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STRUCTURE OF E_3-RINGS

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To the memory of Umberto Gasapina

We give a complete classification of E_3-rings (i.e. non-idempotent rings with set of idempotents E, where every non-idempotent subring containing four idempotents contains the whole E) and prove that the class of non-trivial regular E_3-rings is empty.

Introduction.

In [1] we have defined E_k-rings R (k positive integer) as those non-idempotent rings whose subrings containing $k+1$ idempotents either are idempotent or contain E, the set of idempotents of R. In that paper E_1-rings and E_2-rings are completely described: the first ones are all trivial, i.e. they contain exactly two idempotents, save for rings isomorphic or antiisomorphic to $R = \langle e, f \rangle$ with $2e = 2f = 0$ and $ef = e$, $fe = f$, containing three idempotents. On the other hand, non-trivial E_2-rings (i.e. containing at least four idempotents) are the rings where E is a proper multiplicative subsemigroup, and moreover, either E is commutative of order 4 with identity or $E \setminus 0$ is a singular band of prime order $p > 2$ such that $E \setminus 0 = \{e + ha \mid h = 0, 1, \dots, p-1\}$ for some $e \in E \setminus 0$, $a \in R \setminus 0$.

It seems to be of some interest to go on with the study of E_k-rings for $k > 2$. Here we present a characterization of E_3-rings.

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In what follows Z will denote the centre of the ring R, and $(R, \cdot)$ the multiplicative semigroup of R. The term "subsemigroup" (subgroup) stands for "multiplicative subsemigroup" (multiplicative subgroup). The symbol $\langle a, b, \ldots \rangle$ represents the subring of R generated by the elements $a, b, \ldots$ Non defined terminology and notation may be found in [2] and [3].

According to [1], we say that R is a non-trivial E_3-ring if it satisfies the following conditions:

i) $R \supset E$ and $|E| > 4$

ii) If A is a subring of R, $|A \cap E| > 3$ implies either $A \subseteq E$ or $A \supset E$.

The main object of the note is to prove the following.

Theorem 1. *A non-idempotent ring R is a non-trivial E_3-ring if and only if it satisfies one of the following conditions:*

i) R is a non-commutative E_2-ring with $|E| > 5$;

ii) $|E| = 5$ and $E = \{0, e, f, 2f - e, 3f - 2e\}$;

iii) $|R| = 8$, $E = \{0, e, f, fe, e+fe, e+f+fe\}$ and $R \setminus E = \{e+f, f+fe\}$;

iv) $|R| = 8$, $E = \{0, e, f, e+ef, f+ef, e+f+ef\}$ and $R \setminus E = \{e+f, ef\}$;

v) $|E| = 5$, $E = \{0, e, f, (ef)^2, (fe)^2\}$ and $(ef)^2 = e+ef+efe$, $(fef)^2 = f$;

vi) $|E| = 5$ and $E = \{0, e, f, u, e - f + u\}$;

vii) R is anti-isomorphic to a ring of type iii) or iv) or v).

In preparation for the proof of the theorem we establish the following Lemmas.

Lemma 1. *Let R be a non-trivial E_3-ring. If R contains two idempotents e, f such that*

$$2ef \neq 2efe, \tag{1}$$

then, either R is an E_2-ring with $|E| > 5$, or $|E| = 5$ and $E = \{0, e, f, 2f - e, 3f - 2e\}$.

Proof. If (1) holds, the idempotents $0, e, e + ef - efe, e + 2ef - 2efe$ are distinct, and, since $2e \neq 0$ implies $2e \in R \setminus E$, the subring $\langle e, ef \rangle$ contains E. Consequently, e is a left identity of E. Now, let u be a non-zero idempotent, and suppose $ue \neq e$. It is easy to verify that the subring $\langle e, ue \rangle$ contains the four distinct idempotents $0, e, ue, e - ue$ and the non-idempotent element $2e$. Then, $\langle e, ue \rangle \supset E$ and e is a right identity of E, in contradiction to (1). Thus $ue = e$ for every $u \in E \setminus 0$.

Since $fe = e$, and $E \subset \langle e, f\rangle = \{ie + jf \mid i, j \in \mathbb{Z}\}$, f is a left identity of E. Then, if $uf \neq f$ for some $u \in E \setminus 0$, the subring $\langle f, uf\rangle$, containing $2f \neq 0$ and the distinct idempotents $0, f, uf, f - uf$, contains E, a contradiction, since $\langle f, uf\rangle$ is commutative. Thus, $uf = f$ for any $u \in E \setminus 0$, and we may conclude that $E \setminus 0$ is a right zero semigroup. So (1) becomes

$$2f \neq 2e. \tag{2}$$

Moreover, $E \subset \langle e, f\rangle = \langle e, f - e\rangle = \{me + n(f - e) \mid m, n \in \mathbb{Z}\}$, and every $e + n(f - e)$ is a non-zero idempotent. Conversely, if $me + n(f - e)$ is a non-zero idempotent, we must have $[me + n(f - e)]e = e$, whence $me = e$. Thus we may conclude that

$$E \setminus 0 = \{e + n(f - e) \mid n \in \mathbb{Z}\}. \tag{3}$$

Now, let us prove that either $p(f - e) = 0$ for some prime $p \neq 2$ or $4(f - e) = 0$. Suppose $4(f - e) \neq 0$. Then, the subring $\langle e, 2(f - e)\rangle$, containing the distinct idempotents $0, e, e + 2(f - e), e + 4(f - e)$ and the non-idempotent $2(f - e)$, contains E, whence $f = he + 2k(f - e)$ for some $h, k \in \mathbb{Z}$. This implies $he = e$, whence $(2k - 1)(f - e) = 0$, and $f - e$ has odd finite additive order r. Let $p \neq r$ be a prime factor of r. Since $2p(f - e) \neq 0$, the idempotents $0, e, e + p(f - e), e + 2p(f - e)$ are distinct, so the subring $\langle e, p(f - e)\rangle$ contains E. Hence, $f = se + tp(f - e)$ for some $s, t \in \mathbb{Z}$. This implies $se = e$, whence $(pt - 1)(f - e) = 0$. This means that $pt - 1$ is a multiple of r, contrary to the fact that p divides r. Thus $r = p$. At this point we have shown that either $p(f - e) = 0$ for some odd prime p or $4(f - e) = 0$. In the first case, R is an E_2-ring by Th. 2.1 of [1], and $|E| > 4$ induces $p \geq 5$ and $|E| > 5$. In the second case, since $2(f - e) \neq 0$, by (2), E consists of the five distinct elements $0, e, f, 2f - e, 3f - 2e$.

Lemma 2. *Let R be a non-trivial E_3-ring. If $2xy = 2yx$ for every $x, y \in E$, and there exist two idempotents e, f such that*

$$ef = f \quad \text{and} \quad f \neq fe \neq e, \tag{4}$$

then $|R| = 8$, $E = \{0, e, f, fe, e + fe, e + f + fe\}$ and $R \setminus e = \{e + f, f + fe\}$.

Proof. We first notice that fe is a non-zero idempotent: otherwise, $f = ef = (ef)^2 = 0$, a contradiction. Since $0, e, fe, e - fe$ are distinct idempotents and e is not a right identity of E, the subring Re is idempotent, whence $2e = 0$, and $2f = 2ef = 0$. Then, $\langle e, f\rangle = \{he + kf + jfe\}$ with $h, k, j \in \{0, 1\}$. Moreover, since $\langle e, f\rangle$ contains the non-idempotent $f + fe$, we have $\langle e, f\rangle \supset E$, and it is easily seen that E consists of the six distinct elements $0, e, f, fe, e + fe, e + f + fe$. This implies that, for every $z \in R$, the idempotent $e + ez + eze$, coincides either with e or with $e + f + fe$. Since $ze \in E$, we have $eze = ze$, so we may conclude that either $ez + ze = 0$ or $ez + ze = f + fe$. Suppose $ez + ze = 0$ for some $z \in R \setminus E$. Then, $ez = ze$; moreover, $e + fe$ commutes with both e and f, so it is a central idempotent, and the subring $H = \langle e, e + fe, z\rangle$ is commutative. This is a contradiction, since $\langle e, e + fe, z\rangle$, containing the distinct idempotents $0, e, fe, e + fe$, contains E. Thus, $ez+ze = f+fe$ for every $z \in R\setminus E$. Now, consider the element $z+f$ with $z \in R\setminus E$. If $z+f \in R\setminus E$, we get, by the above, $e(z+f)+(z+f)e = f+fe$, whence $ef = fe$, a contradiction. Thus, $z + f \in E$, and it is immediate that $z + f \in \{e, fe\}$, implying $R \setminus E = \{e + f, f + fe\}$.

Lemma 3. *Let R be a non-trivial E_3-ring. If $2xy = 2yx$ for every $x, y \in E$, and there exist two idempotents e, f such that*

$$(5) \qquad efe \neq ef \neq f \neq e + ef - efe \quad \text{and} \quad (efe)^2 \neq e,$$

then $|R| = 8$, $E = \{0, e, f, e+ef, f+ef, e+f+ef\}$ and $R\setminus E = \{e+f, ef\}$.

Proof. If the non-zero idempotents $e, e + ef - efe, e + (ef)^2 - (efe)^2$ are distinct, the subring eR, containing the non-idempotent $ef - efe$, contains E, which implies $ef = f$, contrary to the hypothesis. Thus we have either $e = e+(ef)^2-(efe)^2$ or $e+ef-efe = e+(ef)^2-(efe)^2$. Since $2ef = 2fe$, each of the two cases leads to $(ef)^2 \in E$. For the same reason we must have $(ef)^2 \in \{0, e, e + ef - efe\}$, which implies $(ef)^2 = 0$, in view of $(efe)^2 \neq e$. Now, consider the non-zero idempotents $e, e + ef - efe, e + fe - efe$. If they are distinct, we have $\langle e, ef, fe\rangle \supset E$, whence $f = \alpha e + \beta ef + \gamma fe + \delta fef + \varepsilon efe + \zeta(fe)^2$, for some integers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$. This implies also $f = (\alpha + \beta + \gamma + \delta)fef$, whence $ef = 0$, contrary to (5). Thus we must have $fe = efe$, implying $fe = (fe)^2 = (fe)^3 = 0$. Since the subring $\langle e, f\rangle$ contains the distinct idempotents $0, e, f, e + ef$ and the non-idempotent ef, it contains E. Consequently, every idempotent v may be expressed in the form $v = \alpha e + \beta f + \gamma ef$ with $\alpha, \beta \in \mathbb{Z}$ and $\gamma \in \{0, 1\}$, in view of $2ef = 2fe = 0$. Therefore, $v = v^2 = \alpha^2 e + \beta^2 f + (\alpha\beta + \alpha\gamma + \beta\gamma)ef$, whence $\alpha^2 e = \alpha e$ and $\beta^2 f = \beta f$. Thus $\alpha e, \beta f \in E$. Since $e \neq ef \neq f$ by the hypotheses, the non-idempotent subrings eR and Rf cannot contain E, hence $\alpha e \in \{0, e, e + ef\}$ and $\beta f \in \{0, f, f+ef\}$. This allows us to conclude that the distinct idempotents of R are $0, e, f, e + ef, f + ef, e + f + ef$.

Now, it remains to show that $R \setminus E = \{e + f, ef\}$. Putting for simplicity $u = e + f + ef$, we may represent E in the more convenient form $E =$

$\{0, e, f, u, u - e, u - f\}$, and we immediately see that u is the identity of E, hence u is central. For every $z \in R \setminus E$, the commutative subring $H_z = \langle e, u, eze \rangle$ contains the distinct idempotents $0, e, u, u - e$. Since H_z cannot contain E, we have $H_z \subset E$, implying $eze \in E$, and $2u = 2e = 2f = 0$. Moreover, since $eR \not\supseteq E$, eze belongs to the subset $\{0, e, e + ef\}$. But, $eze = e + ef$ implies $eze = e$, a contradiction, hence $eze \in \{0, e\}$. Then, if $ez \in R \setminus E$, we have $eze = 0$ and $\langle e, u, ez \rangle \supset E$, whence $f = \alpha e + \beta u + ez$ with $\alpha, \beta \in \{0, 1\}$. From this we deduce $0 = \alpha e + \beta e$ and $f = \beta f$, whence $\alpha = \beta = 1$. Therefore, $ez = ef$. On the other hand, if $ez \in E$, we must have $ez \in \{0, e, e + ef\}$; consequently, in any case, $ez \in \{0, e, e + ef, ef\}$. Finally, consider the element ze: if $ze \in R \setminus E$, we have $\langle e, u, ze \rangle \supset E$, whence $f = \alpha e + \beta u + ze$ with $\alpha, \beta \in \{0, 1\}$. This implies $ef = \alpha e + \beta e + eze = efe$, contrary to the hypothesis. Thus $ze \in E$, and it is immediate that $ze \in \{0, e\}$. Moreover, since the subring $\langle e, u, z \rangle$, which contains E, is not commutative, we have $ez \neq ze$, implying $ez + ze = ef$. Consider now the element $z + f$. If it lies in $R \setminus E$, from the above it follows that $e(z + f) + (z + f)e = ef$, which induces $ef = 0$, a contradiction. Thus $z + f \in E$, whence $z + f \in \{e, u - e\}$, that is $z \in \{e + f, ef\}$. This completes the proof.

Lemma 4. *Let R be a non-trivial E_3-ring. If $2xy = 2yx$ for every $x, y \in E$ and there exist two idempotents e, f such that*

$$(6) \qquad efe \neq ef \neq f \neq e + ef - efe \quad \text{and} \quad (efe)^2 = e,$$

then either R is a ring of the type described in Lemma 2 or $|E| = 5$, $E = \{0, e, f, (ef)^2, (fe)^2\}$ and $e + ef - efe = (ef)^2$, $(fef)^2 = f$.

Proof. If $fe = e$, the relations (6) become $e \neq ef \neq f$, so we are just in the hypotheses of Lemma 2, save for the exchange between e and f. Then we may assume $fe \neq e$. We notice that the subring eR contains the non-idempotent $ef - efe$ and the non-zero idempotents $e, e + ef - efe, (ef)^2$. Since $ef \neq f$, eR cannot contain E; therefore the three idempotents are not distinct, and it is clear that

$$(7) \qquad (ef)^2 = e + ef - efe.$$

If $ef = fef$, we have $e = (efe)^2 = (fe)^2$, whence $fe = e$, contrary to the hypothesis. Thus, the subring Rf contains the non-idempotent $ef - fef$ and the three non-zero idempotents $f, f + ef - fef, (ef)^2$. Since $Rf \not\supseteq E$, in view of $ef \neq e$, these idempotents cannot be distinct, and we immediately see that

$$(8) \qquad (ef)^2 = f + ef - fef.$$

This implies $(efe)^2 = fe + efe - (fe)^2$, that is

$$(fe)^2 = fe + efe - e \tag{9}$$

and

$$(fef)^2 = f. \tag{10}$$

Moreover, by comparing (7) and (8) we get

$$fef = f - e + efe. \tag{11}$$

Since $2xy = 2yx$ for every $x, y \in E$, the relations $(efe)^2 = e$ and $(fef)^2 = f$ imply

$$2e = 2f \tag{12}$$

therefore it is clear that the subring $\langle e, f\rangle$ consists of the elements

$$\alpha e + \beta f + \gamma ef + \delta fe + \varepsilon efe \tag{13}$$

with $\alpha \in \mathbb{Z}$ and $\beta, \gamma, \delta, \varepsilon \in \{0, 1\}$. Now put $\overline{E} = \{0, e, f, (ef)^2, (fe)^2\}$ and let us show that $\overline{E} = E$. In fact, since $E \subset \langle e, f\rangle$, every $u \in E$ has the form (13), so that, from the preceding relations we may deduce $u = u^2 = \zeta e + (\beta^2 + \beta\gamma + \beta\delta + \gamma\delta)f + (\alpha\beta + \alpha\gamma + \beta\gamma + \gamma^2 + \beta\varepsilon + \gamma\varepsilon)ef + (\alpha\beta + \beta\delta + \alpha\delta + \beta\varepsilon + \delta^2 + \delta\varepsilon)fe + (\alpha\delta + \alpha\gamma - \beta\gamma - \beta\delta - \delta^2 - \delta\varepsilon - \gamma^2 - \gamma\varepsilon)efe$ for some $\zeta \in \mathbb{Z}$. There are three distinct cases: $\beta = \gamma$; $\beta = \delta$; $\gamma \neq \beta \neq \delta$ implying $\gamma = \delta$. In the first two cases it is easily seen that $\overline{E} = E$. If $\gamma = \delta = 1$ and $\beta = 0$, we have $u = \eta e + f(\alpha + \varepsilon + 1)(ef + fe)$; if $\gamma = \delta = 0$ and $\beta = 1$, we have $u = \theta e = f + (\alpha + \varepsilon)(ef + fe)$, for some $(\eta, \vartheta \in \mathbb{Z})$.
Thus, in any case,

$$u = he + f + k(ef + fe) \tag{14}$$

for some $h \in \mathbb{Z}$ and $k \in \{0, 1\}$. If $k = 0$, we may suppose h odd; otherwise, $u = (h + 1)f \in \overline{E}$ by the above. Then (14) implies $u = u^2 = h^2e + hef + hfe + f$, whence $(h^2 - h)e + h(ef + fe) = 0$. Consequently, $(h^2 - 1)e + ef + fe = 0$ in view of (12). This implies $ef = efe$, contrary to the hypothesis. If $k = 1$, we find, using relations (7), (9), (10), (11), (12), $u^2 = (h + 3)^2e$, which implies $u \in \overline{E}$. Thus $\overline{E} = E$ as required.

Lemma 5. *Let R be a non-trivial E_3-ring, and let e, f be two non-zero distinct idempotents of R, with*

$$ef = f, \quad fe = e, \quad 2e = 2f.$$

Then, the only idempotents of the subring $\langle e, f\rangle$ are $0, e, f$.

Proof. If $2f = 0$, we have $\langle e, f\rangle = \{0, e, f, e + f\}$ with $(e + f)^2 = 0$ and the statement is true. Now suppose $2f \neq 0$. Then, $\langle e, f\rangle = \{kf, e + kf \mid k \in \mathbb{Z}\}$. By means of an immediate calculation we get

$$kf \in E \quad \text{if and only if} \quad (1-k)f \in E \text{ and}$$
$$kf \in \{0, f\} \quad \text{if and only if} \quad (1-k)f \in \{0, f\}.$$

Therefore, if there exists a non-zero idempotent $kf \neq f$, $(1-k)f$ too is a non-zero idempotent distinct from f. Since $2f \in R\backslash E$, if $kf \neq (1-k)f$, the subring $\langle f\rangle$, containing the distinct idempotents $0, f, kf, (1-k)f$, contains E, contrary to $ef \neq fe$. Thus, $kf = (1-k)f$, which implies $kf = (kf)^2 = k(1-k)f = 0$, a contradiction. Consequently, the only idempotents of the form kf are 0 and f. Analogously, the only idempotents of the form ke are 0 and e. Now, if $e + kf \in E$ for some integer k, we have $e+kf = (e+kf)^2 = (k+1)e+(k+1)kf \in \langle e\rangle$, whence $e + kf \in \{0, e\}$.

Lemma 6. *If e, f are distinct non-zero idempotents of a ring R, and $ef = fe$, the subring $\langle e, f\rangle$ contains at least four distinct idempotents.*

Proof. It is immediate that the four distinct idempotents are $0, e, f, x$ with $x = ef$ when $ef \neq 0, e, f$; $x = e + f$ when $ef = 0$; $x = e - f$ when $ef = f$; $x = f - e$ when $ef = e$.

Lemma 7. *Let R be a non-trivial E_3-ring satisfying the conditions:*

i) $2xy = 2yx$ *for every* $x, y \in E$;

ii) $ef = f$, $fe = e$ *for some distinct* $e, f \in E$;

iii) $xy \neq yx$ *implies either* $xy = y, yx = x$ *or* $xy = x$, $yx = y$ *for every* $x, y \in E$.

Then $|E| = 5$ and $E = \{0, e, f, u, e - f + u\}$.

Proof. Let $u \in E \setminus \{0, e, f\}$. By the hypotheses it follows that one of the following conditions holds:

1) $eu = ue$,
2) $eu = u, ue = e$,
3) $eu = e, ue = u$.

Symmetrically, one of the following holds:

I) $fu = uf$,

II) $fu = u, uf = f$,

III) $fu = f, uf = u$.

We may immediately reduce to examine the following three cases:

1I) $eu = ue$, $fu = uf$,

2II) $eu = u, ue = e$, $fu = u, uf = f$,

3III) $eu = e, ue = u$, $fu = f, uf = u$.

The last leads to the contradiction $f = fu = fue = fe = e$. Now, let us examine the case 1I). If $2e \neq 0$, the commutative subring $\langle e, u\rangle$ contains four distinct idempotents (Lemma 4) and the non-idempotent $2e$. Then it contains E, a contradiction, since $ef \neq fe$. Consequently, $2e = 0$ and $e + u$ is idempotent. Since $(e + u)f \neq f(e + u)$, the hypotheses imply either

$$\begin{cases} (e+u)f = f \\ f(e+u) = e+u \end{cases} \quad \text{or} \quad \begin{cases} (e+u)f = e+u \\ f(e+u) = f \end{cases}.$$

In the first case we have $uf = 0$ and $fu = u$, a contradiction. In the second we get $f + uf = e + u$ and $e + fu = f$, which, in view of $2f = 2e = 0$, lead to the contradiction $u = 0$. Then, it remains to study the case 2II). In this case, $E \setminus 0$ is a right zero-semigroup; moreover, $2e = 2f = 2u$, by the hypotheses. So, each of the subrings $\langle e, f\rangle$, $\langle e, u\rangle$, $\langle f, u\rangle$ has exactly three idempotents, by Lemma 5. On the other hand, the subring $\langle e, f, u\rangle$, containing four distinct idempotents and the non-idempotent $e - f$, contains E. Since $\langle e, f, u\rangle = \{he + kf + ju \mid h, k, j \in \mathbb{Z}\}$, if there exists and idempotent v distinct from $0, e, f, u$, it may be written in the form $v = e - f + ju$ for some odd integer j. Then, we have $e - f + ju = (e - f + ju)^2 = j(e - f) + j^2u$, whence $ju = (ju)^2$. Since $ju \in \langle e, u\rangle \cap \langle f, u\rangle$, it follows from Lemma 5 that either $ju = 0$, or $ju = u$. But, $ju = 0$ implies $v = e - f$, a contradiction. Thus, $ju = u$. Since the element $e - f + u$ is actually idempotent, we may conclude that $E = \{0, e, f, u, e - f + u\}$.

Remark 1. It is immediate to see that, in the statement of Lemma 7, the condition $ef = f$, $fe = e$ may be replaced by the condition $ef = e$, $fe = f$.

Lemma 8. *An E_3-ring whose idempotents are central is trivial.*

Proof. Let $E \subseteq Z$, and let H be a non-idempotent subring of R containing two distinct non-zero idempotents. Then, H contains at least four distinct idempotents, by Lemma 6, and R is an E_2-ring with central idempotents. Hence $|E| = 4$, by Theorem 2.1 of [1].

Proof of Theorem 1. "Only if part". Let R be a non-trivial E_3-ring. If R contains two idempotents e, f with $2ef \neq 2efe$, R is either of type i) or of type ii) by Lemma 1 and by Theorem 2.1 of [1]. To the same conclusion we arrive starting from the hypothesis that $2fe = 2efe$ for some $e, f \in E$. Then, suppose $2xy = 2yx$ for every $x, y \in E$, and consider the following subcases:

A) There exist $e, f \in E$ with $ef \neq efe$ and $e + ef - efe \neq f$. If $ef = f$, R turns out to be a ring of type iii), by Lemma 2. If $ef \neq f$ and $(efe)^2 \neq e$, R is of type iv) by Lemma 3. Finally, if $ef \neq f$ and $(efe)^2 = e$, R is of type v) by Lemma 4.

B) There exist $e, f \in E$ with $fe \neq efe$ and $e + fe - efe \neq f$. It is easily seen that R is anti-isomorphic to one of the rings of case A).

C) For every $e, f \in E$, either

$$ef = efe \qquad \text{or} \qquad e + ef - efe = f,$$

and symmetrically, either

$$fe = efe \qquad \text{or} \qquad e + fe - efe = f.$$

Since the relations $e + ef - efe = f$ and $e + fe - efe = f$ are equivalent respectively to $ef = f$, $fe = e$ and to $ef = e$, $fe = f$, we may conclude that, for every $e, f \in E$ one on the following holds:

1) $ef = fe$,
2) $ef = f$, $fe = e$,
3) $ef = e$, $fe = f$.

If $ef = fe$ for every $e, f \in E$, then $E \subseteq Z$, and R (Lemma 8) should be a trivial E_3-ring, a contradiction. Otherwise, R is a ring of type vi), by Lemma 7 and Remark 1.
"If part". It is immediate.

Remark 2. Neither of the classes of rings described in the statement of Theorem 1 is empty, as we will show by means of the following examples.

Example 1. The ring of all square matrices over the field $\mathbb{Z}_p$ (p prime > 3) of the form $\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}$ is a non-trivial E_3-ring of type i) (see Example 2 of [1]).

Example 2. Let R be the ring of square matrices over the ring $\mathbb{Z}_4$, of the form $\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}$. Since the non-zero idempotents of r are

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad u = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \quad v = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix},$$

with $u = 2f - e$ and $v = 3f - 2e$, R is a non-trivial E_3-ring of type ii).

Example 3. Let $R = \langle e, f \rangle$ the ring generated by the idempotent matrices

$$e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad f = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

over the ring $\mathbb{Z}_2$. It is a routine verification that $efe = e$, $fef = f$, and that the distinct idempotents of R are $0, e, f, ef, fe$. Moreover, since the conditions $(ef)^2 = e + ef + efe$ and $(fef)^2 = f$ are trivially satisfied, and $e + f$ is not idempotent, R turns out to be a non-trivial E_3-ring of type v).

Example 4. Let G be the additive abelian group generated by three elements e, f, g with defining relations $2e = 2f = 2g = 0$, and let S_1, S_2 and S_3 be the multiplicative semigroups respectively defined by the following multiplication tables:

S_1	e	f	g
e	e	f	g
f	g	f	g
g	g	f	g

S_2	e	f	g
e	e	f	g
f	e	f	g
g	e	f	g

S_3	e	f	g
e	e	g	g
f	0	f	0
g	0	g	0

Since $G = \{he + kf + jg\}$ with $h, k, j \in \{0, 1\}$, if we extend these products to all elements of G, making use of the distributive laws, we construct three rings R_1, R_2 and R_3 which are non-trivial E_3-rings of the types iii), vi) and iv) respectively.

In [1] we proved that R is a non-trivial regular E_2-ring if and only if $|R| > 4$ and R is the direct sum of two division rings. It is natural to ask whether something analogous holds for E_3-rings. The answer is negative. In fact, we may establish the following.

Theorem 2. *A non-trivial E_3-ring cannot be regular.*

Proof. Let R be a non-trivial regular E_3-ring. If R satisfies the hypotheses either of Lemma 1 or of Lemma 7, $E \setminus 0$ turns out to be a right-zero semigroup, therefore the element $e - f$ is not regular: in fact, $(e - f)x(e - f) = (e - f)$, with $x(e - f)$ idempotent, implies $e - f = 0$. In the hypotheses of Lemma 2, we have $ef = f$ and $2e = 2f = 0$; moreover, every idempotent has the form $he + kf + jfe$ $(h, k, j \varepsilon \mathbb{Z})$. Since $(f + fe)(he + kf + jfe) = 0$, the element $f + fe$ is not regular. In the hypotheses of Lemma 3, we have $fe = 0 \neq ef$, and $R = \{0, e, f, ef, e + f, e + ef, f + ef, e + f + ef\}$. Therefore, it is clear that $efxef = 0$ for every $x \in R$, so that the element ef is not regular. Finally, in the hypotheses of Lemma 4, if R is of the type described in Lemma 2, the statement is true by the above. Otherwise, let us show that the element $ef - efe$ is not regular. In fact, it is obvious that, if $(ef - efe)x(ef - efe) = ef - efe$, the idempotent $x(ef - efe)$ cannot coincide with $0, e, (ef)^2$. Since $E = \{0, e, f, (ef)^2, (fe)^2\}$, we may have either $x(ef - efe) = f$, implying $fe = 0$ or $x(ef - efe) = (fe)^2$ implying $(fe)^2 = 0$. Since $(efe)^2 = e$, in both cases we get $e = 0$, a contradiction. Thus, in each of the cases examined R contains some non-regular element. Any other case is easily reduced to one of these.

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