STRUCTURE OF $E_3$-RINGS

ADA VARISCO

To the memory of Umberto Gasapina

We give a complete classification of $E_3$-rings (i.e. non-idempotent rings with set of idempotents $E$, where every non-idempotent subring containing four idempotents contains the whole $E$) and prove that the class of non-trivial regular $E_3$-rings is empty.

Introduction.

In [1] we have defined $E_k$-rings $R$ ($k$ positive integer) as those non-idempotent rings whose subrings containing $k + 1$ idempotents either are idempotent or contain $E$, the set of idempotents of $R$. In that paper $E_1$-rings and $E_2$-rings are completely described: the first ones are all trivial, i.e. they contain exactly two idempotents, save for rings isomorphic or antiisomorphic to $R = \langle e, f \rangle$ with $2e = 2f = 0$ and $ef = e, fe = f$, containing three idempotents. On the other hand, non-trivial $E_2$-rings (i.e. containing at least four idempotents) are the rings where $E$ is a proper multiplicative subsemigroup, and moreover, either $E$ is commutative of order 4 with identity or $E \setminus 0$ is a singular band of prime order $p > 2$ such that $E \setminus 0 = \{e + ha \mid h = 0, 1, \ldots, p - 1\}$ for some $e \in E \setminus 0, a \in R \setminus 0$.

It seems to be of some interest to go on with the study of $E_k$-rings for $k > 2$. Here we present a characterization of $E_3$-rings.

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In what follows $Z$ will denote the centre of the ring $R$, and $(R, \cdot)$ the multiplicative semigroup of $R$. The term "subsemigroup" (subgroup) stands for "multiplicative subsemigroup" (multiplicative subgroup). The symbol $\langle a, b, \ldots \rangle$ represents the subring of $R$ generated by the elements $a, b, \ldots$ Non-defined terminology and notation may be found in [2] and [3].

According to [1], we say that $R$ is a non-trivial $E_3$-ring if it satisfies the following conditions:

i) $R \supset E$ and $|E| > 4$

ii) If $A$ is a subring of $R$, $|A \cap E| > 3$ implies either $A \subseteq E$ or $A \supset E$.

The main object of the note is to prove the following.

**Theorem 1.** A non-idempotent ring $R$ is a non-trivial $E_3$-ring if and only if it satisfies one of the following conditions:

i) $R$ is a non-commutative $E_2$-ring with $|E| > 5$;

ii) $|E| = 5$ and $E = \{0, e, f, 2f - e, 3f - 2e\}$;

iii) $|R| = 8$, $E = \{0, e, f, e + e, e + f + f e\}$ and $R \setminus E = \{e + f, f + e\}$;

iv) $|R| = 8$, $E = \{0, e, f, e + ef, f + ef, e + f + ef\}$ and $R \setminus E = \{e + f, ef\}$;

v) $|E| = 5$, $E = \{0, e, f, (ef)^2, (fe)^2\}$ and $(ef)^2 = e + ef + ef e$, $(fe)^2 = f$;

vi) $|E| = 5$ and $E = \{0, e, f, u, e - f + u\}$;

vii) $R$ is anti-isomorphic to a ring of type iii) or iv) or v).

In preparation for the proof of the theorem we establish the following Lemmas.

**Lemma 1.** Let $R$ be a non-trivial $E_3$-ring. If $R$ contains two idempotents $e$, $f$ such that

\begin{equation}
2ef \neq 2efe,
\end{equation}

then, either $R$ is an $E_2$-ring with $|E| > 5$, or $|E| = 5$ and $E = \{0, e, f, 2f - e, 3f - 2e\}$.

**Proof.** If (1) holds, the idempotents $0, e, e + ef - ef e, e + 2ef - 2efe$ are distinct, and, since $2e \neq 0$ implies $2e \in R \setminus E$, the subring $\langle e, ef \rangle$ contains $E$. Consequently, $e$ is a left identity of $E$. Now, let $u$ be a non-zero idempotent, and suppose $ue \neq e$. It is easy to verify that the subring $\langle e, ue \rangle$ contains the four distinct idempotents $0, e, ue, e - ue$ and the non-idempotent element $2e$. Then, $\langle e, ue \rangle \supset E$ and $e$ is a right identity of $E$, in contradiction to (1). Thus $ue = e$ for every $u \in E \setminus 0$. 
Since \( fe = e \), and \( E \subseteq \langle e, f \rangle = \{ ie + jf \mid i, j \in \mathbb{Z} \} \), \( f \) is a left identity of \( E \). Then, if \( uf \neq f \) for some \( u \in E \setminus \{0\} \), the subring \( \langle f, uf \rangle \), containing \( 2f \neq 0 \) and the distinct idempotents \( 0, f, uf, f - uf \), contains \( E \), a contradiction, since \( \langle f, uf \rangle \) is commutative. Thus, \( uf = f \) for any \( u \in E \setminus \{0\} \), and we may conclude that \( E \setminus \{0\} \) is a right zero semigroup. So (1) becomes

\[
2f \neq 2e.
\]

Moreover, \( E \subseteq \langle e, f \rangle = \{ e, f - e \} = \{ me + n(f - e) \mid m, n \in \mathbb{Z} \} \), and every \( e + n(f - e) \) is a non-zero idempotent. Conversely, if \( me + n(f - e) \) is a non-zero idempotent, we must have \( [me + n(f - e)]e = e \), whence \( me = e \). Thus we may conclude that

\[
E \setminus \{0\} = \{ e + n(f - e) \mid n \in \mathbb{Z} \}. \tag*{(3)}
\]

Now, let us prove that either \( p(f - e) = 0 \) for some prime \( p \neq 2 \) or \( 4(f - e) = 0 \). Suppose \( 4(f - e) \neq 0 \). Then, the subring \( \langle e, 2(f - e) \rangle \), containing the distinct idempotents \( 0, e, e + 2(f - e), e + 4(f - e) \) and the non-idempotent \( 2(f - e) \), contains \( E \), whence \( f = he + 2k(f - e) \) for some \( h, k \in \mathbb{Z} \). This implies \( he = e \), whence \( (2k - 1)(f - e) = 0 \), and \( f - e \) has odd finite additive order \( r \). Let \( p \neq r \) be a prime factor of \( r \). Since \( 2p(f - e) \neq 0 \), the idempotents \( 0, e, e + p(f - e), e + 2p(f - e) \) are distinct, so the subring \( \langle e, p(f - e) \rangle \) contains \( E \). Hence, \( f = se + tp(f - e) \) for some \( s, t \in \mathbb{Z} \). This implies \( se = e \), whence \( (pt - 1)(f - e) = 0 \). This means that \( pt - 1 \) is a multiple of \( r \), contrary to the fact that \( p \) divides \( r \). Thus \( r = p \). At this point we have shown that either \( p(f - e) = 0 \) for some odd prime \( p \) or \( 4(f - e) = 0 \).

In the first case, \( R \) is an \( E_2 \)-ring by Th. 2.1 of [1], and \( |E| > 4 \) induces \( p \geq 5 \) and \( |E| > 5 \). In the second case, since \( 2(f - e) \neq 0 \), by (2), \( E \) consists of the five distinct elements \( 0, e, f, 2f - e, 3f - 2e \).

**Lemma 2.** Let \( R \) be a non-trivial \( E_3 \)-ring. If \( 2xy = 2yx \) for every \( x, y \in E \), and there exist two idempotents \( e, f \) such that

\[
ef = f \quad \text{and} \quad f \neq fe \neq e, \tag*{(4)}
\]

then \( |R| = 8 \), \( E = \{ 0, e, f, fe, e + fe, e + f + fe \} \) and \( R \setminus e = \{ e + f, f + fe \} \).

**Proof.** We first notice that \( fe \) is a non-zero idempotent: otherwise, \( f = ef = (ef)^2 = 0 \), a contradiction. Since \( 0, e, fe, e - fe \) are distinct idempotents and \( e \) is not a right identity of \( E \), the subring \( Re \) is idempotent, whence \( 2e = 0 \), and \( 2f = 2ef = 0 \). Then, \( \langle e, f \rangle = \{ he + kf + jfe \} \) with \( h, k, j \in \{0, 1\} \). Moreover, since \( \langle e, f \rangle \) contains the non-idempotent \( f + fe \),
we have \( \langle e, f \rangle \supset E \), and it is easily seen that \( E \) consists of the six distinct elements \( 0, e, f, fe, e+fe, e+f+fe \). This implies that, for every \( z \in R \), the idempotent \( e+ze+eze \), coincides either with \( e \) or with \( e+f+fe \). Since \( ze \in E \), we have \( eze = ze \), so we may conclude that either \( ez+ze = 0 \) or \( ez+ze = f+fe \). Suppose \( ez+ze = 0 \) for some \( z \in R \setminus E \). Then, \( ez = ze \); moreover, \( e+fe \) commutes with both \( e \) and \( f \), so it is a central idempotent, and the subring \( H = \langle e, e+fe, z \rangle \) is commutative. This is a contradiction, since \( \langle e, e+fe, z \rangle \), containing the distinct idempotents \( 0, e, fe, e+fe \), contains \( E \). Thus, \( ez+ze = f+fe \) for every \( z \in R \setminus E \). Now, consider the element \( z+f \) with \( z \in R \setminus E \). If \( z+f \in R \setminus E \), we get, by the above, \( e(z+f)+(z+f)e = f+fe \), whence \( ef = fe \), a contradiction. Thus, \( z+f \in E \), and it is immediate that \( z+f \in \{ e, fe \} \), implying \( R \setminus E = \{ e+f, f+fe \} \).

\textbf{Lemma 3.} Let \( R \) be a non-trivial \( E_3 \)-ring. If \( 2xy = 2yx \) for every \( x, y \in E \), and there exist two idempotents \( e, f \) such that

\begin{equation}
\text{(5)} \quad efe \neq ef \neq e \neq e + ef - efe \quad \text{and} \quad (efe)^2 \neq e,
\end{equation}

then \( |R| = 8 \), \( E = \{ 0, e, f, e + ef, f + ef, e + f + ef \} \) and \( R \setminus E = \{ e + f, f + ef \} \).

\textbf{Proof.} If the non-zero idempotents \( e, e + ef - efe, e + (ef)^2 - (efe)^2 \) are distinct, the subring \( eR \), containing the non-idempotent \( ef - efe \), contains \( E \), which implies \( ef = f \), contrary to the hypothesis. Thus we have either \( e = e + (ef)^2 - (efe)^2 \) or \( e + ef - efe = e + (ef)^2 - (efe)^2 \). Since \( 2ef = 2fe \), each of the two cases leads to \( (ef)^2 \in E \). For the same reason we must have \( (ef)^2 \in \{ 0, e, e + ef - efe \} \), which implies \( (ef)^2 = 0 \), in view of \( (efe)^2 \neq e \). Now, consider the non-zero idempotents \( e, e + ef - efe, e + f - efe \). If they are distinct, we have \( \langle e, ef, fe \rangle \supset E \), whence \( f = \alpha e + \beta ef + \gamma f e + \delta f e + \varepsilon f e + \zeta (fe)^2 \), for some integers \( \alpha, \beta, \gamma, \delta, \varepsilon, \zeta \). This implies also \( f = (\alpha + \beta + \gamma + \delta) ef \), whence \( ef = 0 \), contrary to (5). Thus we must have \( fe = efe \), implying \( fe = (fe)^2 = (fe)^3 = 0 \). Since the subring \( \langle e, f \rangle \) contains the distinct idempotents \( 0, e, f + ef \) and the non-idempotent \( ef \), it contains \( E \). Consequently, every idempotent \( v \) may be expressed in the form \( v = \alpha e + \beta f + \gamma ef \) with \( \alpha, \beta \in \mathbb{Z} \) and \( \gamma \in \{ 0, 1 \} \), in view of \( 2ef = 2fe = 0 \). Therefore, \( v = v^2 = \alpha^2 e + \beta^2 f + (\alpha \beta + \alpha \gamma + \beta \gamma) ef \), whence \( \alpha^2 e = \alpha e \) and \( \beta^2 f = \beta f \). Thus \( \alpha e, \beta f \in E \). Since \( e \neq ef \neq f \) by the hypotheses, the non-idempotent subrings \( eR \) and \( Rf \) cannot contain \( E \), hence \( \alpha e \in \{ 0, e, e + ef \} \) and \( \beta f \in \{ 0, f, f + ef \} \). This allows us to conclude that the distinct idempotents of \( R \) are \( 0, e, f, e + ef, f + ef, e + f + ef \).

Now, it remains to show that \( R \setminus E = \{ e + f, ef \} \). Putting for simplicity \( u = e + f + ef \), we may represent \( E \) in the more convenient form \( E = \)}
\{0, e, f, u, u - e, u - f\}, and we immediately see that \(u\) is the identity of \(E\), hence \(u\) is central. For every \(z \in R \setminus E\), the commutative subring \(H_z = \langle e, u, eze \rangle\) contains the distinct idempotents \(0, e, u, u - e\). Since \(H_z\) cannot contain \(E\), we have \(H_z \subseteq E\), implying \(eze \in E\), and \(2u = 2e = 2f = 0\). Moreover, since \(eR \not\supset E\), \(eze\) belongs to the subset \(\{0, e, e + ef\}\). But, \(eze = e + ef\) implies \(eze = e\), a contradiction, hence \(eze \in \{0, e\}\). Then, if \(ez \in R \setminus E\), we have \(eze = 0\) and \(\langle e, u, ez \rangle \supset E\), whence \(f = \alpha e + \beta u + ez\) with \(\alpha, \beta \in \{0, 1\}\). From this we deduce \(0 = \alpha e + \beta e\) and \(f = \beta f\), whence \(\alpha = \beta = 1\). Therefore, \(ez = ef\). On the other hand, if \(ez \in E\), we must have \(ez \in \{0, e, e + ef\}\); consequently, in any case, \(ez \in \{0, e, e + ef, ef\}\).

Finally, consider the element \(ze\): if \(ze \in R \setminus E\), we have \(\langle e, u, ze \rangle \supset E\), whence \(f = \alpha e + \beta u + ze\) with \(\alpha, \beta \in \{0, 1\}\). This implies \(ef = \alpha e + \beta e + eze = ef e\), contrary to the hypothesis. Thus \(ze \in E\), and it is immediate that \(ze \in \{0, e\}\).

Moreover, since the subring \(\langle e, u, z \rangle\), which contains \(E\), is not commutative, we have \(ez \neq ze\), implying \(ez + ze = ef\). Consider now the element \(z + f\). If it lies in \(R \setminus E\), from the above it follows that \(e(z + f) + (z + f)e = ef\), which induces \(ef = 0\), a contradiction. Thus \(z + f \in E\), whence \(z + f \in \{e, u - e\}\), that is \(z \in \{e + f, ef\}\). This completes the proof.

**Lemma 4.** Let \(R\) be a non-trivial \(E_3\)-ring. If \(2xy = 2yx\) for every \(x, y \in E\) and there exist two idempotents \(e, f\) such that

\[
(6) \quad efe \neq ef \neq f \neq e + ef - ef e \quad \text{and} \quad (efe)^2 = e,
\]

then either \(R\) is a ring of the type described in Lemma 2 or \(|E| = 5\), \(E = \{0, e, f, (ef)^2, (fe)^2\}\) and \(e + ef - ef e = (ef)^2, (fe)^2 = f\).

**Proof.** If \(fe = e\), the relations (6) become \(e \neq ef \neq f\), so we are just in the hypotheses of Lemma 2, save for the exchange between \(e\) and \(f\). Then we may assume \(fe \neq e\). We notice that the subring \(eR\) contains the non-idempotent \(ef - efe\) and the non-zero idempotents \(e, e + ef - efe, (ef)^2\). Since \(ef \neq f\), \(eR\) cannot contain \(E\); therefore the three idempotents are not distinct, and it is clear that

\[
(7) \quad (ef)^2 = e + ef - efe.
\]

If \(ef = fef\), we have \(e = (efe)^2 = (fe)^2\), whence \(fe = e\), contrary to the hypothesis. Thus, the subring \(Rf\) contains the non-idempotent \(ef - fef\) and the three non-zero idempotents \(f, f + ef - fef, (ef)^2\). Since \(Rf \not\supset E\), in view of \(ef \neq e\), these idempotents cannot be distinct, and we immediately see that

\[
(8) \quad (ef)^2 = f + ef - fef.
\]


This implies \((efe)^2 = fe + efe - (fe)^2\), that is
\[
(ef)^2 = fe + efe - e
\]
and
\[
(efe)^2 = f.
\]
Moreover, by comparing (7) and (8) we get
\[
(11) \quad fef = f - e + efe.
\]
Since \(2xy = 2yx\) for every \(x, y \in E\), the relations \((efe)^2 = e\) and \((efe)^2 = f\) imply
\[
(12) \quad 2e = 2f
\]
therefore it is clear that the subring \((e, f)\) consists of the elements
\[
(13) \quad \alpha e + \beta f + \gamma efe + \delta fe + \varepsilon efe
\]
with \(\alpha \in \mathbb{Z}\) and \(\beta, \gamma, \delta, \varepsilon \in \{0, 1\}\). Now put \(\overline{E} = \{0, e, f, (ef)^2, (fe)^2\}\) and let us show that \(\overline{E} = E\). In fact, since \(E \subseteq (e, f)\), every \(u \in E\) has the form (13), so that, from the preceding relations we may deduce \(u = u^2 = \xi e + (\beta^2 + \beta \gamma + \beta \delta + \gamma \delta) f + (\alpha \beta + \alpha \gamma + \beta \gamma + \gamma^2 + \beta \varepsilon + \gamma \varepsilon) efe + (\alpha \beta + \beta \delta + \alpha \delta + \beta \varepsilon + \delta^2 + \delta \varepsilon) f e + (\alpha \delta + \alpha \gamma - \beta \gamma - \beta \delta - \delta^2 - \delta \varepsilon - \gamma^2 - \gamma \varepsilon) efe\) for some \(\xi \in \mathbb{Z}\). There are three distinct cases: \(\beta = \gamma; \beta = \delta; \gamma \neq \beta \neq \delta\) implying \(\gamma = \delta\). In the first two cases it is easily seen that \(\overline{E} = E\). If \(\gamma = \delta = 1\) and \(\beta = 0\), we have \(u = \eta e + f (\alpha + \varepsilon + 1) (efe + f e)\); if \(\gamma = \delta = 0\) and \(\beta = 1\), we have \(u = \theta e = f + (\alpha + \varepsilon) (efe + f e)\), for some \((\eta, \theta \in \mathbb{Z})\).
Thus, in any case,
\[
(14) \quad u = he + f + k (efe + f e)
\]
for some \(h \in \mathbb{Z}\) and \(k \in \{0, 1\}\). If \(k = 0\), we may suppose \(h\) odd; otherwise, \(u = (h + 1) f \in \overline{E}\) by the above. Then (14) implies \(u = u^2 = h^2 e + hef + hfe + f\), whence \((h^2 - h)e + h(efe + f e) = 0\). Consequently, \((h^2 - 1)e + efe + f e = 0\) in view of (12). This implies \(efe = efe\), contrary to the hypothesis. If \(k = 1\), we find, using relations (7), (9), (10), (11), (12), \(u^2 = (h + 3)^2 e\), which implies \(u \in \overline{E}\). Thus \(\overline{E} = E\) as required.
Lemma 5. Let $R$ be a non-trivial $E_3$-ring, and let $e, f$ be two non-zero distinct idempotents of $R$, with

$$ef = f, \quad fe = e, \quad 2e = 2f.$$ 

Then, the only idempotents of the subring $\langle e, f \rangle$ are $0, e, f$.

**Proof.** If $2f = 0$, we have $\langle e, f \rangle = \{0, e, f, e + f\}$ with $(e + f)^2 = 0$ and the statement is true. Now suppose $2f \neq 0$. Then, $\langle e, f \rangle = \{kf, e + kf \mid k \in \mathbb{Z}\}$. By means of an immediate calculation we get

$$kf \in E \quad \text{if and only if} \quad (1 - k)f \in E \quad \text{and} \quad$$

$$kf \in \{0, f\} \quad \text{if and only if} \quad (1 - k)f \in \{0, f\}.$$

Therefore, if there exists a non-zero idempotent $kf \neq f, (1 - k)f$ too is a non-zero idempotent distinct from $f$. Since $2f \in R \setminus E$, if $kf \neq (1 - k)f$, the subring $\langle f \rangle$, containing the distinct idempotents $0, f, kf, (1 - k)f$, contains $E$, contrary to $ef \neq f e$. Thus, $kf = (1 - k)f$, which implies $kf = (kf)^2 = k(1 - k)f = 0$, a contradiction. Consequently, the only idempotents of the form $kf$ are $0$ and $f$. Analogously, the only idempotents of the form $ke$ are $0$ and $e$. Now, if $e + kf \in E$ for some integer $k$, we have $e + kf = (e + kf)^2 = (k + 1)e + (k + 1)kf \in \langle e \rangle$, whence $e + kf \in \{0, e\}$.

Lemma 6. If $e, f$ are distinct non-zero idempotents of a ring $R$, and $ef = fe$, the subring $\langle e, f \rangle$ contains at least four distinct idempotents.

**Proof.** It is immediate that the four distinct idempotents are $0, e, f, x$ with $x = ef$ when $ef \neq 0, e, f; x = e + f$ when $ef = 0; x = e - f$ when $ef = f; x = f - e$ when $ef = e$.

Lemma 7. Let $R$ be a non-trivial $E_3$-ring satisfying the conditions:

i) $2xy = 2yx$ for every $x, y \in E$;

ii) $ef = f, fe = e$ for some distinct $e, f \in E$;

iii) $xy \neq yx$ implies either $xy = y, yx = x$ or $xy = x, yx = y$ for every $x, y \in E$.

Then $|E| = 5$ and $E = \{0, e, f, u, e - f + u\}$.

**Proof.** Let $u \in E \setminus \{0, e, f\}$. By the hypotheses it follows that one of the following conditions holds:

1) $eu = ue$,

2) $eu = u, ue = e$,

3) $eu = e, ue = u$. 
Symmetrically, one of the following holds:

I) \( fu = uf \),

II) \( fu = u, uf = f \),

III) \( fu = f, uf = u \).

We may immediately reduce to examine the following three cases:

II) \( eu = ue, fu = uf \),

2II) \( eu = u, ue = e, fu = u, uf = f \),

3III) \( eu = e, ue = u, fu = f, uf = u \).

The last leads to the contradiction \( f = fu = fue = fe = e \). Now, let us examine the case II). If \( 2e \neq 0 \), the commutative subring \( \langle e, u \rangle \) contains four distinct idempotents (Lemma 4) and the non-idempotent \( 2e \). Then it contains \( E \), a contradiction, since \( ef \neq fe \). Consequently, \( 2e = 0 \) and \( e + u \) is idempotent. Since \( (e + u)f \neq f(e + u) \), the hypotheses imply either

\[
\begin{align*}
(e + u)f &= f \\
(f(e + u)) &= e + u
\end{align*}
\text{ or }
\begin{align*}
(e + u)f &= e + u \\
(f(e + u)) &= f
\end{align*}
\]

In the first case we have \( uf = 0 \) and \( fu = u \), a contradiction. In the second we get \( f + uf = e + u \) and \( e + fu = f \), which, in view of \( 2f = 2e = 0 \), lead to the contradiction \( u = 0 \). Then, it remains to study the case 2II). In this case, \( E \setminus 0 \) is a right zero-semigroup; moreover, \( 2e = 2f = 2u \), by the hypotheses. So, each of the subrings \( \langle e, f \rangle, \langle e, u \rangle, \langle f, u \rangle \) has exactly three idempotents, by Lemma 5. On the other hand, the subring \( \langle e, f, u \rangle \), containing four distinct idempotents and the non-idempotent \( e - f \), contains \( E \). Since \( \langle e, f, u \rangle = \{ he + kf + ju \mid h, k, j \in \mathbb{Z} \} \), if there exists and idempotent \( v \) distinct from \( 0, e, f, u \), it may be written in the form \( v = e - f + ju \) for some odd integer \( j \). Then, we have \( e - f + ju = (e - f + ju)^2 = f(e - f) + j^2u \), whence \( ju = (ju)^2 \). Since \( ju \in \langle e, u \rangle \cap \langle f, u \rangle \), it follows from Lemma 5 that either \( ju = 0 \), or \( ju = u \). But, \( ju = 0 \) implies \( v = e - f \), a contradiction. Thus, \( ju = u \). Since the element \( e - f + u \) is actually idempotent, we may conclude that \( E = \{ 0, e, f, u, e - f + u \} \).

Remark 1. It is immediate to see that, in the statement of Lemma 7, the condition \( ef = f, fe = e \) may be replaced by the condition \( ef = e, fe = f \).

Lemma 8. An \( E_3 \)-ring whose idempotents are central is trivial.

Proof. Let \( E \subseteq Z \), and let \( H \) be a non-idempotent subring of \( R \) containing two distinct non-zero idempotents. Then, \( H \) contains at least four distinct idempotents, by Lemma 6, and \( R \) is an \( E_2 \)-ring with central idempotents. Hence \( |E| = 4 \), by Theorem 2.1 of [1].
Proof of Theorem 1. "Only if part". Let \( R \) be a non-trivial \( E_3 \)-ring. If \( R \) contains two idempotents \( e, f \) with \( 2ef \neq 2efe \), \( R \) is either of type i) or of type ii) by Lemma 1 and by Theorem 2.1 of [1]. To the same conclusion we arrive starting from the hypothesis that \( 2fe = 2efe \) for some \( e, f \in E \). Then, suppose \( 2xy = 2yx \) for every \( x, y \in E \), and consider the following subcases:

A) There exist \( e, f \in E \) with \( ef \neq efe \) and \( e + ef -efe \neq f \). If \( ef = f \), \( R \) turns out to be a ring of type iii), by Lemma 2. If \( ef \neq f \) and \( (efe)^2 \neq e \), \( R \) is of type iv) by Lemma 3. Finally, if \( ef \neq f \) and \( (efe)^2 = e \), \( R \) is of type v) by Lemma 4.

B) There exist \( e, f \in E \) with \( fe \neq efe \) and \( e + fe -efe \neq f \). It is easily seen that \( R \) is anti-isomorphic to one of the rings of case A).

C) For every \( e, f \in E \), either

\[
ef = efe \quad \text{or} \quad e + ef -efe = f,
\]

and symmetrically, either

\[
fef = efe \quad \text{or} \quad e + fe -efe = f.
\]

Since the relations \( e + ef -efe = f \) and \( e + fe -efe = f \) are equivalent respectively to \( ef = f \), \( fe = e \) and to \( ef = e \), \( fe = f \), we may conclude that, for every \( e, f \in E \) one on the following holds:

1) \( ef = fe \),
2) \( ef = f \), \( fe = e \),
3) \( ef = e \), \( fe = f \).

If \( ef = fe \) for every \( e, f \in E \), then \( E \subseteq Z \), and \( R \) (Lemma 8) should be a trivial \( E_3 \)-ring, a contradiction. Otherwise, \( R \) is a ring of type vi), by Lemma 7 and Remark 1.

"If part". It is immediate.

Remark 2. Neither of the classes of rings described in the statement of Theorem 1 is empty, as we will show by means of the following examples.

Example 1. The ring of all square matrices over the field \( \mathbb{Z}_p \) (\( p \) prime > 3) of the form

\[
\begin{bmatrix}
x & 0 \\
y & 0
\end{bmatrix}
\]

is a non-trivial \( E_3 \)-ring of type i) (see Example 2 of [1]).
Example 2. Let $R$ be the ring of square matrices over the ring $\mathbb{Z}_4$, of the form
\[
\begin{bmatrix}
  x & 0 \\
  y & 0
\end{bmatrix}
\]. Since the non-zero idempotents of $r$ are
\[
e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad u = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \quad v = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix},
\]
with $u = 2f - e$ and $v = 3f - 2e$, $R$ is a non-trivial $E_3$-ring of type ii).

Example 3. Let $R = \langle e, f \rangle$ the ring generated by the idempotent matrices
\[
e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]
over the ring $\mathbb{Z}_2$. It is a routine verification that $efe = e$, $fef = f$, and that the distinct idempotents of $R$ are $0, e, f, ef, fe$. Moreover, since the conditions $(ef)^2 = e + ef + efe$ and $(fef)^2 = f$ are trivially satisfied, and $e + f$ is not idempotent, $R$ turns out to be a non-trivial $E_3$-ring of type v).

Example 4. Let $G$ be the additive abelian group generated by three elements $e, f, g$ with defining relations $2e = 2f = 2g = 0$, and let $S_1, S_2$ and $S_3$ be the multiplicative semigroups respectively defined by the following multiplication tables:

\[
\begin{array}{c|ccc}
S_1 & e & f & g \\
\hline
 e & e & f & g \\
 f & g & f & g \\
 g & g & f & g \\
\end{array}
\quad
\begin{array}{c|ccc}
S_2 & e & f & g \\
\hline
 e & e & f & g \\
 f & e & f & g \\
 g & e & f & g \\
\end{array}
\quad
\begin{array}{c|ccc}
S_3 & e & f & g \\
\hline
 e & e & g & g \\
 f & 0 & f & 0 \\
 g & 0 & g & 0 \\
\end{array}
\]

Since $G = \{he + kf + jg\}$ with $h, k, j \in \{0, 1\}$, if we extend these products to all elements of $G$, making use of the distributive laws, we construct three rings $R_1, R_2$ and $R_3$ which are non-trivial $E_3$-rings of the types iii), vi) and iv) respectively.

In [1] we proved that $R$ is a non-trivial regular $E_2$-ring if and only if $|R| > 4$ and $R$ is the direct sum of two division rings. It is natural to ask whether something analogous holds for $E_3$-rings. The answer is negative. In fact, we may establish the following.
Theorem 2. A non-trivial $E_3$-ring cannot be regular.

Proof. Let $R$ be a non-trivial regular $E_3$-ring. If $R$ satisfies the hypotheses either of Lemma 1 or of Lemma 7, $E \setminus 0$ turns out to be a right-zero semigroup, therefore the element $e - f$ is not regular: in fact, $(e - f)x(e - f) = (e - f)$, with $x(e - f)$ idempotent, implies $e - f = 0$. In the hypotheses of Lemma 2, we have $ef = f$ and $2e = 2f = 0$; moreover, every idempotent has the form $he + kf + jfe (h,k,j \in \mathbb{Z})$. Since $(f + fe)(he + kf + jfe) = 0$, the element $f + fe$ is not regular. In the hypotheses of Lemma 3, we have $fe = 0 \neq ef$, and $R = \{0, e, f, ef, e + f, e + ef, f + ef, e + f + ef\}$. Therefore, it is clear that $ef x ef = 0$ for every $x \in R$, so that the element $ef$ is not regular. Finally, in the hypotheses of Lemma 4, if $R$ is of the type described in Lemma 2, the statement is true by the above. Otherwise, let us show that the element $ef - efe$ is not regular. In fact, it is obvious that, if $(ef - efe)x(ef - efe) = ef - efe$, the idempotent $x(ef - efe)$ cannot coincide with $0, e, (ef)^2$. Since $E = \{0, e, f, (ef)^2, (fe)^2\}$, we may have either $x(ef - efe) = f$, implying $fe = 0$ or $x(ef - efe) = (fe)^2$ implying $(fe)^2 = 0$. Since $(ef)^2 = e$, in both cases we get $e = 0$, a contradiction. Thus, in each of the cases examined $R$ contains some non-regular element. Any other case is easily reduced to one of these.

REFERENCES