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AN INTERSECTION PROBLEM IN A FINITE SET

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Fixed a finite set X, a positive integer k and n subsets $A_1, \dots, A_n$ of X such that $|A_i| > \frac{1}{2}|X|$ for $i = 1, \dots, n$, we determine the maximum positive integer $f(k)$ such that if $n \leq f(k)$ there exists a subset $B \subseteq X$, containing at most k elements, such that $B \cap A_i \neq \emptyset$ for $i = 1, \dots, n$.

1. Introduction.

In the following, if A is a finite set and $\mathscr{F}$ is a finite family of finite subsets (members of $\mathscr{F}$), we denote by $|A|$ the number of distinct elements in A and by $|\mathscr{F}|$ the number of distinct members in $\mathscr{F}$. Moreover, if x is a real number, we denote by $\lfloor x \rfloor$ the integer part of x.

In several works problems of intersection and union between subsets of a finite set have been studied.

For example, in [4] it has been proved that if $\mathscr{F}_1, \dots, \mathscr{F}_k$ are families of subsets of an n-element set X such that $A_i \cap A_j \neq \emptyset$ if $A_i, A_j \in \mathscr{F}_l$, for $1 \leq l \leq k$, then

$$\left| \bigcup_{j=1}^{k} \mathscr{F}_j \right| \leq 2^n - 2^{n-k}.$$

In [5] the following result has been proved.

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Let X be a set having at least two elements and $k \geq 3$ an integer; let $\mathscr{F}$ be a family of subsets of X no k of which are pairwise disjoint. Then, if $|X| = mk$ or $|X| = mk - 1$, we have respectively

$$|\mathscr{F}| \leq \sum_{j=m+1}^{mk} \binom{mk}{j} + \binom{mk}{m} \frac{(k-1)}{k}$$

or

$$|\mathscr{F}| \leq \sum_{j=m}^{mk-1} \binom{mk-1}{j}.$$

In this paper we shall consider the following problem: given a finite set X, a fixed positive integer k and n subsets $A_1, \ldots, A_n$ of X such that $|A_i| > \frac{1}{2}|X| \quad (i = 1, \ldots, n)$, which is the maximum positive integer $f(k)$ such that, if $n \leq f(k)$, there exists a subset $B \subseteq X$ containing at most k elements such that $B \cap A_i \neq \emptyset$ for $i = 1, \ldots, n$?

In Theorem 1 it will be proved that $f(k) \geq 2^{k+1} - 2$, while in Section 4 it will be shown that $f(k) < 2^{k+1} - 1$, using an interesting example in the finite projective spaces (this example is due to R. Dvornicich). This implies that $f(k) = 2^{k+1} - 2$.

This problem fits into the theory of intersecting families (for some fundamental results see the references [1], [2] and [3]).

2. Lemmas.

Lemma 1. *Let m be an integer ≥ 0, then*

$$\sum_{i=1}^{\infty} \left\lfloor \frac{m + 2^{i-1}}{2^i} \right\rfloor = m.$$

Proof. Let us proceed by induction on m.

If $m = 1$, we have

$$\sum_{i=1}^{\infty} \left\lfloor \frac{1 + 2^{i-1}}{2^i} \right\rfloor = \left\lfloor \frac{1+1}{2} \right\rfloor + \left\lfloor \frac{1+2}{4} \right\rfloor + \left\lfloor \frac{1+4}{8} \right\rfloor + \cdots = 1.$$

Now suppose $m > 1$ and the assertion be true for all integers less than m. Let us distinguish the cases m even and m odd.

If m is even, we have

$$\begin{aligned}\sum_{i=1}^{\infty}\left\lfloor\frac{m+2^{i-1}}{2^i}\right\rfloor &= \left\lfloor\frac{m+1}{2}\right\rfloor+\sum_{i=2}^{\infty}\left\lfloor\frac{m+2^{i-1}}{2^i}\right\rfloor\\ &= \frac{m}{2}+\left\lfloor\frac{m+2}{4}\right\rfloor+\left\lfloor\frac{m+4}{8}\right\rfloor+\cdots\\ &= \frac{m}{2}+\left\lfloor\frac{\frac{m}{2}+1}{2}\right\rfloor+\left\lfloor\frac{\frac{m}{2}+2}{4}\right\rfloor+\cdots\\ &= \frac{m}{2}+\sum_{k=1}^{\infty}\left\lfloor\frac{\frac{m}{2}+2^{k-1}}{2^k}\right\rfloor\\ &= \frac{m}{2}+\frac{m}{2}\quad\left(\text{by the inductive hypothesis on } \frac{m}{2}\right)\\ &= m.\end{aligned}$$

If m is odd, we obtain

$$\begin{aligned}\sum_{i=1}^{\infty}\left\lfloor\frac{m+2^{i-1}}{2^i}\right\rfloor &= \left\lfloor\frac{m+1}{2}\right\rfloor+\sum_{k=2}^{\infty}\left\lfloor\frac{m+2^{k-1}}{2^k}\right\rfloor\\ &= \frac{m+1}{2}+\sum_{k=1}^{\infty}\left\lfloor\frac{\frac{m}{2}+2^{k-1}}{2^k}\right\rfloor\\ &= \frac{m+1}{2}+\sum_{k=1}^{\infty}\left\lfloor\frac{\frac{m-1}{2}+2^{k-1}}{2^k}\right\rfloor\quad(m \text{ being odd})\\ &= \frac{m+1}{2}+\frac{m-1}{2}\quad\left(\text{by the inductive hypothesis on } \frac{m-1}{2}\right)\\ &= m.\end{aligned}$$

The assertion follows now by induction on m. □

Lemma 2. *Let k be a fixed positive integer. If m is any integer such that $1 \le m < 2^{k+1}$, then*

$$\sum_{i=1}^{k}\left\lfloor\frac{m+2^{i-1}}{2^i}\right\rfloor \ge m-1.$$

Proof. If $1 \le m < 2^{k+1}$, we have

$$\left\lfloor\frac{m+2^{k+1}}{2^{k+2}}\right\rfloor = \left\lfloor\frac{m+2^{k+2}}{2^{k+3}}\right\rfloor = \cdots = 0;$$

therefore, by Lemma 1, it follows that

$$m = \sum_{i=1}^{\infty} \left\lfloor \frac{m + 2^{i-1}}{2^i} \right\rfloor = \sum_{i=1}^{k+1} \left\lfloor \frac{m + 2^{i-1}}{2^i} \right\rfloor$$
$$= \sum_{i=1}^{k} \left\lfloor \frac{m + 2^{i-1}}{2^i} \right\rfloor + \left\lfloor \frac{m + 2^{k}}{2^{k+1}} \right\rfloor .$$

On the other hand, we also have

$$0 < \frac{m + 2^k}{2^{k+1}} < \frac{2^{k+1} + 2^k}{2^{k+1}} = \frac{3}{2};$$

this implies that

$$0 \leq \left\lfloor \frac{m + 2^k}{2^{k+1}} \right\rfloor \leq 1.$$

Then we obtain

$$\sum_{i=1}^{k} \left\lfloor \frac{m + 2^{i-1}}{2^i} \right\rfloor = m - \left\lfloor \frac{m + 2^k}{2^{k+1}} \right\rfloor \geq m - 1. \qquad \square$$

Lemma 3. *Let n be a fixed positive integer and r any integer ≥ 2, then*

$$\left\lfloor \frac{n - \sum_{i=1}^{r-1} \left\lfloor \frac{(n+1)+2^{i-1}}{2^i} \right\rfloor}{2} \right\rfloor + 1 = \left\lfloor \frac{(n+1) + 2^{r-1}}{2^r} \right\rfloor .$$

Proof. If we set n in the form

$$n = m2^{r-1} + h,$$

where m , h are integers and $0 \leq h < 2^{r-1}$, we have

$$\left\lfloor \frac{(n+1)+1}{2} \right\rfloor = m2^{r-2} + \left\lfloor \frac{(h+1)+1}{2} \right\rfloor$$
$$\left\lfloor \frac{(n+1)+2}{2^2} \right\rfloor = m2^{r-3} + \left\lfloor \frac{(h+1)+2}{2^2} \right\rfloor$$
$$\vdots \qquad \vdots$$
$$\left\lfloor \frac{(n+1)+2^{r-2}}{2^{r-1}} \right\rfloor = m + \left\lfloor \frac{(h+1)+2^{r-2}}{2^{r-1}} \right\rfloor$$

Hence

$$n - \sum_{i=1}^{r-1} \left\lfloor \frac{(n+1)+2^{i-1}}{2^i} \right\rfloor = m2^{r-1} + h - m\sum_{j=0}^{r-2} 2^j - \sum_{i=1}^{r-1} \left\lfloor \frac{(h+1)+2^{i-1}}{2^i} \right\rfloor$$

$$= m2^{r-1} - m(2^{r-1}-1) + h - \sum_{i=1}^{r-1} \left\lfloor \frac{(h+1)+2^{i-1}}{2^i} \right\rfloor$$

$$= m + h - \sum_{i=1}^{r-1} \left\lfloor \frac{(h+1)+2^{i-1}}{2^i} \right\rfloor$$

$$= \begin{cases} m + h - \sum_{i=1}^{\infty} \left\lfloor \frac{(h+1)+2^{i-1}}{2^i} \right\rfloor & \text{if } 0 \le h \le 2^{r-1}-2 \\ m + h - \sum_{i=1}^{\infty} \left\lfloor \frac{(h+1)+2^{i-1}}{2^i} \right\rfloor + 1 & \text{if } h = 2^{r-1}-1 \end{cases}$$

$$= \begin{cases} m + h - (h+1) & \text{if } 0 \le h \le 2^{r-1}-2 \\ & \text{(by Lemma 1)} \\ m + h - (h+1) + 1 & \text{if } h = 2^{r-1}-1 \end{cases}$$

$$= \begin{cases} m - 1 & \text{if } 0 \le h \le 2^{r-1}-2 \\ m & \text{if } h = 2^{r-1}-1. \end{cases}$$

Then, to complete our proof, we must show that

$$\left\lfloor \frac{(n+1)+2^{r-1}}{2^r} \right\rfloor = \begin{cases} \left\lfloor \frac{m-1}{2} \right\rfloor + 1 & \text{if } 0 \le h \le 2^{r-1}-2 \\ \left\lfloor \frac{m}{2} \right\rfloor + 1 & \text{if } h = 2^{r-1}-1. \end{cases}$$

Now, if $h = 2^{r-1}-1$, we have

$$\left\lfloor \frac{(n+1)+2^{r-1}}{2^r} \right\rfloor = \left\lfloor \frac{(m+2)2^{r-1}}{2^r} \right\rfloor = \left\lfloor \frac{m}{2} \right\rfloor + 1.$$

If $0 \le h \le 2^{r-1}-2$, then use

$$\left\lfloor \frac{(n+1)+2^{r-1}}{2^r} \right\rfloor = \left\lfloor \frac{m+1}{2} + \frac{h+1}{2^r} \right\rfloor.$$

Let us distinguish the cases $m \equiv 0 \pmod 2$ and $m \equiv 1 \pmod 2$. Then, if $m = 2k+1$ is odd, it implies

$$\left\lfloor \frac{m+1}{2} + \frac{h+1}{2^r} \right\rfloor = k + 1 + \left\lfloor \frac{h+1}{2^r} \right\rfloor = k + 1 = \frac{m-1}{2} + 1;$$

on the other hand, if $m = 2k$ is even, we have

$$\left\lfloor \frac{m+1}{2} + \frac{h+1}{2^r} \right\rfloor = k + \left\lfloor \frac{1}{2} + \frac{h+1}{2^r} \right\rfloor = k = \left\lfloor \frac{m-1}{2} \right\rfloor + 1.$$

This completes our proof. □

3. The main result.

Let us now prove our main result.

Theorem 1. *Let X be a finite set and $A_1, \ldots, A_n$ be distinct subsets of X such that*

$$(1) \qquad |A_i| > \frac{1}{2}|X| \qquad (i = 1, \ldots, n).$$

Let k be a fixed positive integer. Then, if $n \leq 2^{k+1} - 2$, there exists a subset $B \subseteq X$ such that $|B| \leq k$ and $B \cap A_i \neq \emptyset$ for all $i = 1, \ldots, n$.

Proof. If $\mathscr{F}$ is a family of subsets of X and x is an arbitrary element of X, let us denote by $d(\mathscr{F}, x)$ the number of distinct members $A \in \mathscr{F}$ (i.e. subsets of X) such that $x \in A$.

Let $\mathscr{F}_1 = \{A_1, \ldots, A_n\}$. By (1) we have

$$(2) \quad \sum_{x \in X} d(\mathscr{F}_1, x) = \sum_{x \in X} I_{A_1}(x) + \cdots + I_{A_n}(x) = |A_1| + \cdots + |A_n| > \frac{n}{2}|X|,$$

where

$$I_{A_i}(x) = \begin{cases} 1 & \text{if } x \in A_i \\ 0 & \text{otherwise} \end{cases} \qquad (i = 1, \ldots, n).$$

By (2) it follows that there exists an element $x_1 \in X$ such that $d(\mathscr{F}_1, x_1) > \frac{n}{2}$, i.e. $d(\mathscr{F}_1, x_1) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$. This implies that $\mathscr{F}_1$ contains at least

$$\alpha_1 = \left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lfloor \frac{n+2}{2} \right\rfloor = \left\lfloor \frac{(n+1)+1}{2} \right\rfloor$$

distinct members, which we denote by $A_{1,1}, \ldots, A_{1,\alpha_1}$, such that $x_1 \in A_{1,1} \cap \ldots \cap A_{1,\alpha_1}$. Now let us consider the subfamily

$$\mathscr{F}_2 = \mathscr{F}_1 \setminus \{A_{1,1}, \ldots, A_{1,\alpha_1}\} \subset \mathscr{F}_1.$$

$\mathscr{F}_2$ contains exactly

$$n_2 = n - \alpha_1 = n - \left\lfloor \frac{(n+1)+1}{2} \right\rfloor$$

distinct members.

By (1), arguing as before, it follows that $\mathscr{F}_2$ contains at least

$$\alpha_2 = \left\lfloor \frac{n_2}{2} \right\rfloor + 1 = \left\lfloor \frac{n - \left\lfloor \frac{(n+1)+1}{2} \right\rfloor}{2} \right\rfloor + 1 = \left\lfloor \frac{(n+1)+2}{4} \right\rfloor$$

distinct members (different from $A_{1,1}, \ldots, A_{1,\alpha_1}$), say $A_{2,1}, \ldots, A_{2,\alpha_2}$, such that $x_2 \in A_{2,1} \cap \ldots \cap A_{2,\alpha_2}$.

By inductive hypothesis, at the ith step we can determine a subfamily $\mathscr{F}_i$ such that

$$|\mathscr{F}_i| = n_i = n - \alpha_1 - \ldots - \alpha_{i-1},$$

where $\alpha_k = \left\lfloor \frac{(n+1)+2^{k-1}}{2^k} \right\rfloor$, $1 \leq k \leq i-1$.

By (1) we obtain again

$$\sum_{x \in X} d(\mathscr{F}_i, x) > \frac{n_i}{2}|X|;$$

therefore there exists an element $x_i \in X$ such that

$$d(\mathscr{F}_i, x_i) \geq \left\lfloor \frac{n_i}{2} \right\rfloor + 1.$$

This implies that $\mathscr{F}_i$ contains at least

$$\alpha_i = \left\lfloor \frac{n_i}{2} \right\rfloor + 1 = \left\lfloor \frac{n - \left\lfloor \frac{(n+1)+1}{2} \right\rfloor - \left\lfloor \frac{(n+1)+2}{4} \right\rfloor - \ldots - \left\lfloor \frac{(n+1)+2^{i-2}}{2^{i-1}} \right\rfloor}{2} \right\rfloor + 1$$

distinct members of $\mathscr{F}_1$ (all different from $A_{1,1}, \ldots, A_{1,\alpha_1}, A_{2,1}, \ldots, A_{2,\alpha_2}, \ldots$) which contain a common element $x_i \in X$; moreover, by Lemma 3

$$\alpha_i = \left\lfloor \frac{(n+1)+2^{i-1}}{2^i} \right\rfloor.$$

Therefore, if we repeat this procedure s times, we shall obtain at least

$$\sum_{i=1}^{s} \left\lfloor \frac{(n+1)+2^{i-1}}{2^i} \right\rfloor$$

distinct members of $\mathscr{F}_1$, say $A_{1,1}, \ldots, A_{1,\alpha_1}, A_{2,1}, \ldots, A_{2,\alpha_2}, \ldots, A_{s,1}, \ldots, A_{s,\alpha_s}$ (where $\alpha_1 = \left\lfloor \frac{(n+1)+1}{2} \right\rfloor$, $\alpha_2 = \left\lfloor \frac{(n+1)+2}{4} \right\rfloor, \ldots, \alpha_s = \left\lfloor \frac{(n+1)+2^{s-1}}{2^s} \right\rfloor$), such that

$$\begin{array}{c} x_1 \in A_{1,1} \cap \ldots \cap A_{1,\alpha_1} \\ \ldots \\ \ldots \\ x_s \in A_{s,1} \cap \ldots \cap A_{s,\alpha_s}, \end{array} \tag{3}$$

for some elements $x_1, \ldots, x_s$ of X.

By our assumption $n \leq 2^{k+1} - 2$, the inequality $n + 1 < 2^{k+1}$ holds; therefore, by Lemma 2, it follows that

$$\sum_{i=1}^{k} \left\lfloor \frac{(n+1)+2^{i-1}}{2^i} \right\rfloor \geq n. \tag{4}$$

(4) implies that there are no further members in the family $\mathscr{F}_1$ after the kth step. Therefore the procedure can be stopped with some sth step where $s \leq k$. Hence the set $B = \{x_1, \ldots, x_s\}$ satisfies $|B| \leq k$ and $B \cap A_i \neq \emptyset$ for all $i = 1, \ldots, n$. $\square$

4. A fundamental example.

Now it will be proved that the result established in Theorem 1 is best possible; in other words it will be given an example of a finite set X for which, fixed $k \in \mathbb{N}$, and $n = 2^{k+1} - 1 > 2^{k+1} - 2$, there exist n subsets $A_1, \ldots, A_n$ of X, with $|A_i| > \frac{1}{2}|X|$ for all i, such that: for each subset $B \subseteq X$, with $|B| \leq k$, there exists some $i \in \{1, \ldots, n\}$ such that $A_i \cap B = \emptyset$.

Let $\mathbf{F}_q$ be the finite field having q elements, $q = p^h$, where p is a prime number and h is an integer ≥ 1.

Let k be a fixed integer ≥ 2. We denote by $\mathbf{P}(k, q)$ the k-dimensional projective space over the field $\mathbf{F}_q$. The $(k-1)$-dimensional subspaces of $\mathbf{P}(k, q)$ are the hyperplanes.

Now let us consider $X = \mathbf{P}(k, 2)$. Since the points of X are all the $(k+1)$-tuples from the elements 0 and 1, different from $(0, 0, \ldots, 0)$, we have

$$|X| = 2^{k+1} - 1.$$

The number of hyperplanes in X is also $2^{k+1} - 1$, and any hyperplane contains exactly $2^k - 1$ points. Let $n = 2^{k+1} - 1$.

Now take the complementary subsets in X of all hyperplanes of X and call them $A_1, \ldots, A_n$.

Then $|A_i| = n - (2^k - 1) = (2^{k+1} - 1) - (2^k - 1) = (2^{k+1} - 2^k) = 2^k > \frac{1}{2}(2^{k+1} - 1) = \frac{1}{2}|X|$, for $i = 1, \ldots, n$.

Finally, if B is a subset of X such that $|B| \leq k$, there exists some hyperplane in X that contains B; this implies that $B \cap A_i = \emptyset$, for some $i \in \{1, \ldots, n\}$.

This completes the example in $\mathbf{P}(k, 2)$ and shows that the value in $2^{k+1} - 2$ in theorem 1 is best possible.

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REFERENCES

[1] A. Ehrenfeucht - J. Mycielski, *On families of intersecting sets,* J. Combinatorial Theory, (A) 17 (1974), pp. 259-260.

[2] P. Erdös - Ko Chao - R. Rado, *Intersection theorems for systems of finite sets,* Quart J. Math. Oxford, (2) 12 (1961), pp. 313-320.

[3] G.O.H. Katona, *Intersection theorems for systems of finite sets,* Acta Math. Acad. Sci. Hungar., 15 (1964), pp. 329-337.

[4] D.J. Kleitman, *Families of non-disjoint subsets,* J.Combinatorial Theory, 1 (1966), pp. 153-155.

[5] D.J. Kleitman, *Maximum number of subsets of a finite set no k of which are pairwisely disjoint,* J. Combinatorial Theory, 5 (1968), pp. 155-157.

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