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ON THE EXISTENCE OF CYCLES OF EVERY EVEN LENGTH ON GENERALIZED FIBONACCI CUBES

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To the memory of Umberto Gasapina

A new topology for the interconnection of computing nodes in multiprocessors systems is the generalized Fibonacci cube.

It can be embedded as a subgraph in the Boolean cube and it is also a supergraph of other structures. We prove that every edge of such a graph, but few initial cases, belongs to cycles of every even length.

1. Introduction.

A new interconnection topology for parallel processors is the generalized Fibonacci cube, which can be embedded, as graph, in the boolean hypercube.

The k-th order, $k \geq 2$, Fibonacci cube of dimension n, denoted by Γ_n^k, is the graph obtained by removing every vertex in the hypercube Q_{n-k} that has k or more consecutive 1's in its binary address.
Hence Γ_n^k is a subgraph of Q_{n-k}; so it is a bipartite graph and all its cycles have

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even length.
Recall that the k-th order Fibonacci numbers, denoted by $F_n^{[k]}$, are defined by

$$(1) \qquad F_n^{[k]} = F_{n-1}^{[k]} + F_{n-2}^{[k]} + \cdots + F_{n-k}^{[k]}, \quad \text{for } n \geq k,$$

and $F_i^{[k]} = 0$ for $0 \leq i \leq k-2$ and $F_{k-1}^{[k]} = 1$.
Thus each number in the sequence is the sum of the previous k numbers. In general, if we consider k consecutive numbers either there is only one odd integer (the first or the last) or there are two odd numbers and they are consecutive.
The usual Fibonacci numbers are obtained when $k = 2$.
In the generalized Fibonacci number system [1], every nonnegative integer has a unique representation as a sum of distinct k-th order Fibonacci numbers such that no k consecutive Fibonacci numbers are used.
Now we introduce the generalized Fibonacci codes which enable us to give the exact definition of generalized Fibonacci cubes.
Assume that $n \geq k \geq 2$. Let i be an integer where $0 \leq i < F_n^{[k]}$.
The k-th order Fibonacci code $(k-FC)$ of i, denoted by $FC^{[k]}(i)$, is a sequence of $n-k$ 0, 1 numbers $(b_{n-k-1}, \ldots, b_1, b_0)$, where

1. $$\sum_{j=0}^{n-k-1} b_j \cdot F_{j+k}^{[k]} = i$$

2. $$b_j \cdot b_{j+1} \cdots b_{j+k-1} = 0, \text{ for } 0 \leq j \leq n-2k.$$

For example, for $k = 3$ and $n = 7$, the number 5 is represented by the sequence $(0, 1, 0, 1)$.
The k-FC of an integer i can be obtained by the following way: find the greatest $F_j^{[k]}$ that is less than or equal to i, assign 1 to bit $j-k$, then proceed recursively on $i - F_j^{[k]}$; finally, all unassigned bits are set to O's.
The Hamming distance between two binary strings a and b, denoted by $H(a, b)$, is the number of bits where a and b differ.
Now we are able to present the definition of generalized Fibonacci cube, given in [3].

Definition. *The k-th order Fibonacci cube of dimension n, denoted by Γ_n^k is a graph (V_n^k, E_n^k), where $V_n^k = \{0, 1, \ldots, F_n^{[k]} - 1\}$ and $E_n^k = \{(i, j) \mid H(FC^{[k]}(i), FC^{[k]}(j)) = 1, 0 \leq i, j < F_n^{[k]}\}$.*
Define $\Gamma_i^k = (\emptyset, \emptyset)$ for $0 \leq i \leq k-2$, and $\Gamma_{k-1}^k = (\{0\}, \emptyset)$.

For convenience we view each k-FC as a string. Let α and β denote two strings: the concatenation of α and β is denoted by $\alpha.\beta$ or simply $\alpha\beta$. Then α^i represents the consecutive concatenation of i α's, where $i \geq 1$. For a set of strings S, define $\alpha.S = \{\alpha.\beta \mid \beta \in S\}$.
In [3] it was proved that Γ_n^k has a decomposition into $\Gamma_{n-1}^k, \dots, \Gamma_{n-k}^k$, represented by the relation

$$\Gamma_n^k = \Gamma_{n-1}^k \hat{+} \Gamma_{n-2}^k \hat{+} \cdots \hat{+} \Gamma_{n-k}^k . \tag{2}$$

In this decomposition every vertex of Γ_{n-1}^k is a string of length $n-k$ whose first bit is 0, while every vertex of Γ_{n-i}^k is a string whose first i bits are $1, \cdots, 1, 0$, with $i-1$ 1s.
Denote by G_{n-i} the subgraph of Γ_n^k isomorphic to Γ_{n-i}^k in the decomposition (2) and by H_{n-i} the subgraph induced by $G_{n-i}, \dots, G_{n-k}$, $1 \leq i \leq k$.
It is easy to see that every vertex of H_{n-i} is adjacent to only one vertex of G_{n-i+1}.
The contrary does not hold; in fact, the vertices of the subgraph of G_{n-i+1} isomorphic to $\Gamma_{n-i+1-k}^k$ are not adjacent to vertices of H_{n-i}. If v is a vertex of H_{n-i} and v' is the vertex of G_{n-i+1} adjacent to v, we call v' *correspondent* to v in G_{n-i+1}, while v is the *correspondent* of v' in H_{n-i}.
On the other side v' belongs to H_{n-i+1}; then it is adjacent to only one vertex v'' of G_{n-i+2}.
This vertex v'' is said the correspondent of v' in G_{n-i+2} and v' is the correspondent of v'' in H_{n-i+1}.
If $v\omega$ is an edge of H_{n-i} and v', ω' are the correspondent ones of v, ω in G_{n-i+1}, then v' is adjacent to ω' and $v'\omega'$ is called the edge *correspondent* of $v\omega$ in G_{n-i+1}.
A cycle of Γ_n^k is *maximal* if it has length $F_n^{[k]}$ or $F_n^{[k]} - 1$, according to $F_n^{[k]}$ is even or odd respectively.
We say that a graph G is *even-edge-pancyclic* (denoted e.e.-pancyclic) when each edge of G belongs to cycles of every even length.
As Γ_n^k is bipartite, all its cycles are even. So this graph is e.e.-paracyclic when every edge belongs to cycles of every possible length.
The main result of this paper (Theorems 1 and 2) is that Γ_n^k is even-edge pancyclic for $n \geq 7$ when $k = 2$ and $n \geq k+2$, when $k \geq 3$.

2. The Fibonacci cubes.

Let $k = 2$; in this case Γ_n^k is denoted Γ_n and is called the Fibonacci cube of dimension n.

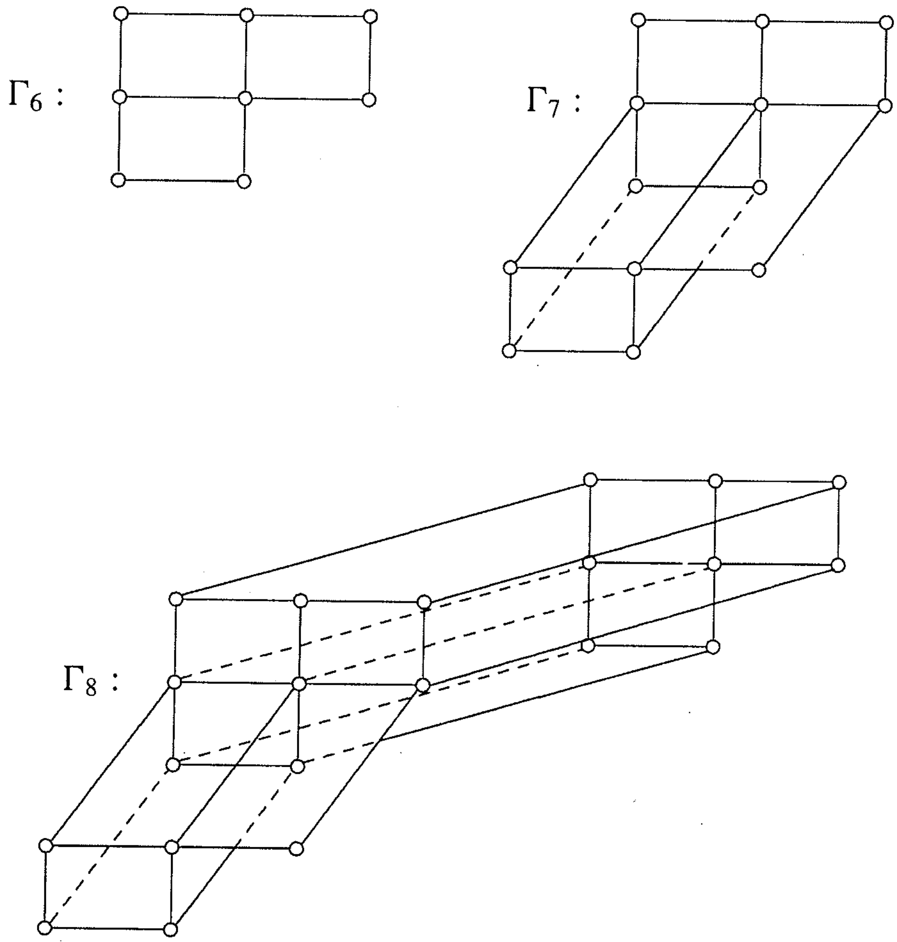

Figure 1

It is easy to see that Γ_6 is not e.e.-pancyclic, while Γ_7 and Γ_8 satisfy the property (see in Figure 1 a drawing of these graphs). The decomposition (2) for the Fibonacci cube is $\Gamma_n = G_{n-1}\hat{+}G_{n-2}$.

Lemma 2.1. *Let $n \geq 8$ and C a maximal cycle of G_{n-1}. Then C contains at least 3 edges having their correspondent ones in G_{n-2}.*

Proof. Denote by G'_{n-2}, G'_{n-3} the subgraphs of G_{n-1} with respect the decomposition (2) and v_1, v_2, v_3 three consecutive vertices of C, where $v_2 \in G'_{n-2}$.
If $v_1 \notin G'_{n-2}$, then it belongs to G'_{n-3}. It implies that $v_3 \in G'_{n-2}$, because a vertex of G'_{n-2} is adjacent to at most one vertex of G'_{n-3}. Thus v_2v_3 is an edge of G'_{n-2} and then has its correspondent one in G_{n-2}.
By the condition on n, $|G'_{n-2}| \geq 5$; then C contains at least other four vertices of G'_{n-2} and hence other two similar edges. □

Lemma 2.2. *Let $n \geq 9$ and F_n odd. If G_{n-1}, G_{n-2} are edge-pancyclic, then also Γ_n is edge-pancyclic.*

Proof. If F_n is odd, then exactly one of the two integers F_{n-1} and F_{n-2} is odd; without loss of generality assume that F_{n-1} is odd.
Let e be an edge of G_{n-2}. By the assumption, it belongs to cycles of every length in G_{n-2}, in particular to an hamiltonian cycle C. Let xy be an edge of C distinct from e and $x'y'$ its correspondent in G_{n-1}. Also $x'y'$ belongs to cycles of every length in G_{n-1}. Let D_h a cycle of length h which contains $x'y'$ with $4 \leq h \leq F_{n-1} - 1$; then e belongs to the cycle $xCyy'x'x$ of length $F_{n-2} + 2$, and to the cycle $xCyy'D_hx'x$ of length $F_{n-2} + h$.
Thus e belongs to cycles of every length.
Now, let e be an edge of G_{n-1}. By the assumption, it belongs to cycles of every length in G_{n-1}: if C is a maximal cycle containing e, then, by Lemma 2.1, it contains at least one edge $xy \neq e$ having correspondent $x'y'$ in G_{n-2}.
Using the above procedure, the result follows.
Finally, assume that $e = xx'$, where $x \in G_{n-2}$ and $x' \in G'_{n-2}$. Then x' is correspondent of x in G_{n-1}; moreover if y adj x in G_{n-2} and y' is correspondent of y in G_{n-1}, then x' adj y'.
Hence e belongs to the 4-cycle $xyy'x'x$. As xy and $x'y'$ belong to e.e.-pancyclic subgraphs, the property follows. □

Denote by G'_{n-2} and G'_{n-3} the subgraphs of G_{n-1} with respect the decomposition of type (2); moreover let L be the subgraph of Γ_n induced by G_{n-2} and G'_{n-2}.

Lemma 2.3. *Assume $n \geq 9$, F_n even and G_{n-2} e.e.-pancyclic, of odd order. Then L is e.e.-pancyclic and for every edge e of L there exists an hamiltonian cycle of L which contains e and at least one edge distinct from e having the correspondent one in G'_{n-3}.*

Proof. If F_n is even, then both of F_{n-1}, F_{n-2} are odd. Using the same procedure of the Lemma 2.2, we obtain that every edge e of L belongs to cycles of even length h, where $4 \leq h \leq F_n - 2$.
Now we prove that e belongs also to an hamiltonian cycle. Assume $F_{n-2} = 2t + 1$, $t > 0$. Let $e \in G_{n-2}$ and C a maximal cycle which it belongs to.
Denote by $v_1, \dots, v_{2t}$ the vertices of C and z the remaining vertex which we can assume adjacent to v_{2t} and to at least another vertex v_h, where $1 < h < 2t$.
Let $C' = \{v'_1, \dots, v'_{2t}\}$ and z' be the correspondent ones in G'_{n-2}. Then e belongs to the hamiltonian cycle:

$$v_1 \cdots v_{2t} z z' v'_{2t} \cdots v'_1 \tag{3}$$

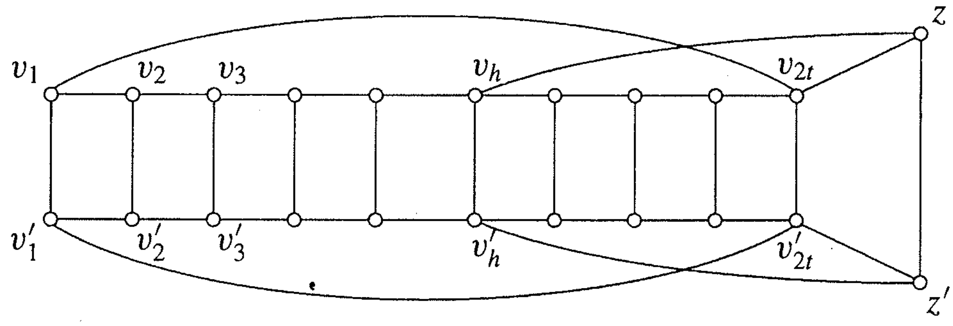

Figure 2

which contains all the edges of C and C', but $v_1 v_{2t}$ and $v_1' v_{2t}'$ (see Figure 2).
A similar result holds when $e \in G_{n-2}'$.
When $e = (v_1, v_{2t})$ we can consider the same cycle C where the vertices are in the different order

(4) $$v_{h-1}, v_{h-2}, \cdots, v_1, v_{2t}, \cdots, v_h$$

and z is adjacent to v_h.
Consider, now, the case in which e connects a vertex of G_{n-2} with one of G_{n-2}'.
Let $C = (v_1, v_2, \ldots, v_{2t})$ denote again a maximal cycle of G_{n-2}, z the remaining vertex and C', z' the correspondent ones in G_{n-2}'. If $e = (z, z')$, then it belongs to the cycle (3).
Assume that $e = (v_i v_i')$, $1 \leq i < 2t$.
It implies that e belongs to the following hamiltonian cycles

(5) $$v_i v_i' v_{i+1}' v_{i+1} \cdots v_{2t} z z' v_{2t}' v_1' v_1 \cdots v_i$$

and

(6) $$v_i v_{i+1} v_{i+1}' v_{i+2}' \cdots v_{2t}' z' z v_{2t} v_1 v_1' \cdots v_i' v_i$$

in the case of i even, and

(7) $$v_i v_i' \cdots v_{2t}' z' z v_{2t} v_1 v_1' v_2' v_2 \cdots v_i$$

and

(8) $$v_i v_{i+1} v_{i+1}' \cdots v_{2t} z z' v_{2t}' v_1' v_1 \cdots v_i' v_i$$

in the case of i odd. Note that the cycles (5), $\cdots$, (8) alternate edges of C, C' and (5), (6) as well (7), (8) contain all the edges C, C'. By Lemma 2.1, C' contains at least three edges having their correspondent ones in G'_{n-3}. Hence, as the cycles (5) and (6) or (7) and (8) contain all the edges of C', then at least one of (5), (6) and of (7), (8) contains one edge, distinct from e, having the correspondent one in G'_{n-3}.
When $e = \{v_{2t}, v'_{2t}\}$, we repeat the same procedure by considering the cycle (4).

□

Theorem 1. *For $n \geq 7$, every edge of the Fibonacci cube Γ_n belongs to cycles of every even length.*

Proof. We proceed by induction, the cases $n = 7, 8$ being satisfied. So assume that $n \geq 9$ and the result holds for every Γ_h, when $h < n$. Let F_n be odd. As by assumption G_{n-1} and G_{n-2} are e.e.-pancyclic, then from Lemma 2.2 the result follows.
Let F_n be even. By Lemma 2.3, L is e.e.-pancyclic, and for every edge $e \in L$ there is an hamiltonian cycle C of L which contains e and an edge $xy \notin e$ having correspondent, say $x'y'$, in G'_{n-3}.
As this subgraph is e.e.-pancyclic by assumption, then $x'y'$ belongs to cycles D_h of every even length h, where $4 \leq h \leq F_{n-3}$. Then e belongs to the cycles $xCyy'x'x$ and $xCyy'D_hx'x$.
If $e \in G'_{n-3}$, it is sufficient to note that every edge of G'_{n-3} has correspondent in G'_{n-2}, that is in L.
This completes the proof by induction. □

3. Generalized Fibonacci cubes.

In this section we assume $k \geq 3$. Clearly Γ_n^k, $k + 1 \leq n \leq 2k - 1$, is isomorphic to the boolean cube Q_{n-k}.
Moreover a decomposition of a graph G into two subgraphs A and B is denoted $G = A\hat{+}B$, as in (2), when A and B are vertex-disjoint and every vertex of B is adjacent to exactly one vertex of A, with the condition that never distinct vertices of B are adjacent to a same vertex of A.

Lemma 3.1. *For $0 \leq i \leq 2$,*

$$\Gamma_{2k+i}^k = Q_{k+i-1} \hat{+} \Gamma_{2(k-1)+i}^{k-1}.$$

Proof. Recall that Γ^k_{2k} is obtained by assigning 0 or 1 before all the strings of Γ^k_{2k-1} and deleting those with k consecutive 1s, that is those in Γ^k_{2k-1} having $k-1$ initial consecutive 1s.
We can represent this situation by writing

$$\Gamma^k_{2k} = 0Q_{k-1} \hat{+} 1Q_{k-1}\,.$$

As Q_{k-1} contains only 1 string with $k-1$ 1s, then $1Q_{k-1}$ is isomorphic to Q_{k-1} minus a vertex. That is

$$1Q_{k-1} = 0(Q_{k-2}) \hat{+} 1Q_{k-2} = \Gamma^{k-1}_{2(k-1)}$$

and $\Gamma^k_{2k} = Q_{k-1} \hat{+} \Gamma^{k-1}_{2(k-1)}$.
Moreover

$$\begin{aligned}\Gamma^k_{2k+1} &= 0(Q_{k-1} \hat{+} \Gamma^{k-1}_{2(k-1)}) \hat{+} 1(Q_{k-1} \hat{+} \Gamma^{k-1}_{2(k-1)}) = \\ &= Q_k \hat{+} (0\Gamma^{k-1}_{2(k-1)} \hat{+} 1\Gamma^{k-1}_{2(k-1)} = \\ &= Q_k \hat{+} \Gamma^{k-1}_{2(k-1)+1}.\end{aligned}$$

In a similar way we have Γ^k_{2k+2}. □

Lemma 3.2. *Γ^k_n, $k+2 \le n \le 2k+2$, is e.e.-pancyclic.*

Proof. For $k+2 \le n \le 2k-1$ Γ_n is isomorphic to the boolean cube Q_{n-k} which is e.e.-pancyclic.
Now consider $n = 2k+i$, where $0 \le i \le 2$. The cases for $k = 3$, in Figure 3 are clearly e.e.-pancyclic.
Consider the case $i = 0$; we proceed by induction on k.
From the condition that an hamiltonian cycle of Q_{k-1} has at least two edges having correspondent in $\Gamma^{k-1}_{2(k-1)}$, then with a proof similar to the one of Lemma 2.2, the result follows.
The same consideration holds for $i = 1, 2$. □

Lemma 3.3. *Let $n \ge 2k+3$; every maximal cycle of G_{n-i} has at least three edges having correspondent one in G_{n-i-1}, $1 \le i < k$.*

Proof. Consider the decomposition $G_{n-i} = G'_{n-i-1} \hat{+} H'_{n-i-2}$; let C be a maximal cycle of G_{n-i} and $v_1v_2v_3$ three consecutive vertices of C, where $v_2 \in G'_{n-i-1}$.
If $v_1 \notin G'_{n-i-1}$, then v_1 is the correspondent of v_2 in H'_{n-i-2}.
It implies that v_3 belongs to G'_{n-i-1} because every vertex of G'_{n-i-1} is adjacent to at most one vertex of H'_{n-i-2}. Thus $v_2v_3 \in G'_{n-i-1}$ and then has its correspondent in G_{n-i-1}. As $|G'_{n-i-1}| \ge 5$, the result follows. □

Lemma 3.4. *Assume that every subgraph G_{n-i}, where $1 \le i \le k$, is e.e.-pancyclic and has order even, but at most one. Then also H_{n-i} is e.e.-pancyclic.*

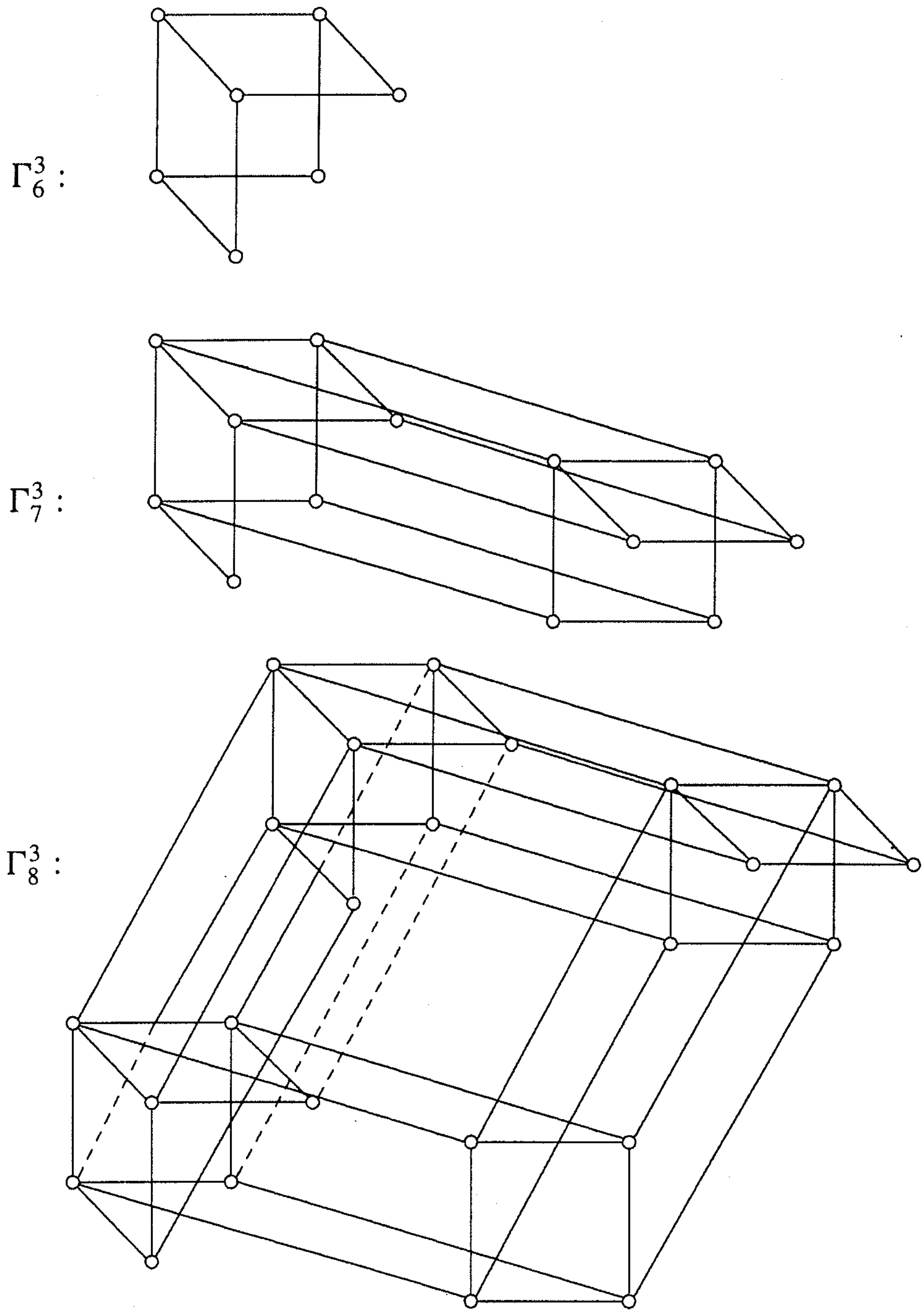

Figure 3

Proof. The proof is by induction on the value of i, the case $i = k$ being satisfied by assumption.
Because $H_{n-i} = G_{n-i} \hat{+} H_{n-i-1}$, then by Lemma 3.3 with a procedure similar to the one of Lemma 2.2 we obtain the result. □

Consider the decomposition $G_{n-1} = G'_{n-2} \hat{+} G'_{n-3} \hat{+} \cdots \hat{+} G'_{n-k-1}$; moreover denote by $\mathbf{L}$ the subgraph induced by G_{n-2} and G'_{n-2} and by H'_{n-3} the subgraph induced by $G'_{n-3}, \ldots, G'_{n-k-1}$.

Lemma 3.5. *Assume that G_{n-2} has odd order and is e.e.-pancyclic. Then also $\mathbf{L}$ is e.e.-pancyclic and every edge e of $\mathbf{L}$ belongs to an hamiltonian cycle which contains at least other two edges, distinct from e, having correspondent one, one in H_{n-3} and the other in H'_{n-3}.*

Proof. The proof is perfectly similar to the one of Lemma 2.3. □

Theorem 2. *For $k \geq 3$ and $n \geq k+2$, every edge of Γ_n^k belongs to cycles of every even length.*

Proof. We proceed by induction, the cases $k+2 \leq n \leq 2k+2$ being satisfied by Lemma 3.2. So assume that $n \geq 2k+3$.
Let F_n^k be odd; then only one of the integers in the sum (1) is odd, say F_{n-1}^k. By Lemma 3.4, the result holds.
Let F_n^k be even. This implies that exactly two consecutive integers in the sum (1) are odd; without loss of generality they can be F_{n-1}^k, F_{n-2}^k. By Lemma 3.4 H_{n-3} is e.e.-pancyclic; also H'_{n-3} is e.e.-pancyclic, because $n-k-1 \geq k+2$. Moreover by Lemma 3.5 $\mathbf{L}$ is e.e.-pancyclic. The situation for Γ_n^k can be represented by the drawing in Figure 4

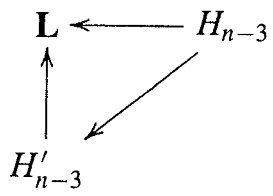

Figure 4

where the arrow from one to another subgraph means that every vertex of the first is adjacent to exactly one vertex of the second one.
Let e be an edge of H'_{n-3}. By Lemma 3.4, it belongs to cycles of every length in such a subgraph. Let C be an hamiltonian cycle which contains e and xy an edge of C distinct from e. Denote by $x'y'$ the edge correspondent of xy in $\mathbf{L}$. By Lemma 3.5 $x'y'$ belongs to cycles of every length. Let D be an hamiltonian cycle which contains $x'y'$ and at least another edge f having correspondent f' in H_{n-3}. By Lemma 3.4 f' belongs to cycles of every length; then the result follows.
It is easy to see that a similar situation holds also if e belongs either to H_{n-3} or to $\mathbf{L}$ or connects vertices of two distinct subgraphs. This completes the proof. □

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