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ON L^1-CONVERGENCE OF REES-STANOJEVIĆ'S SUMS WITH COEFFICIENTS FROM THE CLASS K

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In this paper are considered the modified cosine sums introduced by Rees and Stanojević [2] with coefficients from the class **K**. In addition, is proved that the condition $\lim_{n\to\infty}|a_{n+1}|\log n = 0$ is a necessary and sufficient condition for the L^1-convergence of the cosine series. Also, an open problem about L^1-convergence for the $r-$th derivative of the cosine series is presented.

1. Introduction and Preliminaries

Let us consider the cosine trigonometric series

$$g(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos mx \tag{1}$$

with its partial sums denoted by

$$S_n(x) = \frac{a_0}{2} + \sum_{m=1}^{n} a_m \cos mx$$

and let $\lim_{n\to\infty} S_n(x) = g(x)$.

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A lot of authors investigated the L^1-convergence of the series (1) under different classes of coefficients. Was W. H. Young [9] who showed that condition $\lim_{m\to\infty} a_m \log m = 0$ is necessary and sufficient condition for cosine series with so-called convex ($\triangle^2 a_m \geq 0$, where $\triangle a_m = a_m - a_{m+1}, \triangle^2 a_m = \triangle a_m - \triangle a_{m+1}$) coefficients and A. N. Kolmogorov [1] extended this result to the cosine series with quasi-convex $\left(\sum_{m=1}^{\infty} m \left|\triangle^2 a_{m-1}\right| < \infty\right)$ coefficients.

Likewise, in various papers are introduced different modified cosine and sine sums which we shall make for them a list as follows:

C. S. Rees and Č. V. Stanojević [2] have introduced the following modified cosine sums

$$g_n(x) = \frac{1}{2}\sum_{m=0}^{n} \triangle a_k + \sum_{m=1}^{n}\sum_{j=m}^{n} (\triangle a_j) \cos mx, \tag{2}$$

until Kumari and Ram [6], introduced new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{m=1}^{n}\sum_{j=m}^{n} \triangle \left(\frac{a_j}{j}\right) m \cos mx,$$

and

$$h_n(x) = \sum_{m=1}^{n}\sum_{j=m}^{n} \triangle \left(\frac{a_j}{j}\right) m \sin mx.$$

Later on, N. Hooda, B. Ram and S. S. Bhatia [5] introduced new modified cosine sums as

$$R_n(x) = \frac{1}{2}\left(a_1 + \sum_{m=0}^{n} \triangle^2 a_m\right) + \sum_{m=1}^{n}\left(a_{m+1} + \sum_{j=m}^{n} \triangle^2 a_j\right) \cos mx.$$

Recently K. Kaur, S. S. Bhatia, and B. Ram [4] introduced new modified sine sums as

$$K_n(x) = \frac{1}{2\sin x}\sum_{m=1}^{n}\sum_{j=m}^{n} (\triangle a_{j-1} - \triangle a_{j+1}) \sin mx,$$

and a new class of sequences defined as follows:

A sequence $\{a_m\}$ is said to belongs to the class $\mathbf{K}$, if $a_m \to 0$, $m \to \infty$, and

$$\sum_{m=1}^{\infty} m \left|\triangle^2 a_{m-1} - \triangle^2 a_{m+1}\right| < \infty, \quad (a_0 = 0).$$

The class $\mathbf{K}$ can be generalized in the following manner: A sequence $\{a_m\}$ is said to belongs to the class $\mathbf{K}^r, (r = 0, 1, \dots)$, if $a_m \to 0$, as $m \to \infty$, and

$$\sum_{m=1}^{\infty} m^{r+1} \left|\triangle^2 a_{m-1} - \triangle^2 a_{m+1}\right| < \infty, \quad (a_0 = 0).$$

For $r = 0$, clearly $\mathbf{K} \equiv \mathbf{K}^0$ and $\mathbf{K}^r \subset \mathbf{K}$, $r \geq 1$.

Further, wishing to generalize the class $\mathbf{K}^r$ we define a certain subclass of $\mathbf{K}^r$ as follows:

Let $\alpha := \{\alpha_m\}$ be a positive monotone sequence tending to infinity. A sequence $\{a_m\}$ belongs to the class $\mathbf{K}(\alpha)$ if $a_m \to 0$, as $m \to \infty$, and

$$\sum_{m=1}^{\infty} \alpha_m \left|\triangle^2 a_{m-1} - \triangle^2 a_{m+1}\right| < \infty, \quad (a_0 = 0).$$

If we denote the class $\mathbf{K}(\alpha)$, where $\alpha_m := m^{r+1}$, by $\mathbf{K}^r$, that is, if we introduce the definition $\mathbf{K}^r := \mathbf{K}(m^{r+1})$, we immediately get the generalization of the classes $\mathbf{K}^r$, for any positive integer r.

All of the authors, (see also [7], [8]), cited above studied the L^1-convergence of these cosine and sine sums proving different results. Among others let us formulate the following statements proved in [4]:

Theorem 1.1. *Let the sequence* $\{a_n\}$ *belong to the class* $\mathbf{K}$*, then* $K_n(x)$ *converges to* $g(x)$ *in the* L^1*-norm.*

Corollary 1.2. *If* $\{a_n\}$ *belongs to the class* $\mathbf{K}$*, then the necessary and sufficient condition for the* L^1*-convergence of the cosine series (1) is* $\lim_{n\to\infty} a_n \log n = 0$.

As usually with $D_n(x)$ and $\tilde{D}_n(x)$ we shall denote the Dirichlet and its conjugate kernels defined by $D_n(x) = \frac{a_0}{2} + \sum_{m=1}^{n} \cos mx$, $\tilde{D}_n(x) = \sum_{m=1}^{n} \sin mx$, respectively.

The object of this paper is to prove analogous statements as the Theorem 1.1 and the Corollary 1.2 when we use the modified cosine sums $g_n(x)$ instead of new modified sine sums $K_n(x)$. More precisely, we shall show that Theorem 1.1 and the Corollary 1.2 hold good for modified cosine sums (2) with coefficients from the class $\mathbf{K}$.

Throughout this paper the constants in the O-expression denote positive constants and they may be different in different relations.

2. Main Results

We shall prove the following main result.

Theorem 2.1. *Let the sequence* $\{a_n\}$ *belong to the class* $\mathbf{K}$*, then* $g_n(x)$ *converges to* $g(x)$ *in the* L^1*-norm.*

Proof. During the proof we shall use the same technique as in [4]. We have

$$\begin{aligned}
g_n(x) &= \frac{a_0}{2} + \sum_{m=1}^{n} a_m \cos mx - a_{n+1} D_n(x) \\
&= \frac{1}{2\sin x} \sum_{m=1}^{n} 2a_m \sin x \cos mx - a_{n+1} D_n(x) \qquad (a_0 = 0) \\
&= \frac{1}{2\sin x} \sum_{m=1}^{n} a_m \big[\sin(m+1)x - \sin(m-1)x\big] - a_{n+1} D_n(x) \\
&= \frac{1}{2\sin x} \sum_{m=1}^{n} (a_{m-1} - a_{m+1}) \sin mx \\
&+ a_n \frac{\sin(n+1)x}{2\sin x} + a_{n+1} \frac{\sin nx}{2\sin x} - a_{n+1} D_n(x) \\
&= \frac{1}{2\sin x} \sum_{m=1}^{n} (a_{m-1} - a_{m+1}) \sin mx + (a_n - a_{n+1}) \frac{\sin(n+1)x}{2\sin x}.
\end{aligned} \tag{3}$$

Applying the Abel's transformation in (3) we get

$$\begin{aligned}
g_n(x) = \frac{1}{2\sin x} \sum_{m=1}^{n} (\triangle a_{m-1} - \triangle a_{m+1}) \tilde{D}_m(x) \\
+ (a_n - a_{2n}) \frac{\tilde{D}_n(x)}{2\sin x} + (a_n - a_{n+1}) \frac{\sin(n+1)x}{2\sin x},
\end{aligned} \tag{4}$$

and passing on limit when $n \to \infty$ we obtain

$$\lim_{n\to\infty} g_n(x) = \frac{1}{2\sin x} \sum_{m=1}^{\infty} (\triangle a_{m-1} - \triangle a_{m+1}) \tilde{D}_m(x). \tag{5}$$

In a similiar fashion we can show that

$$\begin{aligned}
S_n(x) = \quad & \frac{1}{2\sin x} \sum_{m=1}^{n} (\triangle a_{m-1} - \triangle a_{m+1}) \tilde{D}_m(x) \\
& + (a_n - a_{2n}) \frac{\tilde{D}_n(x)}{2\sin x} + a_{n+1} \frac{\sin nx}{2\sin x} + a_n \frac{\sin(n+1)x}{2\sin x},
\end{aligned}$$

$$g(x) = \lim_{n\to\infty} S_n(x) = \frac{1}{2\sin x} \sum_{m=1}^{\infty} (\triangle a_{m-1} - \triangle a_{m+1}) \tilde{D}_m(x), \tag{6}$$

and the series

$$\frac{1}{2\sin x} \sum_{m=1}^{\infty} (\triangle a_{m-1} - \triangle a_{m+1}) \tilde{D}_m(x)$$

converges (see [4]).

Therefore $\lim_{n\to\infty} g_n(x) = g(x)$ exists, and from (5) and (6) the following equality

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} S_n(x) = g(x)$$

holds. Hence

$$g(x) - g_n(x) = \frac{1}{2\sin x}\sum_{m=n+1}^{\infty}(\triangle a_{m-1} - \triangle a_{m+1})\tilde{D}_m(x)$$
$$- (a_n - a_{2n})\frac{\tilde{D}_n(x)}{2\sin x} - (a_n - a_{n+1})\frac{\sin(n+1)x}{2\sin x}.$$

Denoting with $\tilde{F}_m(x) = \frac{1}{m+1}\sum_{i=0}^{m}\tilde{D}_i(x)$ the conjugate Fejer kernel, then the use of Abel's transformation gives

$$g(x) - g_n(x) = \frac{1}{2\sin x}\lim_{\ell\to\infty}\Bigg[\sum_{m=n+1}^{\ell-1}(m+1)\left(\triangle^2 a_{m-1} - \triangle^2 a_{m+1}\right)\tilde{F}_m(x)$$
$$+ (\ell+1)(\triangle a_{\ell-1} - \triangle a_{\ell+1})\tilde{F}_\ell(x) - (n+1)(\triangle a_n - \triangle a_{n+2})\tilde{F}_n(x)\Bigg]$$
$$- (a_n - a_{2n})\frac{\tilde{D}_n(x)}{2\sin x} - (a_n - a_{n+1})\frac{\sin(n+1)x}{2\sin x}$$
$$= \frac{1}{2\sin x}\Bigg[\sum_{m=n+1}^{\infty}(m+1)\left(\triangle^2 a_{m-1} - \triangle^2 a_{m+1}\right)\tilde{F}_m(x)$$
$$- (n+1)(\triangle a_n - \triangle a_{n+2})\tilde{F}_n(x)\Bigg] - (a_n - a_{2n})\frac{\tilde{D}_n(x)}{2\sin x}$$
$$- (a_n - a_{n+1})\frac{\sin(n+1)x}{2\sin x}.$$

Thus

$$\|g - g_n\| = O\left(\sum_{m=n+1}^{\infty}(m+1)\left|\triangle^2 a_{m-1} - \triangle^2 a_{m+1}\right|\int_{-\pi}^{\pi}\left|\tilde{F}_m(x)\right|dx\right.$$
$$+ (n+1)\left|\triangle a_n - \triangle a_{n+2}\right|\int_{-\pi}^{\pi}\left|\tilde{F}_n(x)\right|dx$$
$$+ |a_n - a_{2n}|\int_{-\pi}^{\pi}\left|\frac{\tilde{D}_n(x)}{2\sin x}\right|dx$$
$$+ |a_n - a_{n+1}|\int_{-\pi}^{\pi}\left|\frac{\sin(n+1)x}{2\sin x}\right|dx.$$

The first and fourth terms tend to zero as $n \to \infty$ based on facts that

$$\int_{-\pi}^{\pi} \left|\tilde{F}_m(x)\right| dx = \pi$$

and $\{a_m\}$ belongs the class $\mathbf{K}$.

Further, for the second term, denoted by $\Lambda(n)$, for large enough n we obtain

$$\begin{aligned}
\Lambda(n) &= O\Big((n+1)\,|\triangle a_n - \triangle a_{n+2}|\Big) \\
&= O\left((n+1)\Big|\sum_{m=n}^{\infty}\left(\triangle^2 a_m - \triangle^2 a_{m+2}\right)\Big|\right) \\
&= O\left((n+1)\sum_{m=n+1}^{\infty}\left|\triangle^2 a_{m-1} - \triangle^2 a_{m+1}\right|\right) \\
&= O\left(\sum_{m=n+1}^{\infty} m\left|\triangle^2 a_{m-1} - \triangle^2 a_{m+1}\right|\right) = o(1).
\end{aligned}$$

Since $\int_{-\pi}^{\pi}\left|\frac{\tilde{D}_n(x)}{2\sin x}\right| dx = O(n)$ then the third term tends to zero, as well. Indeed, we have

$$\begin{aligned}
|a_n - a_{2n}| \int_{-\pi}^{\pi}\left|\frac{\tilde{D}_n(x)}{2\sin x}\right| dx &= O\left(n\Big|\sum_{m=n}^{\infty}\left(\triangle a_m - \triangle a_{m+2}\right)\Big|\right) \\
&= O\left((n+1)\sum_{m=n+1}^{\infty}\left|\triangle^2 a_{m-1} - \triangle^2 a_{m+1}\right|\right) \\
&= O\left(\sum_{m=n+1}^{\infty} m\left|\triangle^2 a_{m-1} - \triangle^2 a_{m+1}\right|\right) = o(1).
\end{aligned}$$

This completes the proof of our theorem. □

Corollary 2.2. *If $\{a_n\}$ belongs to the class $\mathbf{K}$, then the necessary and sufficient condition for the L^1-convergence of the cosine series (1) is* $\lim_{n\to\infty} |a_{n+1}| \log n = 0$.

Proof. Sufficiency. We can write

$$\begin{aligned}
\|g - S_n\| &\le \|g - g_n\| + \|S_n - g_n\| \\
&= \|g - g_n\| + \left\| a_{n+1}\left(\frac{\sin nx}{2\sin x} + \frac{\sin(n+1)x}{2\sin x}\right)\right\| \\
&= \|g - g_n\| + |a_{n+1}| \int_{-\pi}^{\pi} |D_n(x)|\, dx.
\end{aligned}$$

From the well-known relation $\int_{-\pi}^{\pi} |D_n(x)|\,dx \sim \log n$ and our assumption that $a_n \log n = o(1)$, we obtain $|a_{n+1}| \int_{-\pi}^{\pi} |D_n(x)|\,dx = o(1)$ as $n \to \infty$. Also, according to the Theorem 2.1 $\|g - g_n\| = o(1)$, as $n \to \infty$. This completes the sufficient condition.

Necessity. The following holds

$$\begin{aligned} |a_{n+1}| \log n \sim |a_{n+1}| \int_{-\pi}^{\pi} |D_n(x)|\,dx &= \|a_{n+1} D_n(x)\| \\ &= \|S_n - g_n\| \leq \|S_n - g\| + \|g - g_n\| = o(1) \end{aligned}$$

by our assumption and the Theorem 2.1.

The Corollary 2.2 is proved completely. □

We saw in Section 2 how we extended the class $\mathbf{K}$ to the class $\mathbf{K}^r$, $(r = 0, 1, \ldots)$. Since the condition $\lim_{n\to\infty} |a_{n+1}| \log n = 0$ is a special case of the condition $\lim_{n\to\infty} n^r |a_{n+1}| \log n = 0, (r = 0)$, then the following propositions may be true:

Theorem 2.3. *Let the sequence $\{a_n\}$ belong to the class $\mathbf{K}^r$, then $g_n^{(r)}(x)$ converges to $g^{(r)}(x)$ in the L^1-norm.*

Corollary 2.4. *If $\{a_n\}$ belongs to the class $\mathbf{K}^r$, then the necessary and sufficient condition for the L^1-convergence of the $r-th$ derivative of the cosine series (1) is* $\lim_{n\to\infty} n^r |a_{n+1}| \log n = 0$.

The question is: Do these propositions are true or not? This is still an open problem.

REFERENCES

[1] A. N. Kolmogorov, *Sur l'ordre de grandeur des coefficients de la series de Fourier-Lebesgue*, Bull. Polon. Sci. Ser. Sci. Math. Astron. Phys. (1923), 73–76.

[2] C. S. Rees - Č. V. Stanojević, *Necessary and sufficient condition for intergrability of certain cosine sums*, J. Math. Anal. Appl. 43 (1973), 579–586.

[3] K. Kaur - S. S. Bhatia, *Integrability and L^1-convergence of Ress-Stanojević sums with generalized semiconvex coefficients*, IJMMS 30 (11) (2002), 645–650.

[4] K. Kaur - S. S. Bhatia - B. Ram, *Integrability and L^1-convergence of modified sine sums*, Georgian Math. J. 11 (1) (2004), 99–104.

[5] N. Hooda - B. Ram - S. S. Bhatia, *On L^1-convergence of a modified cosine sum*, Soochow J. of Math. 28 (3) (2002), 305–310.

[6] S. Kumari - B. Ram, *L^1-convergence of modified cosine sums*, Indian J. Pure Appl. Math. 19 (11) (1977), 1101–1104.

[7] X. Z. Krasniqi, *A note on L^1-convergence of the sine and cosine trigonometric series with semi-convex coefficients*, Int. J. Open Probl. Comput. Sci. Math. 2 (2) (2009), 231–239.

[8] N. L. Braha - X. Z. Krasniqi, *On L^1 convergence of certain cosine sums*, Bull. Math. Anal. Appl. 1 (1) (electronic only) (2009), 55–61.

[9] W. H. Young, *On Fourier series of bounded function*, Proc. London Math. Soc. 12 (2) (1913), 41–70.

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