15]), a year after these results were published by H.-J. Hoehnke.

The axioms for semigroups embeddable in inverse semigroups appeared in my diploma work (an analogue of a Master's thesis) defended in 1960 (see [8]) and, two years later, in an expanded form in my Ph.D. thesis [9]. These results have never been published, altough a part of the theory of representations of semigroups by partial transformations was included in Volume II of the Clifford and Preston monograph [1]. However, neither the main results of [10] nor their proofs were included in [1], because my proofs had not been published by that time. I was involved in other projects and published the proofs only in 1977 in a virtually inaccesible (even inside the USSR) semi-periodical of a provincial University (see [17]) that appeared in Russian in 300 copies and ceased to exist a few years after that. Due to poor photoreproduction, formulas in that paper are all but illegibile. No proofs of these results have ever been published in an accesible place in English. This is done for the first time in this present work that contains improved and revised proofs.

Section 1 of this paper characterizes embeddable semigroups by a certain topological property. Section 2 contains another characterization: embeddable semigroups are precisely those that can be ordered in a certain way (in the notation we have already used, the isomorphism $\gamma : \mathbf{S} \to IG(\mathbf{S})$ induces a certain order relation on $\mathbf{S}$, it is turned by γ into the natural order relation on the inverse semigroup $IG(\mathbf{S})$). Section 3 uses these results to present a system of elementary axioms for embeddable semigroups, while Section 4 proves that no finite system of elementary axioms characterizes the class of embeddable semigroups.

1. Semigroups as strong closure spaces.

In the sequel we denote a semigroup in the same way as the set of its elements. Thus, S denotes both a semigroup and the set of all elements of S. As the following Proposition shows, we can consider only the semigroups that have an identity element. As usual, if a semigroup S has no identity element, S^1 denotes S with an external identity 1 adjoined. If S has an identity element, we define $S^1 \doteq S$. See the Remark after the proof of Theorem 5 in Section 3 for more on that subject.

Proposition 1. *A semigroup S is isomorphic to a semigroup of one-to-one partial transformations of a set if and only if S^1 is isomorphic to such a semigroup.*

Proof. This claim is rather obvious (it is tautological for $S^1 = S$), its proof is given for completeness' sake. Suppose that $S^1 \neq S$, that is, S has no identity element. Let S have an isomorphism ι onto a semigroup Φ of one-to-one partial transformations of a set A. Extend ι to a mapping ι^1 of S^1 by defining $\iota^1(1) = \Delta_A$, where Δ_A is the identity transformation of A. Then ι^1 is an isomorphism of S^1 onto a semigroup of one-to-one partial transformations of A. Indeed, ι^1 is one-to-one because, if $\iota^1(s) = \iota^1(1)$ for some $s \in S$, then

$\iota(s) = \Delta_A$, and hence $\Delta_A \in \Phi$ and Φ is a semigroup with identity. Since S is isomorphic to Φ, S is a semigroup with identity, contrary to our assumption. To prove that ι^1 is a homomorphism (and thus an isomorphism), it sufficies to check that $\iota^1(1) = \Delta_A$ is the identity element of $\Phi \cup \{\Delta_A\}$. This is obviously true.

Conversely, if S^1 is isomorphic to a semigroup Φ of one-to-one partial transformations of a set A, then the restriction of this isomorphism to S is an isomorphism of S onto a semigroup of one-to-one partial transformations of A. □

For the sake of simplicity we assume in the sequel that all semigroups considered have an identity element.

A *closure space* is any set S with a set C of subsets of S (called *closed* subsets) such that the intersection of any family of closed subsets is closed. A closure space is called a *Kolmogorov space* (or a T_0-space) if, given any two *distinct* elements of S, there exists a closed set that contains exactly one of these elements.

A subset H of a semigroup S is called *strong* if it satisfies the following condition:

$$xv,\ uv,\ uy \in H \Rightarrow xy \in H \ \text{ for all } u,\ v,\ x,\ y \in S.$$

It is easy to see that any semigroup with the collection of all of its strong subsets forms a *closure space*. If H is a subset of S, the strong closure $\hat{H}$ of H is defined as the smallest strong subset that contains H (that is, $\hat{H}$ is the intersection of all strong subsets that contain H). In particular, $\hat{s}$ denotes the strong closure of a one-element subset $\{s\}$.

The principal result of this section is the following theorem.

Theorem 1. *A semigroup is isomorphically embeddable in an inverse semigroup if and only if its strong subsets form a Kolmogorov closure space.*

Proof. Necessity. Suppose that a semigroup S is embeddable in an inverse semigroup. By Proposition 1, S is isomorphic to a semigroup of one-to-one partial transformations of a set A. Without loss of generality, assume that S is a semigroup of one-to-one partial transformations of A. For any $a, b \in A$ define a subset $H_{a,b} = \{s \in S : s(a) = b\}$ of S. This subset is always strong. Indeed, let $xv, uv, uy \in H_{a,b}$. Then $xv(a) = uv(a) = uy(a) = b$. Thus $u(v(a)) = u(y(a))$. Since u is one-to-one, we obtain $v(a) = y(a)$. It follows that $xy(a) = x(y(a)) = x(v(a)) = xv(a) = b$, and hence $xy \in H_{a,b}$. Thus $H_{a,b}$ is a strong subset of S.

If s and t are two distinct elements of S, then there exist $a, b \in A$ such that either $s(a) = b \neq t(a)$ or $t(a) = b \neq s(a)$. In other words, exactly one of the elements s and t belongs to the strong subset $H_{a,b}$. It follows that S with its strong subsets forms a Kolmogorov closure space.

Sufficiency. We need a few definitions.

(i) For $F \subset S$ and $s \in S$ define $F .\cdot s = \{u \in S : su \in F\}$ and $F \cdot. s = \{u \in S : us \in F\}$.

(ii) If H is a nonempty strong subset of a semigroup S, let $\mathcal{H}$ be the set of all nonempty subsets of S of the form $H \cdot. s$, where $s \in S$.

Thus $\mathcal{H} = \{F : (\exists s \in S) F = H \cdot. s \neq \emptyset\}$. Since $H \cdot. 1 = H \neq \emptyset$, $\mathcal{H}$ is not empty because $H \in \mathcal{H}$. Let $\mathcal{H} = \{H_a : a \in A\}$, where the elements of $\mathcal{H}$ are indexed by some index set A in a one-to-one way (that is, for any two distinct $a, b \in A$, $H_a \neq H_b$).

For every $s \in S$ define a binary relation $P(s) = \{(a, b) \in A \times A : H_a s \subset H_b\}$.

To proceed with our proof we need the following Lemma.

Lemma 1. *P is a (homomorphic) representation of S by one-to-one partial transformations of A.*

Proof. Let $H_a \cap H_b \neq \emptyset$ for some $a, b \in A$. If $H_a = H \cdot. v$, $H_b = H \cdot. y$, and $u \in H_a \cap H_b$ for some $u, v, y \in S$, then $uv, uy \in H$. If $x \in H_a$, then $xv \in H$, and hence $xy \in H$ because H is strong. It follows that $x \in H \cdot. y = H_b$. Thus $H_a \subset H_b$. Interchanging a and b we obtain that $H_b \subset H_a$. Therefore, $H_a = H_b$, and so $a = b$.

Let $(c, a) \in P(s)$ and let $(c, b) \in P(s)$ for some $a, b, c \in A$ and $s \in S$. Then $H_c s \subset H_a$ and $H_c s \subset H_b$. Thus, $H_a \cap H_b \neq \emptyset$, and, as we have just seen, $a = b$. It follows that $P(s)$ is a partial transformation of A for every $s \in S$.

Suppose that $(a, c) \in P(s)$ and $(b, c) \in P(s)$ for some $a, b, c \in A$ and $s \in S$. Then $H_c = H \cdot. t$ for some t and $(a, c) \in P(s) \Rightarrow H_a s \subset H_c \Rightarrow H_a st \subset H \Rightarrow H_a \subset H \cdot. st \Rightarrow H \cdot. st \neq \emptyset$. It follows that $H \cdot. st = H_d$ for some $d \in A$, and hence $H_a \cap H_d = H_a \neq \emptyset$. As we have seen in the first paragraph of the proof of our Lemma, this implies $a = d$. Analogously, $(b, c) \in P(s)$ implies $b = d$. Thus $a = b$ and $P(s)$ is a one-to-one partial transformation for every $s \in A$.

If $(a, b) \in P(s)$ and $(b, c) \in P(t)$, then $H_a s \subset H_b$ and $H_b t \subset H_c$, and hence $H_a st \subset H_b t \subset H_c$, so that $(a, c) \in P(st)$. It follows that $P(t) \circ P(s) \subset P(st)$.

Conversely, if $(a, c) \in P(st)$, then $H_a st \subset H_c$. It follows that $H_a s \subset H_c \cdot. t$, so that $H_c \cdot. t = H_b$ for some $b \in A$. Therefore, $H_a s \subset H_b$ and $H_b t \subset H_c$, whence $(a, b) \in P(s)$ and $(b, c) \in P(t)$. Thus, $(a, c) \in P(t) \circ P(s)$.

It follows that $P(st) \subset P(t) \circ P(s)$, and hence $P(st) = P(t) \circ P(s)$ for all $s, t, \in S$. □

Remark. We assume that P is a homomorphism if $P(st) = P(t) \circ P(s)$ because we read the factors $P(s)$ and $P(t)$ in the product $P(t) \circ P(s)$ from the right to the left, while factors in the product st are read from the left to the right. A reader who prefers the equality $P(st) = P(s) \circ P(t)$ can obtain it changing the definition of $P(s)$ to $(a, b) \in P(s) \Leftrightarrow sH_a \subset H_b$ and considering $\mathcal{H}$ as the set of all nonempty subsets of the form $H \cdot t$.

For every nonempty strong subset H of S we constructed a representation P by one-to-one partial transformations of a set A. Denote P by P_H and A by A_H. Without loss of generality, we may assume that, if H and F are different strong subsets, then A_H and A_F are disjoint. Let A be the union of all sets A_H, and $P(s)$ the union of all binary relations $P_H(s)$ for all nonempty strong subsets H. It is easy to see that P is a representation of S by one-to-one partial transformations of A. Now we prove that P is injective.

Let $P(s) = P(t)$ for distinct $s, t \in S$. Since S is a Kolmogorov closure space, there is a strong subset H that contains precisely one of the elements s and t. Without loss of generality, let $s \in H$ and $t \notin H$. Then $P_H(s) = P(s) \cap (A_H \times A_H) = P(t) \cap (A_H \times A_H) = P_H(t)$ for all H. Clearly, $1s = s \in H$, and so $1 \in H \cdot . s$. Thus $H \cdot . s = H_a$ and $H = H_b$ for some $a, b \in A_H$. Also, $H_a s \subset H = H_b$, that is, $(a, b) \in P_H(s) = P_H(t)$. It follows that $t = 1t \in H_a t \subset H_b = H$, contrary to our assumption about H. Thus $s = t$ and P is an isomorphism of S onto a semigroup of one-to-one partial transformations of A. □

2. Strong quasi order relation on semigroups.

Let S be a semigroup. Define the following binary relation $\prec$ on it: $s \prec t \Leftrightarrow t \in \hat{s}$. Since $s \in \hat{s}$, we obtain $s \prec s$. If $s \prec t$ and $t \prec u$, then $t \in \hat{s}$ and $u \in \hat{t}$. Since $\hat{t}$ is the least strong subset of S that contains t, it follows that $\hat{t} \subset \hat{s}$, and hence $u \in \hat{s}$ and $s \prec u$. Thus $\prec$ is both reflexive and transitive, so it is a quasi order relation on S. We call it the *strong* quasi order of S. Sometimes instead of $\prec$ we write $\hat{\zeta}$, thus $s \prec t \Leftrightarrow (s, t) \in \hat{\zeta}$.

Strong quasi order has various remarkable properties. Here are four of them.

(1) *$\hat{\zeta}$ is stable (that is, compatible with multiplication): $s \prec t$ and $u \prec v$ imply $su \prec tv$ for all $s, t, u, v \in S$.*

(2) *If $e \in S$ is an idempotent element, then $es \prec s$ and $se \prec s$ for all $s \in S$.*

(3) *If S has a zero element 0, then $0 \prec s$ for all $s \in S$.*

(4) *If S is an inverse semigroup, then $\prec$ coincides with the natural order relation of S.*

Proof. (1) If H is a strong subset, then $H .\cdot s$ and $H \cdot. s$ are strong for every $s \in S$. Indeed, if $xv, uv, uy \in H .\cdot s$, then $(sx)v, (su)v, (su)y \in H$, which implies $(sx)y \in H$; and hence $xy \in H .\cdot s$ and $H .\cdot s$ is strong. Analogously, $H \cdot. s$ is strong.

Thus $su \in \widehat{su}$ implies $s \in \widehat{su} \cdot. u$ for any $s, u \in S$. The subset $\widehat{su} \cdot. u$ is strong, and $\hat{s}$ is the least strong subset that contains s. Therefore, $\hat{s} \subset \widehat{su} \cdot. u$, and hence $\hat{s}u \subset \widehat{su}$. If $s \prec t$, then $t \in \hat{s}$, and hence $tu \in \hat{s}u \subset \widehat{su}$. It follows that $su \prec tu$. Analogously, $tu \prec tv$. Thus $su \prec tv$.

(2) Indeed, $1 \cdot es = e \cdot es = e \cdot s \in \widehat{es}$, and hence $s = 1s \in \widehat{es}$. It follows that $es \prec s$. The inequality $se \prec s$ is proved analogously.

(3) Since $0^2 = 0$ and $0s = 0$, the inequality $0 \prec s$ follows from (2).

(4) Let $\leq$ be the natural order relation on S. If $s \leq t$, then, by (2), $s = ss^{-1}t \prec t$ because ss^{-1} is an idempotent.

Conversely, let $s \prec t$. Then $t \in \hat{s}$. Let $H = \{u \in S : s \leq u\}$. If $xv, uv, uy \in H$, then $s \leq xv$, $s \leq uv$, and $s \leq uy$. It follows that $s = ss^{-1}s \leq (xv)(uv)^{-1}(uy) = x(vv^{-1}u^{-1}u)y \leq xy$ because $vv^{-1}u^{-1}u$ is an idempotent of S. Therefore, H is a strong subset of S. Also, $s \in H$, and hence $t \in \hat{s} \subset H$ so that $s \leq t$. □

Theorem 2. *A semigroup is embeddable in an inverse semigroup if and only if its strong quasi order relation $\hat{\zeta}$ is an order relation (that is, $\hat{\zeta}$ is antisymmetric).*

Proof. Let a semigroup S be embeddable in an inverse semigroup. Suppose that $s \prec t$ and $t \prec s$ for some $s, t \in S$. Then $t \in \hat{s}$ and $s \in \hat{t}$. If H is a strong subset of S such that $s \in H$, then $\hat{s} \subset H$, and hence $t \in H$. Analogously, $t \in H \Rightarrow s \in H$ for every strong subset H. Thus, s and t belong to the same strong subsets of S. By Theorem 1, $s = t$ and $\prec$ is antisymmetric.

Conversely, let $\prec$ be antisymmetric. If $s, t \in S$ and s and t belong to the same strong subsets of S, then $t \in \hat{s}$ and $s \in \hat{t}$. Therefore, $s \prec t$ and $t \prec s$, which implies $s = t$. Thus, strong subsets of S form a Kolmogorov closure space and, by Theorem 1, S is embeddable in an inverse semigruop. □

For every nonnegative integer n we define a binary relation ζ_n on S by induction on n:

(1) $\zeta_0 = \Delta_S$, the equality relation on S;

(2) If ζ_n has been defined, then $(s, t) \in \zeta_{n+1}$ precisely when there exist $u, v, x, y \in S$ such that $(s, xv), (s, uv), (s, uy) \in \zeta_n$ and $t = xy$.

Theorem 3. $\hat{\zeta} = \cup\{\zeta_n : n = 0, 1, 2, \ldots\}$.

The proof consists of a sequence of lemmas. Let $\zeta_\omega = \cup\zeta_n$.

Lemma 2. $\zeta_\omega \subset \hat{\zeta}$.

Proof. (Induction on n). Since $\hat{\zeta}$ is reflexive, $\zeta_0 = \Delta_S \subset \hat{\zeta}$. Suppose that $\zeta_n \subset \hat{\zeta}$ for some $n \geq 0$. If $(s, t) \in \zeta_{n+1}$, then $(s, xv), (s, uv), (s, uy) \in \zeta_n$ and $t = xy$ for suitable $u, v, x, y \in S$, and hence $(s, xv), (s, uv), (s, uy) \in \hat{\zeta}$, so that $xv, uv, uy \in \hat{s}$, whence $t = xy \in \hat{s}$ and $(s, t) \in \hat{\zeta}$. Thus $\zeta_{n+1} \subset \hat{\zeta}$. It follows that $\zeta_n \subset \hat{\zeta}$ for all n, and hence $\zeta_\omega \subset \hat{\zeta}$. □

Lemma 3. *ζ_n is reflexive for all $n > 0$.*

Proof. (Induction on n). The claim holds for $\zeta_0 = \Delta_S$. Suppose that it holds for ζ_n. If $x = u = 1$ and $v = y = s$, then $(s, xv) = (s, uv) = (s, uy) = (s, s) \in \zeta_n$, and hence $(s, s) = (s, xy) \in \zeta_{n+1}$. □

Lemma 4. *If $n < m$, then $\zeta_n \subset \zeta_m$.*

Proof. Suppose that $(s, t) \in \zeta_n$. If $x = u = 1$, $v = s$ and $y = t$, then, by Lemma 3, $(s, xv) = (s, uv) = (s, s) \in \zeta_n$ and $(s, t) = (s, uy) \in \zeta_n$, so that $(s, t) = (s, xy) \in \zeta_{n+1}$. Thus $\zeta_n \subset \zeta_{n+1}$. The claim $\zeta_n \subset \zeta_m$ follows by induction on $m - n$. □

Lemma 5. $\hat{\zeta} \subset \zeta_\omega$.

Proof. Let $\langle s \rangle = \{t : (s, t) \in \zeta_\omega\}$. If $xv, uv, uy \in \langle s \rangle$, then $(s, xv) \in \zeta_k$, $(s, uv) \in \zeta_m$, and $(s, uy) \in \zeta_n$ for some $k, m, n \geq 0$. Let $p = \max\{k, m, n\}$. By Lemma 4, $\zeta_i \subset \zeta_p$ for $i = k, m, n$. It follows that $(s, xv), (s, uv), (s, uy) \in \zeta_p$, and hence $(s, xy) \in \zeta_{p+1} \subset \zeta_\omega$. Therefore, $xy \in \langle s \rangle$, that is, $\langle s \rangle$ is a strong subset of S. Since $s \in \langle s \rangle$ and $\hat{s}$ is the smallest strong subset that contains s, we obtain $\hat{s} \subset \langle s \rangle$. If $(s, t) \in \hat{\zeta}$, then $t \in \hat{s} \subset \langle s \rangle$, and hence $(s, t) \in \zeta_\omega$. □

Theorem 3 follows from Lemmas 2 and 5. □

Remark. A quasi order relation ζ on a semigroup S is called *steady* if $(z, xy) \in \zeta$ follows from $(z, xv), (z, uv), (z, uy) \in \zeta$ for any $u, v, x, y, z \in S$. It follows from our proof of Lemma 5 that $\hat{\zeta} \subset \zeta$ for any steady quasi order ζ. Thus *$\hat{\zeta}$ is the least steady quasi order relation on any semigroup.*

Theorem 4. *Let S be a semigroup with identity and $s, t \in S$. Then $(s, t) \in \zeta_n$ if and only if there exist $u^i_j, v^i_j, x^i_j, y^i_j \in S$, where i and j are indices (i is not an exponent) such that $1 \leq i \leq n$, $1 \leq j \leq 3^{i-1}$,*

$$\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{3^{i-1}} \begin{pmatrix} x^i_j v^i_j & = & x^{i+1}_{3j-2} y^{i+1}_{3j-2} & \wedge \\ u^i_j v^i_j & = & x^{i+1}_{3j-1} y^{i+1}_{3j-1} & \wedge \\ u^i_j y^i_j & = & x^{i+1}_{3j} y^{i+1}_{3j} & \end{pmatrix},$$

$s = x_{-1}^1 y_{-1}^1$, and $t = x_1^1 y_1^1$. Here we assume that $\wedge$ is the conjunction sign and $\bigwedge_{i=1}^{n}$ means "for all i from 1 to n". Also, $x_{-1}^1 = x_j^{n+1}$ and $y_{-1}^1 = y_j^{n+1}$ for all j.

Proof. Define $(s, t) \in \alpha_n$ if s and t satisfy the condition of Theorem 3. First we prove by induction on n that $\alpha_n = \zeta_n$. Indeed, $\alpha_0 = \Delta_S = \zeta_0$. Suppose that $\alpha_{n-1} = \zeta_{n-1}$ for some $n \geq 1$. Let $(s, t) \in \alpha_n$. Then the equalities of Theorem 3 hold. Let $\Phi(u, v, x, y, i, j)$ denote the formula

$$\begin{pmatrix} x_j^i v_j^i & = & x_{3j-2}^{i+1} y_{3j-2}^{i+1} & \wedge \\ u_j^i v_j^i & = & x_{3j-1}^{i+1} y_{3j-1}^{i+1} & \wedge \\ u_j^i y_j^i & = & x_{3j}^{i+1} y_{3j}^{i+1} & \end{pmatrix}.$$

For $i = 1$ the formula $\bigwedge_{j=1}^{3^{i-1}} \Phi(u, v, x, y, i, j)$ becomes

$$\begin{array}{rcl} x_1^1 v_1^1 & = & x_1^2 y_1^2 \\ u_1^1 v_1^1 & = & x_2^2 y_2^2 \\ u_1^1 y_1^1 & = & x_3^2 y_3^2 \end{array}.$$

Also, the formula $\bigwedge_{i=2}^{n} \bigwedge_{j=1}^{3^{i-1}} \Phi(u, v, x, y, i, j)$ can be written as

$$\bigwedge_{i=2}^{n} \bigwedge_{j=1}^{3^{i-2}} \Phi(u, v, x, y, i, j) \wedge \bigwedge_{i=2}^{n} \bigwedge_{j=3^{i-2}+1}^{2\cdot 3^{i-1}} \Phi(u, v, x, y, i, j) \wedge$$
$$\bigwedge_{i=2}^{n} \bigwedge_{j=2\cdot 3^{i-2}+1}^{3^{i-1}} \Phi(u, v, x, y, i, j).$$

By the induction hypothesis, the formula

$$x_1^1 v_1^1 = x_1^2 y_1^2 \wedge \bigwedge_{i=2}^{n} \bigwedge_{j=1}^{3^{i-2}} \Phi(u, v, x, y, i, j) \wedge x_{-1}^1 y_{-1}^1 = s$$

means $(s, x_1^1 v_1^1) \in \zeta_{n-1}$, the formula

$$u_1^1 v_1^1 = x_1^2 y_1^2 \wedge \bigwedge_{i=2}^{n} \bigwedge_{j=3^{i-2}+1}^{2\cdot 3^{i-2}} \Phi(u, v, x, y, i, j) \wedge x_{-1}^1 y_{-1}^1 = s$$

means $(s, u_1^1 v_1^1) \in \zeta_{n-1}$, and the formula

$$u_1^1 y_1^1 = x_3^2 y_3^2 \wedge \bigwedge_{i=2}^{n} \bigwedge_{j=2\cdot 3^{i-2}+1}^{3} \Phi(u, v, x, y, i, j) \wedge x_{-1}^1 y_{-1}^1 = s$$

means $(s, u_1^1 v_1^1) \in \zeta_{n-1}$. Here we used Theorem 4 for ζ_{n-1} making some change in the sets of values of the indices. For example, i runs not from 1 to $n-1$, as in Theorem 4, but for 2 to n. Respectively, j runs not from 1 to 3^{i-1}, as in the theorem, but from 1 to 3^{i-2}, or from $3^{i-2}+1$ to $2\cdot 3^{i-2}$, or from $2\cdot 3^{i-2}+1$ to 3^{i-1}.

Thus the formula in Theorem 4 is equivalent to

$$(s, x_1^1 v_1^1) \in \zeta_{n-1} \wedge (s, u_1^1 v_1^1) \in \zeta_{n-1} \wedge (s, v_1^1 y_1^1) \in \zeta_{n-1},$$

which, by our definition of ζ_n, is equivalent to $(s, x_1^1 y_1^1) \in \zeta_n$. Since $x_1^1 y_1^1 = t$, we see that $(s, t) \in \alpha_n \Leftrightarrow (s, t) \in \zeta_n$, i.e. $\alpha_n = \zeta_n$. □

3. Elementary axioms for semigroups embeddable in inverse semigroups.

Lemma 6. *A semigroup S is embeddable in an inverse semigroup if and only if ζ_n is antisymmetric on S for every n.*

Proof. By Theorem 2, the embeddability of S means that $\hat{\zeta}$ is antisymmetric, which, by Theorem 4, means that $(s, t) \in \zeta_m$ and $(t, s) \in \zeta_n$ imply $s = t$ for all m and n. In particular, this is true for $m = n$, and so each ζ_n is antisymmetric. Conversely, if each ζ_n is antisymmetric, $(s, t) \in \zeta_m$ and $(t, s) \in \zeta_n$, let p be the larger of the numbers m and n. Then $(s, t), (t, s) \in \zeta_p$ and, since ζ_p is antisymmetric, $s = t$. □

Let A_n denote the formula

$$\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{3^{i-1}} \begin{pmatrix} x_j^i v_j^i = x_{3j-2}^{i+1} y_{3j-2}^{i+1} \wedge x_{-j}^i v_{-j}^i = x_{2-3j}^{i+1} y_{2-3j}^{i+1} \wedge \\ u_j^i v_j^i = x_{3j-1}^{i+1} y_{3j-1}^{i+1} \wedge u_{-j}^i v_{-j}^i = x_{1-3j}^{i+1} y_{1-3j}^{i+1} \wedge \\ u_j^i y_j^i = x_{3j}^{i+1} y_{3j}^{i+1} \wedge u_{-j}^i y_{-j}^i = x_{-3j}^{i+1} y_{-3j}^{i+1} \end{pmatrix} \Rightarrow x_{-1}^1 y_{-1}^1 = x_1^1 y_1^1,$$

where, as in Theorem 4, we assume that $x_{-1}^1 = x_j^{n+1}$ and $y_{-1}^1 = y_j^{n+1}$ for all j. Also, we assume that A_n is a universal formula (that is, all variables occuring in it are bounded by universal quantifiers). Thus, A_n is a quasi identity (from a certain finite set of equalities there follows a final equality). If we look at the hypothesis in A_n disregarding its second column, we obtain, by Theorem 4, that $(x_{-1}^1 y_{-1}^1, x_1^1 y_1^1) \in \zeta_n$, while the second column of the hypothesis, also by Theorem 4, means that $(x_1^1 y_1^1, x_{-1}^1 y_{-1}^1) \in \zeta_n$. Therefore, A_n means merely that ζ_n is antisymmetric. The following theorem is now an easy corollary to Theorem 2.

Theorem 5. *A semigroup is embeddable in an inverse semigroup if and only if it satisfies quasi identities A_n for all positive integers n.*

A semigroup S is called *rectangular* (see [22]) if $xv = uv = uy$ imply $xy = xv$ for all $u, v, x, y \in S$. It is easily seen that $\zeta_0 = \zeta_1 = \Delta_S$ in a rectangular semigroup. It follows by induction on n that $\zeta_n = \Delta_S$ for all n. By Theorem 4, $\hat{\zeta} = \Delta_S$. In fact, rectangular semigroups are characterized by the equality $\hat{\zeta} = \Delta_S$. Thus, *every rectangular semigroup is embeddable in an inverse semigroup.* Obviously, every semigroup with one-sided (left or right) cancellative law is rectangular. Thus, *right cancellative and left cancellative semigroups are embeddable in inverse semigroups.*

A semigroup S with zero 0 is called *0-rectangular* if $xv = uv = uy \neq 0$ always imply $xy = xv$. It is not difficult to see that $\zeta_n = \Delta_S \cup (\{0\} \times S)$ in a 0-rectangular semigroup for all $n \geq 1$, and hence $\hat{\zeta} = \Delta_S \cup (\{0\} \times S)$ (that is, 0 is the least element of S and no two distinct nonzero elements of S are comparable with respect to $\hat{\zeta}$). Thus, *0-rectangular semigroups are embeddable in inverse semigroups.* A semigroup S is called *right 0-cancellative* if $xv = uv \neq 0$ imply $x = u$. Left 0-cancellative semigroups are defined dually. Obviously, both right and left 0-cancellative semigroups are 0-rectangular, and so *right 0-cancellative semigroups and left 0-cancellative semigroups are embeddable in inverse semigroups.*

Also, it is easy to check that every monogenic (that is, generated by a single element) semigroup is embeddable in an inverse semigroup.

In particular, free semigroups are cancellative, and hence embeddable in inverse semigroups. It follows that the class **C** of semigroups embeddable in inverse semigroups is not a variety. Indeed, a variety is closed under homomorphisms, and every semigroup is a homomorphic image of a free semigroup. Yet not every semigroup is embeddable in an inverse semigroup. However, **C** is a quasi variety, because this class is defined by a system of quasi identities.

Here is the axiom A_1 written out explicity:

$$\begin{array}{l} x_1^1 v_1^1 = x_{-1}^1 y_{-1}^1 \wedge x_{-1}^1 v_{-1}^1 = x_1^1 y_1^1 \wedge \\ u_1^1 v_1^1 = x_{-1}^1 y_{-1}^1 \wedge u_{-1}^1 v_{-1}^1 = x_1^1 y_1^1 \wedge \Rightarrow x_{-1}^1 y_{-1}^1 = x_1^1 y_1^1 \\ u_1^1 y_1^1 = x_{-1}^1 y_{-1}^1 \wedge u_{-1}^1 y_{-1}^1 = x_1^1 y_1^1 \end{array}$$

or, equivalently,

$$x_1^1 v_1^1 = u_1^1 v_1^1 = u_1^1 y_1^1 \wedge x_{-1}^1 v_{-1}^1 = u_{-1}^1 v_{-1}^1 = u_{-1}^1 y_{-1}^1 \Rightarrow x_1^1 v_1^1 = x_{-1}^1 v_{-1}^1.$$

Every semigroup that satisfies A_1 has commuting idempotents. Indeed, let S be a semigroup with identity 1 and let e and f be idempotent elements of S. Substituting $x_1^1 = v_1^1 = x_{-1}^1 = y_{-1}^1 = ef$, $u_1^1 = efe$, $u_{-1}^1 = e$, $v_{-1}^1 = y_1^1 = f$ in A_1 we see that the hypothesis of A_1 holds. Thus, $(ef)^2 = ef \cdot ef = x_{-1}^1 \cdot y_{-1}^1 = x_1^1 \cdot y_1^1 = ef \cdot f = ef$. Thus, S is a semigroup with weak involutive property (see [14]), and the idempotents of S form a subsemigroup. Using that and applying A_1 once more for $x_1^1 = u_{-1}^1 = ef$, $v_1^1 = e$, $u_1^1 = x_{-1}^1 = efe$, $v_{-1}^1 = f$, $y_1^1 = y_{-1}^1 = 1$, we obtain $efe = x_{-1}^1 \cdot y_{-1}^1 = x_1^1 \cdot y_1^1 = ef \cdot f = ef$. Applying A_1 again we obtain analogously $efe = fe$. Therefore, $ef = fe$ and idempotents of S commute.

Now we write A_2 in explicit form:

$$\begin{array}{rr}
& x_1^2 v_1^2 = x_{-1}^1 y_{-1}^1 \wedge x_{-1}^2 v_{-1}^2 = x_1^1 y_1^1 \wedge \\
& u_1^2 v_1^2 = x_{-1}^1 y_{-1}^1 \wedge u_{-1}^2 v_{-1}^2 = x_1^1 y_1^1 \wedge \\
x_1^1 v_1^1 = x_1^2 y_1^2 \wedge x_{-1}^1 v_{-1}^1 = x_{-1}^2 y_{-1}^2 \wedge & u_1^2 y_1^2 = x_{-1}^1 y_{-1}^1 \wedge u_{-1}^2 y_{-1}^2 = x_1^1 y_1^1 \wedge \\
& x_2^2 v_2^2 = x_{-1}^1 y_{-1}^1 \wedge x_{-2}^2 v_{-2}^2 = x_1^1 y_1^1 \wedge \\
& u_2^2 v_2^2 = x_{-1}^1 y_{-1}^1 \wedge u_{-2}^2 v_{-2}^2 = x_1^1 y_1^1 \wedge \quad \Rightarrow \\
u_1^1 v_1^1 = x_2^2 y_2^2 \wedge u_{-1}^1 v_{-1}^1 = x_{-2}^2 y_{-2}^2 \wedge & u_2^2 y_2^2 = x_{-1}^1 y_{-1}^1 \wedge u_{-2}^2 y_{-2}^2 = x_1^1 y_1^1 \wedge \\
& x_3^2 v_3^2 = x_{-1}^1 y_{-1}^1 \wedge x_{-3}^2 v_{-3}^2 = x_1^1 y_1^1 \wedge \\
& u_3^2 v_3^2 = x_{-1}^1 y_{-1}^1 \wedge u_{-3}^2 v_{-3}^2 = x_1^1 y_1^1 \wedge \\
u_1^1 y_1^1 = x_3^2 y_3^2 \wedge u_{-1}^1 y_{-1}^1 = x_{-3}^2 y_{-3}^2 \wedge & u_3^2 y_3^2 = x_{-1}^1 y_{-1}^1 \wedge u_{-3}^2 y_{-3}^2 = x_1^1 y_1^1
\end{array}$$

$$\Rightarrow x_{-1}^1 y_{-1}^1 = x_1^1 y_1^1 .$$

Loking at it we get a pretty good idea of the structure of A_n for any n.

Remark. We have considered semigroups with identity for simplicity only. By Proposition 1, a semigroup S that possesses no identity is embeddable in an inverse semigroup if and only if S^1 satisfies the axioms A_n. This amounts to S satisfying axioms A_n, in which some of the variables may be replaced by "empty symbols" (using Lyapin's term from [3]), that is, certain variables are merely erased.

4. Semigroups embeddable in inverse semigroups are not finitely axiomatizable.

We have obtained an infinite set of quasi identities that describe the class of semigroups embeddable in inverse semigroups. A natural question is whether this class of semigroups can be described by a finite set of elementary conditions. (A formula is called elementary if it consists of equalities of semigroup products connected by prepositional connectives – conjuction, disjunction, implication, equivalence, and negation – with quantifiers bounding individual variables, that is, variables that take values in the set of the *elements* of the semigroup).

It is easy to see that if $m < n$, then every semigroup that satisfies A_n has to satisfy A_m too. Indeed, $\zeta_m \subset \zeta_n$, and hence, if A_n holds, which means that ζ_n is antisymmetric, then ζ_m is antisymmetric too, and so A_m holds. In particular, it follows that every finite set of our axioms $\{A_{i_1}, A_{i_2}, \dots, A_{i_n}\}$ is equivalent to a single axiom A_p, where $p = \max\{i_1, i_2, \dots, i_n\}$. In a sense, our set of axioms is equivalent to a single infinitely long axiom A_ω, where ω is the first infinite ordinal number. Unfortunately, A_ω is not an elementary formula.

Theorem 6. *The class of semigroups embeddable in inverse semigroups cannot be characterized by any finite system of elementary axioms.*

Proof. Suppose that the set $\mathbf{A}$ of our axioms A_i is not equivalent to any of its finite subsets. Here is a standard argument showing that the class $\mathbf{C}$ of all embeddable semigroups is not finitely axiomatizable. If $\mathbf{B}$ is any finite set of elementary axioms characterizing $\mathbf{C}$, then each of the axioms $B_j \in \mathbf{B}$ expresses an elementary property shared by all semigroups from $\mathbf{C}$. By Godel's Completeness Theorem for the First Order Predicate Calculus (see any texbook in mathematical logic), B_j can be formally derived from the formulas $\mathbf{A}$. The derivation has finite length, so it uses only finitely many formulas from $\mathbf{A}$. Since $\mathbf{B}$ is finite, all formulas in $\mathbf{B}$ can be derived from a finite subset $\mathbf{A}'$ of $\mathbf{A}$.

On the other hand, each of the formulas in $\mathbf{A}$ can be derived from the formulas $\mathbf{B}$. It follows that each of the formulas in $\mathbf{A}$ follows from the formulas in $\mathbf{A}'$. Thus $\mathbf{A}$ is equivalent to its finite subset $\mathbf{A}'$, which contradicts our assumption about $\mathbf{A}$.

Therefore, what we really have to prove is that $\mathbf{A}$ is not equivalent to any of its finite subsets. Thus, Theorem 6 follows from Proposition 2, which forms the main content of this section.

Proposition 2. *For every positive integer n axiom A_n does not imply axiom A_{n+1}.*

Proof. (By constructing a semigroup S_{n+1} that satisfies A_n but does not satisfy A_{n+1}). For a fixed positive integer n, consider the alphabet Φ_{n+1} consisting of letters $u_j^i, v_j^i, x_j^i, y_j^i$ for $i = 1, 2, \ldots, n+1$ and $j = \pm 1, \pm 2, \ldots, \pm 3^{i-1}$. For convenience sake, we also consider the symbols $x_j^{n+2} = x^1_{-\operatorname{sign} j}$ and $y_j^{n+2} = y^1_{-\operatorname{sign} j}$, where sign j is 1 or -1 for $j > 0$ and $j < 0$, respectively. Individual variables running over Φ_{n+1} are denoted by lower case letters (possibly with indices) from the first half of the roman alphabet. Words in the alphabet Φ_{n+1} are denoted by upper case roman letters, words consisting of a single letter are denoted in the same way as that letter. Λ is the empty word, and $|A|$ denotes the length of a word A.

Let S_{n+1} be the semigroup with identity generated by the elements of Φ_{n+1} subject to the defining relations that appear as eqalities in the hypothesis of the axiom A_{n+1}. Words that appear in the left-(right-)hand sides of the defining relations are called *left (right) transformed pairs*. A left and a right transformed pairs are called *corresponding* if they form one of the defining realtions. We need a few properties of the system of defining relations.

Property 1. *No transformed pair, except right transformed pairs $x_1^1 y_1^1$ and $x_{-1}^1 y_{-1}^1$ (these two pairs will be called* special*) appears twice in the defining relations.*

Corollary 1. *No right transformed pair is a left transformed pair.*

Corollary 2. *For every left transformed pair there corresponds precisely one right transformed pair.*

Property 2. *If $ab = cd$ and $ef = cg$ are two distinct defining relations, then cd is a special transformed pair. If $ab = cd$ and $ef = gd$ are two distinct defining relations, then cd is a special transformed pair.*

Property 3. *No transformed pair begins (ends) with a letter that is the final (initial) letter of any transformed pair.*

All these properties follow in an obvious way from the external form of the defining relations.

Let A be a word in Φ_{n+1}. Replacing all left transformed pairs that are segments of A by the corresponding right transformed pairs, we obtain a word $\overline{A}$. It follows from Property 1 that $\overline{A}$ contains no occurrences of left transformed pairs. Corollary 2 to Property 1 together with Properties 2 and 3 imply that the word $\overline{A}$ does not depend on the order in which left transformed pairs appearing in A are changed. We call $\overline{A}$ the *canonical form* of A. Clearly, two words, A

and B, represent the same element of S_{n+1} precisely when $\overline{A} = \overline{B}$. Therefore, without loss of generality, we may assume that the elements of S_{n+1} are all words A in canonical form (that is, $A = \overline{A}$) with the following multiplication: $A \cdot B = \overline{AB}$. Obviously, special transformed pairs have canonical form, and hence they represent different elements of S_{n+1}. Comparing the defining relations of S_{n+1} with the axiom A_{n+1}, we see that A_{n+1} fails in S_{n+1}.

Next we look into a problem that is by far more difficult: proving that A_n holds in S_{n+1}.

In the sequel $A = B$ means that A and B represent the same element of S_{n+1}, while $A \equiv B$ means that A and B are two coinciding words. Also, we write S instead of S_{n+1} (and hence $A \in S$ means $A = \overline{A}$). We need further properties of S.

Property 4. *Let $AabB = CcdD$, where $Aa, bB, CcdD \in S$, and $|A| = |C|$. Then*

(1) $A = C$ and $B = D$, and

(2) $ab \equiv cd$ or $ab = cd$ is a defining relation.

Proof. If ab is not a left transformed pair, then $AabB \equiv \overline{AabB} \equiv CcdD$, which implies (1) and (2). If ab is a left transformed pair and $ab = ef$ is the corresponding defining relation, then $AabB = AefB \equiv CcdD$, which implies (1) and $ef \equiv cd$, that is, $ab = cd$ is a defining relation. □

Property 5. *Let the following assumptions be true:*

(1) $\overline{XV} \equiv A_1aB_1$, $\overline{UV} \equiv A_2aB_2$, $\overline{UY} \equiv A_3aB_3$;

(2) $|A_1| = |A_2| = |A_3|$ and $|B_1| = |B_2| = |B_3|$;

(3) if one of the words A_ia ($1 \le i \le 3$) ends with a transformed pair, then this transformed pair is not special and all three words A_ia end with it;

(4) if one of the words a_iB ($1 \le i \le 3$) begins with a transformed pair, then this transformed pair is not special and all three words aB_i begin with it.

Under these assumptions there exist words A and B such that

(5) $|A| = |A_1|$ and $|B| = |B_1|$;

(6) $\overline{XY} \equiv AaB$;

(7) if A_ia end with a transformed pair, then Aa ends with the same transformed pair;

(8) if aB_i begin with a transformed pair, then aB begins with the same transformed pair.

Proof. Proof of Property 5 is split into two cases.

Case 1. Let $|A_1| \geq |X|$. Then it is easy to calculate that $|Y| \geq |aB_3|$. If $A_1 \equiv \Lambda$, then $|X| = 0$, $X \equiv U \equiv \Lambda$, $\overline{XV} \equiv aB_3$, and hence conditions (5-8) hold.

Let $A_1 \not\equiv \Lambda$. Then A_2 and A_3 have positive length, and hence they are nonempty words. Let $A_i \equiv C_i a_i$ for $1 \leq i \leq 3$ and consider two subcases.

(a) Let $|Y| > |aB_3|$. Then $|Y| \geq |a_3 a B_3|$, whence $\overline{X}\,\overline{Y} \equiv Da_3aB_3$, where Da_3 is a suitable word. If a_3a is a transformed pair, then it is right because the word A_3a ends with it. Thus $\overline{XY} \equiv \overline{D}a_3aB_3$ and the requirements (5-8) hold for $A \equiv \overline{D}a_3$ and $B \equiv B_3$. If a_3a is not a transformed pair, then $\overline{XY} \equiv \overline{Da_3aB_3}$, $A \equiv \overline{Da_3}$, and $B \equiv B_3$.

(b) Suppose that $|Y| = |aB_3|$. Let

(1) $$\overline{X} \equiv X'c_1,\ \overline{V} \equiv c_2V',\ \overline{U} \equiv U'c_3,\ \overline{Y} \equiv c_4Y'.$$

Applying Property 4 we obtain $C_1 \equiv X'$, $C_2 \equiv C_3 \equiv U'$, $B_1 \equiv B_2 \equiv V'$, $B_3 \equiv Y'$ and

(2) $$c_1c_2 = a_1a,\ c_3c_2 = a_2a,\ c_3c_4 = a_3a.$$

Consider formulas (2). It follows from Condition (1) of Property 5 that each of them is either trivial (that is, $=$ can be replaced by $\equiv$) or a defining relation. If the first equality in (2) is trivial, then $c_1 \equiv a_1$ and $c_2 \equiv a$, that is, the second equality has the form $c_3a = a_2a$. If follows from the form of our defining relations that $c_3a = a_2a$ cannot be one of them, and hence $c_3a \equiv a_2a$. Thus $c_3 \equiv a_2$. If the third of the equalities in (2) is a defining relation, then, by Condition (3) of Property 5, the right-hand sides of the equalities in (2) coincide, whence $a_3 \equiv a_2$ and the third equality has the form $a_2c_4 = a_2a$ and so it cannot be a defining relation. Therefore, $a_2c_4 \equiv a_2a$, and hence $c_1 \equiv c_3$ and $c_2 \equiv c_4$. In an analogous way we can prove that the triviality of any of the equalities in (2) implies all three of these equalities are trivial.

If all equalities in (2) are nontrivial, i.e., all of them are defining realtions, then, by Condition (3) of Property 5, the right-hand sides of the equalities in (2) coincide. They are not special transformed pairs, and hence, by Property 2, all relations in (2) coincide, that is, $c_1 \equiv c_3$ and $c_2 \equiv c_4$.

Thus, $c_1 \equiv c_3$ and $c_2 \equiv c_4$ in all cases. It follows that $\overline{X}\,\overline{Y} \equiv \overline{X'c_1}\,\overline{c_2Y'}$ and $\overline{XY} \equiv \overline{X'c_1c_2Y'} \equiv \overline{X'a_1aY'} \equiv A_1aB_3$, so that requirements (5-8) are satisfied.

Case 2. Let $|A_1| < |X|$. Then $|B_3| > |Y|$. This case is dual to Case 1 with respect to the involution that replaces each word by the word consisitng of the same letters written in the opposite order. □

Property 6. *Let* $\overline{XV} \equiv A_i x_{j_i}^{i_1} y_{j_1}^{i_1} B_1$, $\overline{UV} \equiv A_2 x_{j_2}^{i_2} y_{j_2}^{i_2} B_2$, $\overline{UY} \equiv A_3 x_{j_3}^{i_3} y_{j_3}^{i_3} B_3$ $\operatorname{sign} j_1 = \operatorname{sign} j_2 = \operatorname{sign} j_3$, $|A_1| = |A_2| = |A_3|$, $|B_1| = |B_2| = |B_3|$, *and* $i_1, i_2, i_3 \geq m$, *where* m *is a fixed integer such that* $m \geq 2$. *Then* $\overline{XY} \equiv A x_j^i y_j^i B$, *where* A *and* B *are words such that* $|A| = |A_1|$, $|B| = |B_1|$ *and* $i \geq m-1$, $\operatorname{sign} j = \operatorname{sign} j_1$.

Remark. It follows from $i_1 \geq m$ that the case $i_1 = n+2$ is possible, and hence our convention about the letters x_j^{n+2} and y_j^{n+2} is applied.

Proof. Case 1. Let $|X| \leq |A_1|$. It follows that $|x_{j_3}^{i_3} y_{j_3}^{i_3} B_3| \leq |Y|$. By Property 3, the word $A_3 x_{j_3}^{i_3}$ cannot end with a transformed pair, and hence $\overline{X}\,\overline{Y} \equiv C x_{j_3}^{i_3} y_{j_3}^{i_3} B_3$, where C is a suitable word. Therefore $\overline{XY} \equiv \overline{C} c_{j_3}^{i_3} y_{j_3}^{i_3} B_3$ and our claim holds.

Case 2. Let $|X| \geq |A_1 x_{j_1}^{i_1} y_{j_1}^{i_1}|$. Arguing as in Case 1, we obtain $\overline{X}\,\overline{Y} \equiv \overline{A_i x_{j_1}^{i_1} y_{j_1}^{i_1}} D$ and $\overline{XY} \equiv \overline{A_1 x_{j_1}^{i_1} y_{j_1}^{i_1} D}$, which implies our claim.

Case 3. Let $|X| = |A_1| + 1$. We use notations from (1). By Property 4, $A_1 \equiv X'$, $A_2 \equiv A_3 \equiv U'$, $B_1 \equiv B_2 \equiv V'$, $B_3 \equiv Y'$ and

$$c_1 c_2 = x_{j_1}^{i_1} y_{y_1}^{i_1}, \quad c_3 c_2 = x_{j_2}^{i_2} y_{j_2}^{i_2}, \quad c_3 c_4 = x_{j_3}^{i_3} y_{j_3}^{i_3}, \tag{3}$$

where each of the equalities in (3) is either trivial or a defining relation. Clearly, $\overline{XY} \equiv X' \overline{c_1 c_4} Y' \equiv A_i \overline{c_1 c_4} B_3$. It suffices to prove that $\overline{c_1 c_4}$ is $x_j^i y_j^i$ and $i \geq m-1$, $\operatorname{sign} j = \operatorname{sign} j_1$.

If any two of the formulas in (3) coincide, then $c_1 \equiv c_3$ or $c_2 \equiv c_4$, whence $\overline{c_1 c_4} \equiv \overline{c_3 c_4} \equiv x_{j_3}^{i_3} y_{i_3}^{i_3}$ or $\overline{c_1 c_4} \equiv \overline{c_1 c_2} \equiv x_{j_1}^{i_1} y_{j_1}^{i_1}$ and our Property holds.

Let no two formulas in (3) coincide. If three different defining relations correspond to these formulas, then it follows from their external form that they are $x_j^i v_j^i = x_{3j-2}^{i+1} v_{3j-2}^{i+1}$, $u_j^i v_j^i = x_{3j-1}^{i+1} v_{3j-1}^{i+1}$, and $u_j^i y_j^i = x_{3j}^{i+1} v_{3j}^{i+1}$ (for definiteness' sake we assumed that $j > 0$). It follows that $\overline{c_1 c_4} \equiv x_j^i y_j^i$ and $i \geq m-1$ because one of our conditions was $i+1 \geq m$.

Let defining relations correspond to the first two formulas in (3), while the third formula is trivial. If $i_3 = n+2$, then $c_3 c_4 \equiv x^1_{-\operatorname{sign} j_3} y^1_{-\operatorname{sign} j_3}$, whence $c_3 c_2 = x^1_{-\operatorname{sign} j_3} c_2$. Therefore, the second formula in (3) has the form $x^1_{-\operatorname{sign} j_3} c_2 = x_{j_2}^{i_2} y_{j_2}^{i_2}$. It follows from the external form of the defining relations that $\operatorname{sign} j_2 = -\operatorname{sign} j_3$, contrary to our assumption. Thus $i_3 \neq n+2$.

Therefore, $c_3 c_2 = x_{j_3}^{i_3} c_2 = x_{j_3}^{i_3} v_{j_3}^{i_3}$, whence $c_1 c_2 \equiv c_1 v_{j_3}^{i_3}$. However, $c_1 c_2 \neq c_3 c_2$, since otherwise, by Corollary 2 to Property 1, the first two formulas of (3) coincide, contrary to our assumption. Thus $c_1 c_2 \equiv u_{j_3}^{i_3} v_{j_3}^{i_3}$. It follows that $\overline{c_1 c_4} \equiv u_{j_3}^{i_3} y_{j_3}^{i_3} \equiv x_{3j_3}^{i_3+1} y_{3j_3}^{i_3+1}$ and Property 6 holds.

If we assume that one of the first two formulas in (3) is trivial, while the remaining two formulas are defining relations, then an analogous argument proves that Property 6 holds.

Let the first formula in (3) be a defining relation, while the remaining two formulas are trivial. Then first formula in (3) must have the form $c_1 y^{i_2}_{j_2} \equiv x^{i_1}_{j_1} y^{i_1}_{j_1}$, since $c_3 c_2 \equiv x^{i_2}_{j_2} y^{i_2}_{j_2}$. Therefore, $c_1 y^{i_2}_{j_2} \equiv u^{i_2}_{j_2} y^{i_2}_{j_2}$ and $i_1 = i_2 + 1$, as it follows from the external form of the defining relations. Also, $c_3 c_4 \equiv x^{i_3}_{j_3} y^{i_3}_{j_3}$, whence $x^{i_2}_{j_2} \equiv x^{i_3}_{j_3}$, so that $i_2 = i_3$ and $j_2 = j_3$. It follows that $c_1 c_4 \equiv u^{i_2}_{j_2} y^{i_3}_{j_3} \equiv u^{i_2}_{j_2} y^{i_2}_{j_2}$. Therefore, $\overline{c_1 c_4} \equiv \overline{u^{i_2}_{j_2} y^{i_2}_{j_2}} \equiv x^{i_1}_{3j_2} y^{i_2}_{3j_2}$ (because $i_1 = i_2 + 1$), and hence Property 6 holds.

An analogous argument shows that Property 6 holds when not the first, but the second or the third formulas of (3) are defining relations, while the remaining two formulas are trivial.

Finally, suppose that all three equalities in (3) are trivial. Then $c_1 c_2 \equiv x^{i_1}_{j_1} y^{i_1}_{j_1}$ and $c_3 c_2 \equiv x^{i_2}_{j_2} y^{i_2}_{j_2}$, whence $y^{i_1}_{j_1} \equiv y^{i_2}_{j_2}$, and so $i_1 = i_2$ and $j_1 = j_2$. Also, $c_3 c_4 \equiv x^{i_3}_{j_3} y^{i_3}_{j_3}$, and hence $x^{i_2}_{j_2} \equiv x^{i_3}_{j_3}$, so that $i_2 = i_3$ and $j_2 = j_3$. Thus all three equalities in (3) coincide, and we are back to the case we have already considered.

Obviuously, Cases 1, 2, and 3 exhaust all the possibilities. □

Property 7. *Axiom A_n holds in S.*

Proof. Suppose that the hypothesis of A_n holds for elements $U^j_i, V^j_i, X^j_i, Y^j_i \in S$, $i = 1, \ldots, n$; and $j = \pm 1, \ldots, \pm 3^{i-1}$, that is,

$$(4)\qquad \bigwedge_{i=1}^{n}\bigwedge_{j=1}^{3^{i-1}} \begin{pmatrix} X^i_j \cdot V^i_j = X^{i+1}_{3j-2} \cdot Y^{i+1}_{3j-2} \wedge X^i_{-j} \cdot V^i_{-j} = X^{i+1}_{2-3j} \cdot Y^{i+1}_{2-3j} \wedge \\ U^i_j \cdot V^i_j = X^{i+1}_{3j-1} \cdot Y^{i+1}_{3j-1} \wedge U^i_{-j} \cdot V^i_{-j} = X^{i+1}_{1-3j} \cdot Y^{i+1}_{1-3j} \wedge \\ U^i_j \cdot Y^i_j = X^{i+1}_{3j} \cdot Y^{i+1}_{3j} \wedge U^i_{-j} \cdot Y^i_{-j} = X^{i+1}_{-3j} \cdot Y^{i+1}_{-3j} \end{pmatrix},$$

where $X^{n+1}_j \equiv X^1_{-\operatorname{sign} j}$ and $Y^{n+1}_j \equiv Y^1_{-\operatorname{sign} j}$. We have to prove that $\overline{X^1_{-1}} \cdot \overline{Y^1_{-1}} = \overline{X^1_1} \cdot \overline{Y^1_1}$ or, equivalently, $\overline{X^1_{-1} Y^1_{-1}} = \overline{X^1_1 Y^1_1}$.

First of all we rewrite (4) in the form

$$(5)\qquad \bigwedge_{i=1}^{n}\bigwedge_{j=1}^{3^{i-1}} \begin{pmatrix} \overline{X^i_j V^i_j} = \overline{X^{i+1}_{3j-2} Y^{i+1}_{3j-2}} \wedge \overline{X^i_{-j} V^i_{-j}} = \overline{X^{i+1}_{2-3j} Y^{i+1}_{2-3j}} \wedge \\ \overline{U^i_j V^i_j} = \overline{X^{i+1}_{3j-1} Y^{i+1}_{3j-1}} \wedge \overline{U^i_{-j} V^i_{-j}} = \overline{X^{i+1}_{1-3j} Y^{i+1}_{1-3j}} \wedge \\ \overline{U^i_j Y^i_j} = \overline{X^{i+1}_{3j} Y^{i+1}_{3j}} \wedge \overline{U^i_{-j} Y^i_{-j}} = \overline{X^{i+1}_{-3j} Y^{i+1}_{-3j}} \end{pmatrix}.$$

Let $\overline{X^1_{-1} Y^1_{-1}} \equiv A^0 B^1 A^1 \ldots B^i A^i \ldots B^s A^s$, where $B_1, \ldots, B^i, \ldots, B^s$ are special transformed pairs, while the words $A^0, A^1, \ldots, A^S$ contain no occurrences of special transformed pairs. Obviously, $\overline{X_{-1}^1 Y_{-1}^1}$ can be always written in such a form. Let $A_i \equiv CaD$. Introducing the words $A \equiv A^0 B^1 \dots B^i C$ and $B \equiv D \dots B^s A^s$ we obtain $\overline{X_{-1}^1 Y_{-1}^1} \equiv AaB$. It follows from (5) that $\overline{X_j^n V_j^n} \equiv AaB$, $\overline{U_j^n V_j^n} \equiv AaB$, and $\overline{U_j^n Y_j^n} \equiv AaB$. Clearly, all assumptions of Property 5 are fulfilled. By Property 5, $\overline{X_j^n Y_j^n} \equiv A_0 a B_0$, where $|A_0| = |A|$, $|B_0| = |B|$, $A_0 a$ does not end and aB_0 does not begin with a special transformed pair, and if Aa ends (aB begins) with a transformed pair, then $A_0 a$ end (aB_0 begins) with the same transformed pair. This is true for every $j > 0$. This and formulas (5) imply $\overline{X_j^{n-1} V_j^{n-1}} \equiv A_1 a B_1$, $\overline{U_j^{n-1} V_j^{n-1}} \equiv A_2 a B_2$, and $\overline{U_j^{n-1} Y_j^{n-1}} \equiv A_3 a B_3$, where all assumptions of Property 5 hold. By Property 5, $\overline{X_j^{n-1} Y_j^{n-1}} \equiv A_4 a B_4$, where $|A_1| = |A_4|$, $|B_1| = |B_4|$ and the remaining claims of Property 5 hold too. Applying Property 5 once more we obtain $\overline{X_j^{n-2} Y_j^{n-2}} \equiv A_5 a B_5$, where $A_5 a B_5$ satisfies all the claims of Property 5.

Applying Property 5 n times we see that $\overline{X_1^1 Y_1^1}$ has the form $A_6 a B_6$, where $|A_6| = |A|$ and $|B_6| = |B|$. Since a is an arbitrary letter chosen in the subword A^i of the word $\overline{X_{-1}^1 Y_{-1}^1}$, we see that $\overline{X_1^1 Y_1^1} \equiv A^0 C^1 A^1 \dots C^i A^i \dots C^s A^s$, where C^i are two-lettered words. If C^i is not a special transformed pair, then, applying Property 5 to $\overline{X_1^1 Y_1^1}$ sufficiently many times, we obtain $\overline{X_{-1}^1 Y_{-1}^1}$ and see that $B^i \equiv C^i$, which is impossible because B^i is a special transformed pair. Therefore, the words C^i are special transformed pairs.

Write $\overline{X_{-1}^1 Y_{-1}^1}$ in the form $Ax_1^1 y_1^1 B$, where $x_1^1 y_1^1 \equiv B^i$. Then (5) implies $\overline{X_j^n V_j^n} \equiv Ax_1^1 y_1^1 B$, $\overline{U_j^n V_j^n} \equiv Ax_1^1 y_1^1 B$, and $\overline{U_j^n Y_j^n} \equiv Ax_1^1 y_1^1 B$. Applying Property 6, we obtain $\overline{X_j^n Y_j^n} \equiv A_1 x_{j_1}^{i_1} y_{j_1}^{i_1} B_1$, where $|A_1| = |A|$, $|B_1| = |B|$, and $i_1 \geq n+1$, $j_1 < 0$. Applying Property 6 again, we obtain in the same way that $\overline{X_j^{n-1} Y_j^{n-1}} \equiv A_2 x_{j_2}^{i_2} y_{j_2}^{i_2} B_2$ with $|A_2| = |A_1|$, $|B_2| = |B_1|$, $i_2 \geq n$. Applying Property 6 n times, we obtain $\overline{X_1^1 Y_1^1} \equiv A_3 x_{j_3}^{i_3} y_{j_3}^{i_3} B_3$, where $|A_3| = |A|$, $|B_3| = |B|$, $i_3 \geq 2$ and $j_3 < 0$. However, it is obvious that $x_{j_3}^{i_3} y_{j_3}^{i_3} \equiv C^i$. Since C^i is a special transformed pair, we have $C^i \equiv x_{j_3}^{n+2} y_{j_3}^{n+2}$, that is, $C^i \equiv B^i$.

If $B^i \equiv x_{-1}^1 y_{-1}^1$ then, arguing in exactly the same way, we prove that $C^i \equiv B^i$. Therefore, $\overline{X_{-1}^1 Y_{-1}^1} \equiv \overline{X_1^1 Y_1^1}$. $\square$

REFERENCES

[1] A.H. Clifford - G.B. Preston, *The Algebraic Theory of Semigroups,* Vol. II, A.M.S., Providence, R.I., 1964.

[2] P. Dubreil, *Contribution à la théorie des demi-groupes,* Mém. Acad. Sci. Inst. France, 63 (1941), pp. 1-52.

[3] E.S. Lyapin, *Semigroups,* Moscow, Fizmatgiz, 1960 (in Russian); English translation A.M.S., Providence, R.I., 1974.

[4] A. Malcev, *On the immersion of an algebraic ring into a field,* Math. Annalen, 113 (1937), pp. 686-691.

[5] A.I. Malcev, *On the embedding of associative systems in groups I,* Matem. Sbornik, 6 (48) (1939), pp. 331-336 (in Russian with a German summary).

[6] A.I. Malcev, *On the embedding of associative systems in groups II,* Matem. Sbornik, 8 (50) (1940), pp. 251-264 (in Russian with a German summary).

[7] G.B. Preston, *Inverse semi-groups,* J. London Math. Soc., 29 (1954), pp. 396-403.

[8] B.M. Schein, *Immersion of semigroups in generalized groups,* Diploma work, Saratov State University, 1960 (in Russian).

[9] B.M. Schein, *Abstract theory of semigroups of one-to-one transformlations,* Ph. D. Dissertation, Saratov State University, 1962.

[10] B.M. Schein, *A system of axioms for semigroups embeddable in generalized groups,* Doklady Akad. Nauk SSSR, 134 (1960), pp. 1030-1034 (in Russian); English translation in: Soviet Math. Doklady, 1 (1960), pp. 1180-1183.

[11] B.M. Schein, *Embedding semigroups in generalized groups,* Matem. Sbornik 55 (1961), pp. 379-400 (in Russian); English translation in: Translations of the Amer. Math. Soc., (2) 139 (1988), pp. 93-116.

[12] B.M. Schein, *Representation of generalized groups,* Izvestiya Vyssh. Uchebn. Zaved., Matematika, 3 (1962), pp. 164-176 (in Russian).

[13] B.M. Schein, *On transitive representation of semigroups,* Uspekhi Matem. Nauk, 18 (1963), no 2, pp. 215-222 (in Russian).

[14] B.M. Schein, *On the theory of generalized groups and generalized grouds,* Theoriya Polugrupp i Ee Prilozh., Saratov, Saratov Univ. Press, 1 (1965), pp. 286-324 (in Russian); English translation in: Translations of the Amer. Math. Soc., (2) 113 (1979), pp. 89-122.

[15] B.M. Schein, *On the theory of radicals of semigroups,* Summaries of Talks at the 8th All-Union Colloquium in General Algebra, Riga, 1967, pp. 131-132 (in Russian).

[16] B.M. Schein, *Free inverse semigroups are not finitely presentable,* Acta Math. Acad. Sci. Hungar., 26 (1975), pp. 41-52 (in Russian).

[17] B.M. Schein, *Embedding semigroups in inverse semigroups,* Algebra i Teoriya Chisel, Nal'chik, Kabardino-Balkarsk, State University, 2 (1977), pp. 147-163 (in Russian).

[18] B.M. Schein, *Cosets in groups and semigroups,* Proceed. Confer. in Semigroups with Applic. (Oberwolfach, 21-28 July 1991), Singapore, World Sci. Publ. Co., 1993, pp. 180-196.

[19] B.M. Schein, *Semigroups of cosets in semigroups: variations on a Dubreil theme,* Collect. Math., 46 (1995), pp. 171-182.

[20] A. Suschkewitsch, *On the extension of a semigroup to an entire group,* Zapiski Khar'kov. Matem. Tovar., (4) 12 (1935), pp. 81-86 (in Ukrainian).

[21] G. Thierrin, *Demi-groupes inversés et rectangulaires,* Bull. Cl. Sci. Acad. Roy. Belgique, 41 (1955), pp. 83-92.

[22] G. Tallini, *Sulla struttura algebrica delle trasformazioni tra parti di un insieme,* Ann. Mat. Pura ed Appl., (4) 71 (1966), pp. 295-322.

[23] V.V. Wagner, *Generalized groups,* Doklady Akad. Nauk SSSR, 84 (1952), pp. 1119-1122 (in Russian).

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