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AN EXTENDED BARBANIS HAMILTONIAN

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A simple extension of the Barbanis hamiltonian is considered in the form

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 - \frac{1}{2}x^2\left[(h+k)y - hky^2\right].$$

The associated microcanonical distribution function and microcanonical partition function are given in the form of the Appell functions of the first and fourth kinds and the confluent double hypergeometric function.

1. Introduction.

Over the past two decades, considerable interest has been shown in non-linear systems with two degrees of freedom. One of the cases which has been specifically studied is that defined by the Barbanis hamiltonian

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + kx^2y. \tag{1.1}$$

See Barbanis (1966).

The microcanonical distribution function of this system was worked out as a Gauss hypergeometric function by Caboz and Poletti (1983). This may presented in the form

$$Z(E) = d/dE \int_D dp_x dp_y dx dy, \tag{1.2}$$

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where D is the region interior to

$$H - E = 0 \tag{1.3}$$

or, in this case

$$p_x^2 + p_y^2 + x^2 + y^2 + 2kx^2y - 2E = 0, \tag{1.4}$$

where E is the energy of the system.

Interest is also shown in the microcanonical partition function of the system, $Z_c(s)$, which is given by

$$Z_c(s) = \int_c^\infty \exp(-sE)Z(E)\,dE. \tag{1.5}$$

In this study, an extension of the Barbanis hamiltonian is considered, namely

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 - \frac{1}{2}x^2\left[(h+k)y - hky^2\right] \tag{1.6}$$

in which the coefficient of x^2 is now quadratic in y.

2. The microcanonical distribution function of the extended Barbanis hamiltonian.

We consider the hypervolume

$$V(E) = \iiiint_R dp_x dp_y dx dy \tag{2.1}$$

where R is interior to

$$p_x^2 + p_y^2 + x^2 + y^2 - x^2[(h+k)y - hky^2] - 2E = 0. \tag{2.2}$$

A first integral of (2.1) is

$$V(E) = \pi \iint_S \left\{2E - x^2 - y^2 + x^2[(h+k)y - hky^2]\right\} dxdy, \tag{2.3}$$

over the region S, interior to

$$x^2 + y^2 - x^2[(h+k)y - hky^2] - 2E = 0. \tag{2.4}$$

Integrate with respect to x and obtain

$$V(E) = \pi \int_{-\sqrt{2E}}^{\sqrt{2E}} \left\{(2E - y^2)x - [1 - (h+k)y + hky^2]x^3/3\right\} dy. \tag{2.5}$$

On the boundary of S,

$$x^2 = (2E - y^2)[1 - (h+k)y + hky^2]^{-1}\, dy \tag{2.6}$$

and the microcanonical distribution is the hyperelliptic integral

$$Z(E) = 4\pi \int_{-\sqrt{2E}}^{\sqrt{2E}} (2E - y^2)^{\frac{1}{2}}[1 - (h+k)y + hky^2]^{-\frac{1}{2}} dy. \tag{2.7}$$

In the present context, it is convenient to evaluate (2.7) as a double hypergeometric function in the form of an Appell function of the first kind. See Exton (1976) page 23 for example.
Noting that

$$1 - (h+k)y + hky^2 = (1 - hy)(1 - ky), \tag{2.8}$$

we may write (2.7) as

$$Z(E) = 32\pi E(1 + 2E/h)^{-\frac{1}{2}}(1 + 2E/k)^{-\frac{1}{2}} \cdot \tag{2.9}$$

$$\cdot \int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}(1 - z_1 t)^{-\frac{1}{2}}(1 - z_2 t)^{-\frac{1}{2}} dt$$

by an elementary change of variable and when

$$z_1 = 2\sqrt{2E}/(\sqrt{2E} + h) \text{ and } z_2 = 2\sqrt{2E}/(\sqrt{2E} + k). \tag{2.10}$$

The formula

$$F_1(a, b, b^1; c; x, y) \tag{2.11}$$

$$= \frac{P(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1tx)^{-b}(1-ty)^{-b'}dt$$

is now employed. The Appell function of the first kind is given by

$$(2.12) \qquad F_1(a,b,b';c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a,m+n)(b,m)(b',n)}{(c,m+n)m!n!} x^m y^n,$$

convergent for $|x| < 1$ and $|y| < 1$. The Pochammer symbol (a, m) is defined as

$$(2.13) \quad (a,m) = a(a+1)\dots(a+m-1) = \Gamma(a+m)/\Gamma(a); \ (a,0) = 1.$$

We may then write

$$(2.14) \qquad Z(E) = 4\pi^2 E(1+\sqrt{2E}/h)^{-\frac{1}{2}} (1+\sqrt{2E}/k)^{-\frac{1}{2}} \cdot$$

$$\cdot F_1\left(3/2, \frac{1}{2}, \frac{1}{2}; 3; z, z_2\right).$$

This form of the result lends itself quite conveniently for numerical evaluation because the analytic continuation of the function F_1 has been studied in some detail by Olsson (1964).
On the other hand, (2.14) does not readily yield a Laplace transform with respect to E which is required to deduce the microcanonical partition function.

In the case of one variable, where the Barbanis hamiltonian itself is concerned, a well-known quadratic transformation of the Gauss function may be used.

This is to be found in Erdélyi (1953) Vol. I page 111, and is

$$(2.15) \ {}_2F_1(a,b;2b;x) = \left(1-\frac{1}{2}x\right)^{-a} {}_2F_1\left(\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; b+\frac{1}{2}; \frac{x^2}{(2-x)^2}\right).$$

See also Caboz (1983). Transformations of this type for double hypergeometric functions are much less well-known, and a suitable generalization is obtained below.

3. A quadratic transformation for double hypergeometric functions.

Srivastava and Karlsson (1985) page 305 and 320 respectively give the two transformations of Appell functions as follows:

$$F_1(a, b, b; 2a; x, y) = (x/y)^b F_2(2b, a, b; 2a, 2b; x, 1 - x/y) \tag{3.1}$$

(Appell et Kampé de Fériet (1926) page 35) and

$$F_2(2a, a, b; 2a, 2b; 2X, 2Y) \tag{3.2}$$
$$= (1-X-Y)^{-2b} F_4(b, b+\frac{1}{2}; a+\frac{1}{2}, b+\frac{1}{2}; X^2(1-X-Y)^{-2}, Y^2/(1-X-Y)^{-2}),$$

(Bailey (1953) page 239), where

$$F_4(a, b; c, c^1; x, y) = \sum_{m,n=0}^{\infty} \frac{(a, m+n)(b, m+n)}{(c, m)(c^1, n)m!n!} x^m y^n, \tag{3.3}$$

convergent for $|\sqrt{x}| + |\sqrt{y}| < 1$.
See Appell et Kampé de Fériet (1926), for example.
If (3.1) and (3.2) are combined, we have after some algebra,

$$F_1(a, b, b; 2a; x, y) = [4xy/(x + y - xy)^2] \cdot \tag{3.4}$$
$$F_4(b, b + \frac{1}{2}; a + \frac{1}{2}, b + \frac{1}{2}; x^2y^2/(x + y - xy)^2, (x - y)^2/(x + y - xy)^2),$$

provided that both sides of this expression are convergent.
This last formula is the quadratic transformation sought.

4. The microcanonical partition function.

The right-hand member of (3.4) is of the same form as the Appell function on the right of (2.14), so that

$$Z(E) = 8\pi^2 hk/(h+k)E\, F_4\Big(\frac{1}{2}, 1; 2, 1; 8E/(h+k)^2, (h-k)^2/(h+k)^2\Big), \tag{4.1}$$

provided that, for convergence,

$$\left|2\sqrt{E}/(h + k)\right| + |(h - k)/(h + k)| < 1. \tag{4.2}$$

We see that this last criterion does not hold if either h or k vanishes. The form of the microcanonical distribution function (4.1) lends itself conveniently to calculating its Laplace transform with respect to E. If either h or k should vanish, then a separate limiting process is required. this is not developed because the ordinary Barbanis hamiltonian then arises.

Consider the Laplace transform with respect to E of

$$F = EF_4\left(\frac{1}{2}, 1; 2, 1; XE, Y\right). \tag{4.3}$$

This function possesses the following double Barnes integral representation:

$$F = \frac{\pi^{-\frac{1}{2}}}{(2\pi i)^2} \int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \frac{\Gamma(\frac{1}{2}+u+v)\Gamma(1+u+v)\Gamma(-u)\Gamma(-v)}{\Gamma(2+u)\Gamma(1+v)} \cdot \tag{4.4}$$

$$\cdot (-X)^u(-Y)^v E^{u+1} du dv,$$

see Appell et Kampé de Fériet (1926) page 40.
Hence

$$\mathscr{L}(F) = \frac{\pi^{-\frac{1}{2}}}{(2\pi i)^2} \int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \frac{\Gamma(\frac{1}{2}+u+v)\Gamma(1+u+v)\Gamma(-u)\Gamma(-v)}{\Gamma(2+u)\Gamma(1+v)} \cdot \tag{4.5}$$

$$\cdot (-X)^u(-Y)^v \left[\int_0^\infty \exp(-sE)E^{u+1}dE\right] du dv,$$

since all the integrals concerned are absolutely convergent.
The right-hand member of (4.5) may thus be written as

$$\frac{\pi^{-\frac{1}{2}}}{(2\pi i)^2 s^2} \int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \frac{\Gamma(\frac{1}{2}+u+v)\Gamma(1+u+v)\Gamma(-u)\Gamma(-v)}{\Gamma(1+v)} \cdot \tag{4.6}$$

$$\cdot \left(\frac{-X}{s}\right)^u (-Y)^v du dv = \frac{\pi^{-\frac{1}{2}}}{2\pi i s^2} \int_{-i\infty}^{i\infty} \Gamma(-u)\Gamma\left(\frac{1}{2}+u\right)\Gamma(1+u) \cdot$$

$$\cdot (-X/s)^u {}_2F_1\left(\frac{1}{2}+u, 1+u; 1; Y\right) du =$$

$$= \pi^{-\frac{1}{2}} s^{-2} \sum_{n=0}^{\infty} \frac{Y^n}{n!n!} G_{1,2}^{2,1}\left(-\frac{s}{X}\middle| \begin{matrix} 1 \\ \frac{1}{2}+n, \quad 1+n \end{matrix}\right).$$

The properties of the Mejer G-function are discussed in Chapter 5 of Erdélyi (1953) Vol. I, where it is pointed out that the inner G-function of (4.6) can be written as the sum of the two confluent hypergeometric functions

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+n\right)(s/X)^{\frac{1}{2}+n} \quad {}_1F_1\left(\frac{1}{2}+n; \frac{1}{2}; -s/X\right) \tag{4.7}$$

$$+\Gamma\left(-\frac{1}{2}\right)\Gamma(1+n)(s/X)^{1+n} \quad {}_1F_1(1+n; 3/2; -s/x).$$

If this result is inserted into (4.6), and recalling that the microcanonical partition function is given by

$$Z_c(s) = \mathscr{L}[Z(E)], \tag{4.8}$$

it is clear that, after some reduction

$$Z_c(s) = 2\sqrt{2}\pi^{\frac{5}{2}} s^{-\frac{3}{2}} h^{\frac{1}{2}} k^{\frac{1}{2}} \Psi_2\left(\frac{1}{2}, \frac{1}{2}, 1; -s(h+k)^2/8, s(h-k)^2/8\right) \tag{4.9}$$

$$-2\pi^2 s^{-1} h^{\frac{1}{2}} k^{\frac{1}{2}} (h+k)\Psi_2(1; 3/2, 1; -s(h+k)^2/8, 3(h,k)^2/8),$$

with the above restriction (4.2). The confluent double hypergeometric function (or Humbert function) is given by

$$\Psi_2(a; c, c^1; x, y) = \sum_{m,n=0}^{\infty} \frac{(a, m+n)x^m y^n}{(c, m)(c^1, n)m!n!}. \tag{4.10}$$

This function is discussed by Appell et Kampé de Fériet (1926) Chapter 8 for example. See also Humbert (1920-21).

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