GLOBAL SOLUTION OF REAL VISCOSOUS COMPRESSIBLE TRIPOLAR HEAT CONDUCTIVE FLUID ON FINITE CHANNEL

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The global existence of a weak solution of viscous compressible real tripolar heat conductive fluid of initial boundary problem is proved.

1. Introduction.

In the classical Navier-Stokes theory, the viscosity of fluids is modelled by the dependence of stress on the first spatial gradients of velocity. It is well-known that the corresponding mathematical theory contains a number of unsolved problems, in particular there is no existence theory for the global solution of compressible fluids for the full system, only for the isentropique case (see [4]), and no uniqueness theory. There are many results of the existence and uniqueness of the classical solution but only under the assumption of smallness of data or for small time (see Matsumura-Nishida [6], [7], Valli [18], Solonnikov, Kazhikov [16], Tani [17], Zajaczkowski [19]).

From experimental results it follows that we can consider a stronger mechanism of dissipation than the one proposed by Stokes. The fundamental postulate assumes that the dependence of stress on higher order gradients of velocity has to occur in flows of viscous fluids. These problems have led Nečas,
Šilhavý and Novotný to study the physical background and mathematical aspects of the dynamics of multipolar fluids. In the paper [12] by Nečas, Šilhavý an axiomatic theory of viscous multipolar fluids is described. Further works by Nečas, Novotný, Šilhavý, see for example [11], Nečas, Novotný [9] and Nečas [13], are devoted to the study of the problem of adequate existence theory and qualitative properties of equations (regularity, uniqueness, etc.). In compressible motion the pressure \( p \) is not an unknown but a known function which can be determined by \( p = p(\rho, \theta) \). A change in pressure will promote thermal molecular agitation. From the axioms of thermodynamics we know that

\[
p = \rho^2 \frac{\partial (\varepsilon_1 - \theta \eta)}{\partial \rho},
\]

where \( \eta \) is the specific entropy, \( \varepsilon_1 \) is the internal energy and \( \theta \) is the temperature.

The pressure of ideal fluids is governed by the state equation

\[
p = R\rho \theta.
\]

For more realistic fluids the state equation has the form

\[
p = R\rho \theta + B_1(\rho)\rho^2 + B_2(\rho)\rho^3 + \ldots,
\]

where \( B_1, B_2 \) are virial coefficients which are functions of density. We consider the state equation in the easier form

(1.1) \[
p = R\rho \theta + B_1\rho^2,
\]

where \( B_1 \) is a virial coefficient, \( \rho \) is density, \( \theta \) is a temperature and \( R \) is a gas constant and assume that \( B_1 \) is a constant. We recall the equations governing the general motion of compressible multipolar fluids

(1.2) \[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0,
\]

(1.3) \[
\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + p(\rho, \theta)\delta_{ij} - \tau_{ij}^v) = \rho F_i,
\]

(1.4) \[
\frac{\partial}{\partial t}(c_v\rho \theta) + \frac{\partial}{\partial x_j}(c_v\rho \theta v_j) + R\rho \theta \frac{\partial v_j}{\partial x_j} - \lambda \frac{\partial^2 \theta}{\partial x_j \partial x_j} =
\]
\[ = \tau_{ij}^v(v)e_{ij}(v) + \sum_{m=1}^{k-1} \frac{\partial}{\partial x_j} \left( \tau_{i_1...i_mj}^v(v) \frac{\partial^m v_i}{\partial x_{i_1}...\partial x_{i_m}} \right), \]

\[(1.5) \quad \rho(x, 0) = \rho_0, \ v(x, 0) = v_0, \ \theta(x, 0) = \theta_0, \]

\[(1.6) \quad v = 0 \quad \text{on} \ \partial \Omega \times I, \]

\[(1.7) \quad \frac{\partial \theta}{\partial v} = 0 \quad \text{on} \ \partial \Omega \times I, \]

\[(1.8) \quad [[v, w]] = 0 \quad \text{on} \ \partial \Omega \times I, \quad \text{for every} \ w \in V \cap W^{2k,2}(\Omega, \mathbb{R}^N), \]

where \( \Omega \subset \mathbb{R}^N \), \( \Omega \) is a sufficiently smooth bounded domain \((N = 2, 3)\), \( c_v \) (specific heat at constant volume) and \( \lambda \) (heat conductivity) are positive constants and \( F \) denotes the density of external forces. We solve this problem in \( Q_T = \Omega \times (0, T) \). \((V, [[., .]]) \) will be specified later, \( k = 3 \).

The aim of this paper is to prove the existence and the uniqueness of multipolar fluids for one type of "real" fluids. The article is divided into several parts. Section 2 deals with the formulation of the problem. Section 3 is devoted to a priori estimates. In Sections 4 and 5 we prove the existence and the uniqueness of the weak solution for homogeneous and nonhomogeneous Dirichlet conditions.


We are interested in compressible heat conductive tripolar real fluids. We consider a standard stress tensor \( \tau_{ij} \) where

\[(2.1) \quad \tau_{ij} = -p\delta_{ij} + \tau_{ij}^v \]

and higher order stress tensors

\[(2.2) \quad \tau_{i_1...i_mj}^v; \ 1 \leq m \leq k - 1, \ k = 3 \quad \text{and} \quad \tau_{i_1...i_kj} = 0. \]

The tensors \( \tau_{ij}^v, \tau_{i_1...i_mj}^v, 1 \leq m \leq k - 1 \), depend linearly on the spatial gradients of velocity up to order \( 2k - 1 \); the coefficients of linear dependence are real
constants such that \( \tau_{ij}^v, \frac{\partial}{\partial x_j} \tau_{ij}^v \) are symmetric and \( \tau_{ii_1\ldots i_m}^v \) are symmetric in \( i_1 \ldots i_m \). Due to the Clausius-Duhem inequality [12] we suppose that

\[
(2.3) \quad \tau_{ij}^v(v)e_{ij}(v) + \sum_{m=1}^{k-1} \frac{\partial}{\partial x_j} \left( \tau_{i_1\ldots i_m}^v(v) \frac{\partial^m v_i}{\partial x_{i_1} \ldots \partial x_{i_m}} \right) \geq 0.
\]

Let \( V = W^{1,2}_0(\Omega, \mathbb{R}^N) \cap W^{k,2}(\Omega, \mathbb{R}^N) \). We assume that there exists a symmetric bilinear \( V \)-coercive form \((w, w))\) such that

\[
(2.4) \quad ((w, w)) = \int_{\partial \Omega} \left( \sum_{m=1}^{k-1} A_{ij_1\ldots i_m}^m \frac{\partial^m v_i}{\partial x_{i_1} \ldots \partial x_{i_m}} \frac{\partial^m w_j}{\partial x_{j_1} \ldots \partial x_{j_m}} \right) dx,
\]

\[
(2.5) \quad (v, v) \geq \alpha \|v\|_{W^{k,2}(\Omega, \mathbb{R}^N)}^2, \quad \alpha > 0,
\]

and that for \( v, w \in V \cap W^{k,2}(\Omega, \mathbb{R}^N) \) we have

\[
(2.6) \quad ((v, w)) = -\int_{\Omega} \frac{\partial}{\partial x_j} (\tau_{ij}^v(v)) w_i dx + [[v, w]],
\]

where

\[
[[v, w]] = \sum_{m=1}^{k-1} \int_{\partial \Omega} \tau_{i_1\ldots i_m}^v(v) \frac{\partial^m v_i}{\partial x_{i_1} \ldots \partial x_{i_m}} \frac{\partial^m w_j}{\partial x_{j_1} \ldots \partial x_{j_m}} v_j dS.
\]

We define

\[
(2.7) \quad \langle (v, w) \rangle = \tau_{ij}^v(v)e_{ij}(v) + \sum_{m=1}^{k-1} \frac{\partial}{\partial x_j} \left( \tau_{i_1\ldots i_m}^v(v) \frac{\partial^m v_i}{\partial x_{i_1} \ldots \partial x_{i_m}} \right).
\]

From (2.3) it follows that the form \( \langle (v, w) \rangle \) includes derivatives of \( v \) and \( w \) respectively at most of order \( k \).

The weak formulation of (1.3), (1.4) reads

\[
(2.8) \quad \int_{Q_T} \frac{\partial}{\partial t} (\rho \varphi_i) \eta_i \ dx \ dt - \int_{Q_T} \rho \varphi_i v_j \varphi_{i,j} \ dx \ dt + \int_0^T ((v, \varphi)) \ dx \ dt
\]

\[
- \int_{Q_T} R \rho \varphi_{i,t} \ dx \ dt - \int_{Q_T} B_1 \rho^2 \varphi_{i,i} \ dx \ dt = \int_{Q_T} \rho F_i \ dx \ dt;
\]

\[
(2.9) \quad -c_v \int_{Q_T} \rho \frac{\partial \eta}{\partial t} \ dx \ dt - c_v \int_{Q_T} \rho \theta \varphi_0 \eta(0) \ dx + \lambda \int_{Q_T} \frac{\partial \varphi_i}{\partial x_j} \frac{\partial \eta}{\partial x_j} \ dx \ dt
\]

\[
- R \int_{Q_T} \rho \frac{\partial v_j}{\partial x_j} \eta \ dx \ dt - c_v \int_{Q_T} \rho \theta v_j \frac{\partial \eta}{\partial x_j} \ dx \ dt = \int_{Q_T} \langle (v, v) \rangle \eta \ dx \ dt
\]

for every \( \eta \in C_0^\infty(Q_T), \eta(T) = 0, \varphi \in C_0^\infty(Q_T), \varphi(T) = 0, \varphi(t) \in W^{3,2}(\Omega), \eta(t) \in W^{1,2}(\Omega), t \in I \).

The orthogonal basis \( \{ w^r \}^{+\infty}_{r=1} \) in \( V \) is given by solving the following eigenvalue problem

\[
( (v, w^r) ) = \lambda_r \int_{\Omega} v_i w_i^n \, dx
\]

for every \( v \in V \) (\( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \)). From the regularity of the elliptic systems (see [10]) we obtain

\[
w^r \in C^\infty(\overline{Q}, \mathbb{R}^N).
\]

Let \( \gamma = (\gamma_1, \ldots, \gamma_n) \in C^1(\overline{I}, \mathbb{R}^N) \) (\( n = 1, 2, \ldots \)) and put

\[
v^n(x, t) = \sum_{r=1}^{n} \gamma_r(t) w^r.
\]

Let

\[
\rho_0 \in C^d(\overline{\Omega}), \quad \rho_0 > 0 \quad \text{in} \quad \Omega, \quad d = 1, 2, \ldots,
\]

\[
v_0 \in C(\overline{Q}_T),
\]

\[
\theta_0 \in L^2(\Omega), \quad \theta > 0 \quad \text{a.e. in} \quad \Omega,
\]

\[
F \in L^\infty(\overline{Q}_T, \mathbb{R}^N).
\]

We refer to [9] that there exists a solution

\[
\rho^n \in C^1(\overline{Q}_T), \quad \rho^n > 0 \quad \text{in} \quad Q_T, \quad v^n = \sum_{r=1}^{n} \gamma_r(t) w^r, \quad \gamma = (\gamma_1, \ldots, \gamma_n),
\]

\( \gamma \in C^1(\overline{I}, \mathbb{R}^N), \theta^n \in C^1(\overline{Q}_T) \cap C(I, C^2(\overline{Q})), \theta^n \geq 0 \) in \( \overline{Q}_T \), satisfying the equations

\[
\frac{\partial \rho^n}{\partial t} + \frac{\partial}{\partial x_i} (\rho^n v^n_i) = 0;
\]

\[
\frac{\partial}{\partial t} (c_0 \rho^n \theta^n) + \frac{\partial}{\partial x_j} (c_0 \rho^n \theta^n v^n_j) - \lambda \frac{\partial^2 \theta^n}{\partial x_j \partial x_j} =
\]

\[
= -R \rho^n \theta^n \frac{\partial v^n_j}{\partial x_j} + ( (v^n, v^n) );
\]
(3.8) can be written in the equivalent form

\[
\frac{\partial}{\partial t} \left( c_v \rho^n \theta^n + \frac{1}{2} \rho^n \left| v^n \right|^2 \right) + \frac{\partial}{\partial x_j} \left( c_p \rho^n \theta^n v^n_j + \frac{1}{2} \rho^n \left| v^n \right|^2 v^n_j \right) \\
+ \frac{\partial}{\partial x_i} (B_1 \rho^n) v_i - \frac{\partial}{\partial x_j} \left( \tau_{ij}^v (v^n_i) v^n_j + \sum_{m=1}^{k} \tau_{i_1 \ldots i_m}^v (v^n) \frac{\partial^m v^n_j}{\partial x_{i_1} \ldots \partial x_{i_m}} \right) \\
= \lambda \frac{\partial^2 \theta^n}{\partial x_j \partial x_j} + \rho^n F_i v^n_i;
\]

and

\[
\int_{\Omega} \left( \rho^n \frac{\partial v^n_i}{\partial t} + \rho^n v^n_j \frac{\partial v^n_i}{\partial x_j} + \frac{\partial}{\partial x_j} \left\{ R \rho^n \theta^n + B_1 \rho^n \right\} \delta_{ij} - \rho^n F_i \right) w^r_i \, dx = \\
= -((v^n, w^r)), r = 1, \ldots, n.
\]

We have the following initial conditions

\[
(3.10) \quad \rho^n(0) = \rho_0,
\]

\[
(3.11) \quad \theta^n(0) = \theta_0^n, \quad \frac{\partial \theta^n}{\partial v} = 0 \text{ on } \partial \Omega \times I,
\]

where $\theta_0^n \in C^2(\bar{Q})$, $\theta_0^n > 0$ in $\bar{\Omega}$, $\partial \theta_0^n / \partial v = 0$ on $\partial \Omega$, $\theta_0^n \rightarrow \theta_0$ strongly in $L^2(\Omega)$,

\[
(3.12) \quad \int_{\Omega} v^n_i(0) w^r_i \, dx = \int_{0} v_{0i} w^r_i \, dx.
\]

We solve the continuity equation by the method of characteristic to have

\[
(3.13) \quad \rho^n(t, x) = \rho_0(x) \exp \left( - \int_{0}^{t} \frac{\partial}{\partial x_j} v^n_j(\tau, x^n(\tau)) \, d\tau \right),
\]

where $x = x^n(0), y = x^n(t)$; the characteristic $x^n(\tau)$ are solutions to the problem

\[
(3.14) \quad \dot{x}^n(t) = v^n(t, x^n(\tau)), \tau \in (0, T), x^n(0) = x \in \bar{\Omega}.
\]
\[ x \rightarrow x^n(\tau) \text{ is a diffeomorphism from } \overline{\Omega} \text{ onto } \overline{\Omega} \text{ (for more details see [9]). Now,}\]

integrating (3.7) over \( Q_T \), using (1.6) we get

\[ \int_{\Omega_t} \rho^n \, dx = \int_{\Omega_0} \rho_0 \, dx. \]

Multiplying (3.7) by \( \rho_n \) we obtain

\[ (3.15)' \quad \int_{\Omega} (\rho^n)^2 \, dx = -\int_{\Omega} (\rho^n)^2 \nabla v. \]

(i) We multiply (3.9) by \( \gamma_r \), add from \( r = 1 \) to \( n \) and integrate over \( I = (0, T) \).

(ii) We integrate (3.8) over \( Q_T \).

Adding (i), (ii), using (3.15)' it follows that

\[ \frac{1}{2} \int_{\Omega_t} \rho^n |v^n|^2 \, dx - \frac{1}{2} \int_{\Omega_0} \rho_0 |v_0|^2 \, dx + \int_0^T ((v^n, v^n)) \, dt + \]

\[ + B_1 \int_{\Omega_t} \rho_0^n \, dx - B_1 \int_{\Omega_0} \rho_0^2 \, dx - \int_{Q_T} (R \rho^n \theta^n) \frac{\partial v^n_i}{\partial x_i} \, dx \, dt = \int_{Q_T} \rho^n F_i v^n_i \, dx \, dt. \]

Integrating (3.8)' over \( Q_T \), using (3.15)', we have

\[ \int_{Q_t} c_v \rho^n \theta^n - \int_{Q_0} c_v \rho_0^n \theta_0^n + B_1 \int_{\Omega_t} \rho_0^2 - B_1 \int_{\Omega_0} \rho_0^2 + \int_{\Omega_t} \frac{1}{2} \rho^n |v^n|^2 - \]

\[ - \int_{\Omega_0} \rho^n |v^n|^2 = \int_{Q_t} \rho^n F_i v^n_i. \]

From (3.15), (3.16), (3.17) we get the following estimates

\[ \| \rho^n \|_{L^\infty(I, L^1(\Omega))} \leq K_2, \quad K_2 > 0, \]

\[ \| \rho^n \theta^n \|_{L^\infty(I, L^1(\Omega))} \leq K_1, \quad K_1 > 0, \]

\[ \| \rho^n \|_{L^\infty(I, L^1(\Omega))} \leq K_3, \quad K_3 > 0, \]

\[ \| v^n \|_{L^2(I, W^{k,2}(\Omega))} \leq K_2, \]

\[ \| \rho^n \|_{L^\infty(I, L^2(\Omega))} \leq K_1. \]

Similarly as in [9] we obtain
Lemma 3.2. Let \( k = 3, 1 \leq q < +\infty \) (\( N = 2 \)), \( 1 \leq q \leq 6 \) (\( N = 3 \)), \( 0 < \alpha < 1/2 \) and let (3.3) - (3.6) be satisfied. Then

\[
\| \rho^n \|_{L^\infty(I, W^{1,q}(\Omega))} \leq K_5,
\]

(3.23)

\[
\left\| \frac{\partial \rho^n}{\partial t} \right\|_{L^2(I, L^2(\Omega))} \leq K_5.
\]

(3.24)

\[
\| \rho^n \|_{C^{0,\alpha}(I, W^{1,4}(\Omega))} \leq K_5, 0 < \alpha < 1/2.
\]

(3.24)'

Also,

\[
\theta^n \geq 0 \quad \text{a.e. in } Q_T
\]

(3.25) holds.

Proof. See [9].

Now we assume that \( v_0 \in W^{k,2}(\Omega, \mathbb{R}^N) \cap W^{1,2}_0(\Omega, \mathbb{R}^N) \). Using \( \varphi = \partial v^n / \partial t \) as the test function in (2.8) we obtain

\[
\int_{Q_T} \frac{\partial}{\partial t} (\rho^n \varphi \partial v^n) \frac{\partial v^n_i}{\partial t} + \frac{\partial}{\partial x_j} (\rho^n \varphi \partial v^n_j + p(\rho^n, \theta^n) \delta_{ij} - \tau_{ij}^\nu(v^n)) \frac{\partial v^n_i}{\partial t} =
\]

\[
= \int_{Q_T} \rho^n F_i \frac{\partial v^n_i}{\partial t}.
\]

From (3.26) it follows that

\[
\int_{Q_T} \rho^n \frac{\partial \varphi \partial v^n}{\partial t} + \rho^n \varphi \frac{\partial v^n}{\partial t} \frac{\partial v^n_i}{\partial t} + \frac{\partial}{\partial x_i} (R \rho^n \theta^n) \frac{\partial v^n_i}{\partial t} + \frac{\partial}{\partial x_i} (B_1 \rho^n) \frac{\partial v^n_i}{\partial t}
\]

\[
+ \int_0^T \left( (v^n, \frac{\partial v^n}{\partial t}) \right) = \int_{Q_T} \rho^n F_i \frac{\partial v^n_i}{\partial t},
\]

\[
\int_{\Omega} \frac{\partial}{\partial x_i} (R \rho^n \theta^n) \frac{\partial v^n_i}{\partial t} = R \int_{\Omega} \rho^n \frac{\partial \theta^n}{\partial x_i} \frac{\partial v^n_i}{\partial t} + \int_{\Omega} \frac{\partial \rho^n}{\partial x_i} \theta^n \frac{\partial v^n_i}{\partial t},
\]

\[
\left| \int_{\Omega} \frac{\partial \rho^n}{\partial x_i} \theta^n \frac{\partial v^n_i}{\partial t} \right| \leq \frac{1}{\varepsilon} \left( \int_{\Omega} \left| \frac{\partial \rho^n}{\partial x_i} \right|^6 \right)^{1/6} \left( \int_{\Omega} \left| \theta^n \right|^3 \right)^{1/3} + \varepsilon \int_{\Omega} \left| \frac{\partial v^n_i}{\partial t} \right|,
\]
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\[
\left| \int_{\Omega} \rho^n \frac{\partial \theta^n}{\partial x_i} \frac{\partial v_i^n}{\partial t} \right| \leq \frac{c}{\varepsilon_1} \left\| \theta^n \right\|^2_{W^{1,2}(\Omega)} + c \varepsilon_1 \left\| \frac{\partial v^n}{\partial t} \right\|^2_{L_2(\Omega)},
\]

\[
\left| \int_{\Omega} 2 \rho^n \frac{\partial \rho^n}{\partial x_i} \frac{\partial v_i^n}{\partial t} \right| \leq c \left\| \frac{\partial v^n}{\partial t} \right\|_{L^2(\Omega)} \left\| \rho^n \right\|_{L^\infty(\Omega)} \left\| \rho^n \right\|_{W^{1,2}(\Omega)},
\]

\[
\left| \int_{\Omega} \rho^n v_j^n \frac{\partial v_i^n}{\partial t} \frac{\partial v_i^n}{\partial x_j} \right| \leq \left\| \rho^n \right\|_{L^\infty(\Omega)} \left\| v^n \right\|_{W^{1,2}(\Omega)} \left\| \frac{\partial v^n}{\partial t} \right\|_{L^2(\Omega)} \left\| v^n \right\|_{W^{1,3}(\Omega)}.
\]

And thus we have

\[
(3.28) \quad \left\| \frac{\partial v^n}{\partial t} \right\|^2_{L^2(\Omega, R^n)} + \sup_{t \in [0,T]} ((v^n(t), v^n(t))) \leq K_5 (1 + \left\| \theta^n \right\|^2_{L^2(I, W^{1,2}(\Omega))}).
\]

From the regularity of the elliptic system we get

\[
(3.29) \quad \left\| v^n \right\|_{L^2(I, W^{3,2}(\Omega))} \leq K_5 \left( 1 + \left\| \theta^n \right\|_{L^2(I, W^{1,3}(\Omega))} \right).
\]

**Remark.** This estimate plays an essential role in the proof of the uniqueness.

**Lemma 3.3.**

\[
(3.30) \quad \left\| \theta^n \right\|^2_{L^2(I, W^{1,2}(\Omega))} + \left\| \theta^n \right\|_{L^\infty(I, L^2(\Omega))} \leq K_6, \quad K_6 \geq 0.
\]

**Proof.** We multiply (3.8) by \( \theta^n \) and integrate over \( \Omega \).

\[
(3.31) \quad \int_{\Omega} \frac{\partial}{\partial t}(c_v \rho^n \theta^n) \theta^n + \frac{\partial}{\partial x_j}(c_v \rho^n \theta^n v_j^n) \theta^n - \int_{\Omega} \lambda \frac{\partial^2 \theta^n}{\partial x_j \partial x_j} \theta^n
\]

\[
= \int_{\Omega} R \rho^n (\theta^n)^2 \frac{\partial v_j^n}{\partial x_j} + \int_{\Omega} \langle (v^n, v^n) \rangle \theta^n.
\]

We rewrite the left-hand side of (3.31). We obtain

\[
\int_{\Omega} \left( c_v \frac{\partial \rho^n}{\partial t} (\theta^n)^2 + c_v \rho^n \frac{\partial \theta^n}{\partial t} \theta^n + c_v \frac{\partial (\rho^n v_j^n)}{\partial x_j} (\theta^n)^2 + c_v \frac{\partial \theta^n}{\partial x_j} \rho^n \theta^n v_j^n \right) dx
\]

\[
+ \int_{\Omega} \lambda \frac{\partial \theta^n}{\partial x_j} \frac{\partial \theta^n}{\partial x_j} dx \quad \text{(3.7)} \quad \frac{1}{2} \int_{\Omega} \left( c_v \rho^n \frac{\partial}{\partial t} (\theta^n)^2 + c_v \rho^n v_j^n \frac{\partial}{\partial x_j} (\theta^n)^2 \right) dx
\]

\[
+ \int_{\Omega} \lambda \frac{\partial \theta^n}{\partial x_j} \frac{\partial \theta^n}{\partial x_j} dx = \frac{d}{dt} \int_{\Omega} c_v \frac{\partial \rho^n}{\partial t} (\theta^n)^2 dx + \int_{\Omega} \frac{\partial \theta^n}{\partial x_j} \frac{\partial \theta^n}{\partial x_j} dx.
\]
Thus we obtain

\begin{equation}
\frac{d}{dt} \left( \frac{1}{2} c_v \int \rho^n (\theta^n)^2 dx \right) + \lambda \int_{\Omega} \frac{\partial \theta^n}{\partial x_j} \frac{\partial \theta^n}{\partial x_j} dx = R \int_{\Omega} \rho^n \frac{\partial ^2}{\partial x_j} (\theta^n)^2 dx + \int_{\Omega} \langle (\nabla^2 v^n, \nabla v^n) \rangle \theta^n dx.
\end{equation}

We know that the first term on the right-hand side is bounded by \( \| \theta^n \|_{L^2(\Omega)}^2 \). The term \( \int_{\Omega} \langle (\nabla v^n, \nabla v^n) \rangle \theta^n dx \) is more complicated. We use the relation

\begin{equation}
\| v^n \|_{C^1(\overline{\Omega}, \mathbb{R}^N)} \leq \varepsilon_1 \| v^n \|_{W^{2,2}(\Omega, \mathbb{R}^N)} + \overline{K}_1 (\varepsilon_1) \| v^n \|_{L^2(\Omega, \mathbb{R}^N)}
\end{equation}

for every \( \varepsilon_1 > 0 (\overline{K}_1 (\varepsilon_1) > 0) \), see [3].

\begin{equation}
\left( \int_{\Omega} \langle (\nabla v^n, \nabla v^n) \rangle^2 dx \right) \| \theta^n \|_{L^2(\Omega)} \leq c_2 \varepsilon_1 \| v^n(t) \|_{W^{2,2}} \| \theta^n \|_{L^2} + \overline{K}_2 (\varepsilon_1) \| \theta^n \|_{L^2}, \ c_2, \ \overline{K}_2 (\varepsilon_1) > 0.
\end{equation}

Hence, due to (3.29) and the Young inequality, (3.34) on the right-hand side is bounded by

\begin{equation}
K (\varepsilon_1) \| v^n(t) \|_{W^{2,2}}^2 \| \theta^n \|_{L^2} + \varepsilon_1 \| \theta^n \|_{W^{1,2}^2} + K (\varepsilon_1), \ K (\varepsilon_1) > 0.
\end{equation}

Now we use Gronwall's lemma and obtain (3.30).

Thus we have

**Lemma 3.4.** Let \( v_0 \in W^{k,2}(\Omega, \mathbb{R}^N) \cap W^{1,2}_0(\Omega, \mathbb{R}^N) \). Then

\begin{equation}
\left\| \frac{\partial v^n}{\partial t} \right\|_{L^2(Q, \mathbb{R}^N)}^2 + \sup_{[0, T]} \langle (v^n, v^n) \rangle \leq K_{16}
\end{equation}

\begin{equation}
\| v^n \|_{L^2(I, W^{2,2}(\Omega, \mathbb{R}^N))} \leq K_{16}, \ K_{16} > 0.
\end{equation}

**Lemma 3.5.**

\begin{equation}
\left\| \frac{\partial}{\partial t} (\rho^n \theta^n) \right\|_{L^2(I, W^{-1,2}(\Omega, \mathbb{R}^N))} \leq K_{17},
\end{equation}

\begin{equation}
\| \rho^n \theta^n \|_{L^2(I, W^{1,2}(\Omega, \mathbb{R}^N))} \leq K_{17}, \ K_{17} > 0.
\end{equation}
Proof. The estimate (3.38) is a consequence of Lemma 3.3 and (3.23). (3.37) follows from (3.8).

**Lemma 3.6.** Let $\theta_0 \in W^{1,2}(\Omega)$. Then

$$
(3.39) \quad \left\| \frac{\partial \theta^n}{\partial t} \right\|_{L^2(Q_T)} + \left\| \theta^n \right\|_{L^\infty(I, W^{1,2}(\Omega))} \leq K_{18}, \quad K_{18} > 0,
$$

$$
(3.40) \quad \left\| \theta^n \right\|_{L^2(I, W^{2,2}(\Omega))} \leq K_{18}.
$$

Proof. Let us choose a sequence $\theta^n_0 \in C^2(\overline{\Omega})$, $\theta^n_0 > 0$, $\theta^n_0 \rightharpoonup \theta_0$ strongly in $W^{1,2}(\Omega)$, $\partial \theta^n_0 / \partial \nu = 0$ on $\partial \Omega$. Multiplying (3.8) by $\partial \theta^n / \partial t$ and integrating over $Q_t = (0, t) \times \Omega$, $t \in I$, we get

$$
(3.41) \quad c_v \int_{Q_T} \rho^n \left( \frac{\partial \theta^n}{\partial t} \right)^2 \, dx \, dt + \frac{\lambda}{2} \int_{\Omega} \frac{\partial \theta^n}{\partial x_j} \frac{\partial \theta^n}{\partial x_j} \, dx =
$$

$$
- c_v \int_{Q_T} \rho^n v^n \frac{\partial \theta^n}{\partial x_j} \frac{\partial \theta^n}{\partial t} \, dx \, dt - \int_{\Omega_T} \rho \rho^n \theta^n \frac{\partial v^n}{\partial x_j} \frac{\partial \theta^n}{\partial t} \, dx \, dt +
$$

$$
+ \int_{Q_T} \langle \langle v^n, v^n \rangle \rangle \frac{\partial \theta^n}{\partial t} \, dx \, dt + \frac{\lambda}{2} \int_{\Omega_0} \frac{\partial \theta^n_0}{\partial x_j} \frac{\partial \theta^n_0}{\partial x_j} \, dx.
$$

The right-hand side of (3.41) is bounded by $K_{20}(1 + \left\| \frac{\partial \theta^n}{\partial t} \right\|_{L^2(Q_T)})$, $K_{20} > 0$. We use the same technique for the estimate of the term $\langle \langle v^n, v^n \rangle \rangle \frac{\partial \theta^n}{\partial t}$ as in Lemma 3.3. Applying Young’s inequality we verify (3.39), (3.40) is a consequence of the regularity properties to elliptic equations.

### 4.1. The global existence of a weak solution.

**Lemma 4.1.** Let $\tilde{B}_0$, $\tilde{B}$, $\tilde{B}_1$ be Banach spaces, $\tilde{B}_0$, $\tilde{B}_1$ reflexive such that $\tilde{B}_0 \subset \subset \tilde{B} \subset \tilde{B}_1$ (CC is a compact imbedding), let $1 \leq p_0, p_1 < +\infty$. Then

$L^{p_0}(I, \tilde{B}) \subset \subset \{ g; \, g \in L^{p_0}(I, \tilde{B}_0), \partial g / \partial t \in L^{p_1}(I, \tilde{B}_1) \}$.

Proof. See [3].
Lemma 4.2. Let the assumptions (3.2)-(3.6), \( v_0 \in W^{k,2}(\Omega, \mathbb{R}^N) \cap W^{1,2}(\Omega, \mathbb{R}^N) \) be satisfied. Then one can choose a subsequence of \( \{(\rho^n, v^n, \theta^n)\}_{n=1}^{\infty} \) (denoted \( \{(\rho^n, v^n, \theta^n)\} \) again) such that

(i) \( \rho^n \to \rho \) strongly in \( L^p(Q_T), \ 1 < p < +\infty, \rho > \varepsilon > 0 \) a.e. in \( Q_T \);

(ii) \( v^n \to v \) strongly in \( L^2(I, W^{2k-1,2}(\Omega, \mathbb{R}^N)) \),

\[
D^i v^n \to D^i v \text{ weakly in } L^2(Q_T, \mathbb{R}^N) \quad (i = 1, \ldots, 2k);
\]

(iii) \( D^i \theta^n \to D^i \theta \) weakly in \( L^2(Q_T) \);

(iv) \( \theta \geq 0 \) a.e. in \( Q_T \),

\[
\frac{\partial \rho^n}{\partial t} \to \frac{\partial \rho}{\partial t} \text{ strongly in } L^2(Q_T);
\]

(v) \( \frac{\partial \rho^n}{\partial t} \to \frac{\partial \rho}{\partial t} \text{ weakly in } L^2(Q_T) \);

(vi) \( \frac{\partial v^n}{\partial t} \to \frac{\partial v}{\partial t} \text{ weakly in } L^2(Q_T) \);

(vii) \( \rho^n v^n \to \rho v \) strongly in \( L^2(Q_T) \);

(viii) \( \rho^n \theta^n \frac{\partial v_j^n}{\partial x_j} \to \rho \theta v_j \frac{\partial v_j}{\partial x_j} \text{ weakly in } L^2(Q_T) \);

(ix) \( \rho^n v_i v_j^n \to \rho v_i v_j \text{ weakly in } L^2(Q_T) \);

(x) \( \int_{Q_T} \langle (v^n, v^n) \rangle \phi dx d\tau \to \int_{Q_T} \langle (v, v) \rangle \phi dx d\tau, \phi \in C^1(\overline{Q_T}). \)

Proof.

(i) this assertion follows from Lemma 4.1, where we put \( \widetilde{B}_0 = W^{1,6}, \widetilde{B} = L^p, \widetilde{B}_1 = L^q, 1 < p < +\infty, \rho_0 = p, p_1 = 2, 1 \leq q \leq 6 \ (N = 3), \]

\( 1 \leq q < +\infty \ (N = 2) \);

(ii) the first two assertions follow from Lemma 4.1 with \( \widetilde{B}_0 = W^{2k,2}, \widetilde{B} = W^{2k-1,2}, \widetilde{B}_1 = L^2, p_0 = p_1 = 2 \) and \( \widetilde{B}_0 = W^{k,2}, \widetilde{B} = W^{k-1,2}, \widetilde{B}_1 = L^2, \rho_0 = p, p_1 = 2 \) respectively; the last assertion of (ii) follows from the boundedness of \( \{v^n\}_{n=1}^{\infty} \) in \( L^2(I, W^{2k,2}(\Omega, \mathbb{R}^N)) \);

(iii) is a consequence of Lemma 3.3;

(iv) we obtain that \( \rho^n \theta^n \to a \) (we use Lemma 4.1 with \( \widetilde{B} = L^2, \widetilde{B}_1 = W^{-1,2}, \widetilde{B}_0 = W^{1,2}, p_0 = p_1 = 2 \) and that \( a = \rho \theta \) we get form (i), (iii), first assertion of (iv) holds due to (3.25);

(v) is a consequence of (3.24);

(vi) is a consequence of (3.35);

(vii) is a consequence of (i) and (ii);

(viii) is a consequence of (ii) and (iv);

(ix) follows from (ii), (i), (vii);

(x) follows from (ii).
Due to Lemma 4.1 we can pass to the limit in (3.7), (3.8), (3.9) to obtain the following theorems

**Theorem 4.1 (weak solution).** Let \( \rho_0 \in C^d(\Omega) \), \( d = 1, 2, \ldots, \rho_0 > 0 \) in \( \Omega \), \( \theta_0 \in L^2(\Omega) \), \( \theta_0 > 0 \) a.e. in \( \Omega \), \( v_0 \in W^{k,2}(\Omega, \mathbb{R}^N) \cap W^{1,2}_0(\Omega, \mathbb{R}^N) \). Let \( 1 \leq q < +\infty \) \((N = 2)\) or \( 1 \leq q \leq 6 \) \((N = 3)\), \( 0 < \alpha < 1/2 \) then there exists \((\rho, v, \theta)\) such that

\[
(4.1) \rho \in L^\infty(I, W^{1,q}(\Omega)) \cap C^{0,\alpha}(\bar{I}, L^q(\Omega)), \quad \rho > \varepsilon \text{ a.e. in } Q_T \text{ for some } \varepsilon > 0,
\]

\[
(4.2) \frac{\partial \rho}{\partial t} \in L^\infty(I, L^q(\Omega)),
\]

\[
(4.3) v \in L^\infty(I, W^{k,2}(\Omega, \mathbb{R}^N) \cap W^{1,2}_0(\Omega, \mathbb{R}^N)) \cap L^2(I, W^{2k,2}(\Omega, \mathbb{R}^N)),
\]

\[
(4.4) \frac{\partial v}{\partial t} \in L^2(Q_T, \mathbb{R}^N),
\]

\[
(4.5) \theta \in L^\infty(I, L^2(\Omega)) \cap L^2(I, W^{1,2}(\Omega)), \quad \theta \geq 0 \text{ a.e. in } Q_T,
\]

such that (1.2),(1.3) holds a.e. in \( Q_T \) and (2.12) is fulfilled; if \( d > 1 \) then

\[
(4.6) \rho \in L^\infty(I, W^{p,q}(\Omega)) \cap C^{0,\alpha}(\bar{I}, W^{p-1,q}(\Omega)),
\]

\[
(4.7) \frac{\partial \rho}{\partial t} \in L^2(I, W^{p-1,q}(\Omega)),
\]

where \( p = \min(d, 4) \).

**Theorem 4.2 (strong solution).** Let the assumptions of Theorem 4.1 be satisfied and let \( \theta \in W^{1,2}(\Omega) \). Then there exists \((\rho, v, \theta)\) satisfying (4.1)-(4.5) and

\[
(4.8) \theta \in L^2(I, W^{2,2}(\Omega)) \cap L^\infty(I, W^{1,2}(\Omega)),
\]

\[
(4.9) \frac{\partial \theta}{\partial t} \in L^2(Q_T),
\]

such that the equations (1.2)-(1.4) are satisfied a.e. in \( Q_T \).

### 4.2. Uniqueness.

**Theorem 4.3.** Let the assumptions of Theorem 4.1 with \( d \geq 2 \) be satisfied. Then in the class (4.1)-(4.9) there exists at most one solution \((\rho, v, \theta)\) satisfying (1.2)-(1.4) a.e. in \( Q_T \).

**Proof.** Let \((\rho, v, \theta), (\bar{\rho}, \bar{v}, \bar{\theta})\) be two solutions with the same initial conditions. Then \((\xi, w, \eta) = (\rho - \bar{\rho}, v - \bar{v}, \theta - \bar{\theta})\) satisfies

\[
(4.10) \quad \frac{\partial}{\partial t} \xi + \frac{\partial}{\partial x_j}(\xi v_j) + \frac{\partial}{\partial x_j}(\bar{\rho} w_j) = 0 \quad \text{a.e. in } Q_T,
\]

\[
(4.11) \quad \bar{\rho} \frac{\partial w_i}{\partial t} + \frac{\partial v_i}{\partial x_j} + \bar{\rho} \frac{\partial w_i}{\partial x_j} + \bar{\rho} \frac{\partial v_i}{\partial x_j} - \frac{3}{\partial x_j}((\tau_{i,j}(w))) +
\]
\[ R \frac{\partial}{\partial x_i} (\xi \partial_i) + R \frac{\partial}{\partial x_i} (\bar{\rho} \partial_i) + B_1 \frac{\partial}{\partial x_i} (\rho^2 - \bar{\rho}^2) + \xi \frac{\partial v_i}{\partial t} = \xi F_i \quad \text{a.e. in } Q_T, \]

\[ c_v \bar{\rho} \frac{\partial \eta}{\partial t} + c_v \bar{\theta} \frac{\partial \eta}{\partial x_j} + c_v \bar{\rho} \frac{\partial \eta}{\partial x_j} + c_v \bar{\rho} \frac{\partial \bar{\theta}}{\partial x_j} w_j - \]

\[ \lambda \frac{\partial^2 \eta}{\partial x_j \partial x_j} + R \bar{\rho} \frac{\partial v_j}{\partial x_j} \eta + R \bar{\rho} \bar{\theta} \frac{\partial w_j}{\partial x_j} + R \theta \frac{\partial v_j}{\partial x_j} \xi + \]

\[ + c_v \bar{\theta} \frac{\partial \partial_t}{\partial t} = (\langle v, v \rangle) - (\langle \bar{v}, \bar{v} \rangle) \quad \text{a.e. in } Q_T. \]

From (4.10) one obtains the estimates

\[ \frac{\partial}{\partial t} \| \xi(t) \|_{W^{1,2}(\Omega)}^2 \leq K(\varepsilon_1) a_1(t) \| \xi(t) \|_{W^{1,2}(\Omega)}^2 + \varepsilon_1 \| w(t) \|_{W^{1,2}(\Omega, \mathbb{R}^N)}^2, \]

\[ K_4(\varepsilon_1) > 0, \]

for every \( \varepsilon_1 > 0 \), \( a_1 \in L^\infty(I) \). From (4.11) we obtain

\[ \frac{\partial}{\partial t} \left( \int_\Omega \bar{\rho} \ | w \ |^2 \ dx \right) + \| w(t) \|_{W^{1,2}(\Omega, \mathbb{R}^N)}^2 \leq \]

\[ \leq K(\varepsilon_1) a(t) \left( \| \xi(t) \|_{W^{1,2}(\Omega)}^2 + \| w(t) \|_{L^2(\Omega, \mathbb{R}^N)}^2 + \| \eta(t) \|_{L^2(\Omega)}^2 \right) + \varepsilon_1 \| w(t) \|_{W^{1,2}(\Omega, \mathbb{R}^N)}^2 \]

\[ (K(\varepsilon_1) > 0) \text{ for a.e. } t \in I \text{ and every } \varepsilon_1 > 0, \text{ where} \]

\[ a = \left( 1 + \| \frac{\partial \bar{\rho}}{\partial t} \|_{L^\infty(\Omega)} + \| \bar{\rho} \|_{W^{1,6}(\Omega)} + \| \rho \|_{W^{1,6}(\Omega)} \right)^2 \cdot \]

\[ \cdot \left( 1 + \| \bar{v} \|_{W^{2,2}(\Omega, \mathbb{R}^N)} + \| v \|_{W^{2,2}(\Omega, \mathbb{R}^N)} + \| \frac{\partial \bar{\eta}}{\partial t} \|_{L^2(\Omega, \mathbb{R}^N)} + \right) \]

\[ + \left\| \frac{\partial \theta}{\partial t} \right\|_{L^2(\Omega, \mathbb{R}^N)}^2 \left( 1 + \| \bar{\theta} \|_{W^{1,2}(\Omega)} + \| \theta \|_{W^{1,2}(\Omega)} \right)^2, \]

hence \( a \in L^1(I) \). From (4.12) after some computation we get the estimate

\[ \frac{\partial}{\partial t} \left( \int_\Omega \rho \eta^2 dx \right) + \| \nabla \eta(t) \|_{L^2}^2 \leq \]

\[ \bar{K}_2(\varepsilon_1) a(t) \left( \| \xi(t) \|_{W^{1,2}(\Omega)} + \| w(t) \|_{L^2(\Omega, \mathbb{R}^N)} + \| \eta(t) \|_{L^2} \right) \]

\[ + \varepsilon_1 \left( \| w(t) \|_{W^{1,2}(\Omega, \mathbb{R}^N)} + \| \nabla \eta(t) \|_{L^2}^2 \right) \]

for a.e. \( t \in I \) and every \( \varepsilon_1 > 0 \). From (4.13)-(4.15), after applying Gronwall's lemma, we obtain \( \xi = 0, w = 0, \eta = 0 \) a.e. in \( I \).
5. Non-homogeneous condition.

Now we are interested in the case of non-homogeneous boundary conditions. We consider that $\Omega$ is a finite channel, where $\partial \Omega = \Gamma_{\text{inp}} \cup \Gamma_{\text{out}} \cup \Gamma_r$. Besides the initial conditions

$$
(5.1) \quad \rho(0) = \rho_0, \quad v(0) = v_0, \quad \theta(0) = \theta_0
$$

we consider the following boundary conditions:

$$
(5.2) \quad \begin{align*}
    v &= v_0 \text{ on } \Gamma_{\text{inp}} \cup \Gamma_{\text{out}}, \\
    v &= 0 \text{ on } \Gamma_r, \\
    \int_{\Gamma_{\text{inp}}} vv &> 0, \\
    \int_{\Gamma_{\text{out}}} vv &< 0,
\end{align*}
$$

where $v$ is the outer normal,

$$
\rho = \rho_0 \text{ on } \Gamma_{\text{inp}},
\rho > 0.
$$

Further, we consider that $\theta_0 > 0$; and that there exist constants $m$ and $M$ such that

$$
(5.2)' \quad 0 \leq m \leq \theta(x,t) \leq M \text{ on } \partial \Omega \times I \quad \text{(see [5]).}
$$

We consider that $v_0$ is such a function that there exists the extension to the entire cylinder. The function is denoted again $v_0$. So we can write $v = v_0 + z$. (This means $z = 0$ on $\partial \Omega \times I$) Moreover, we assume the boundary conditions

$$
(5.3) \quad [[z, w]] = 0 \text{ on } \partial \Omega \times I, \text{ for every } w \in V \cap W^{2k,2}(\Omega, \mathbb{R}^N).
$$
5.1. The formulation of the problem.

We consider the system (1.2)-(1.4). We state the weak formulation of the problem as follows

\begin{equation}
\tag{5.5}
\int_{Q_T} \rho v_i \frac{\partial \varphi}{\partial t} + \int_{\Omega} \rho_0 v_i \varphi(0) + \int_{Q_T} p(\rho, \theta) \frac{\partial \varphi}{\partial x_i} + \\
+ \int_{0}^{T} ((v, \varphi)) = \int_{Q_T} \rho F_i \varphi,
\end{equation}

\[ \varphi(T) = 0, \ \varphi \in C^\infty(Q_T), \ \varphi = 0 \text{ on } \partial \Omega \times I, \ \varphi(t) \in W^{3,2}(\Omega); \]

\begin{equation}
\tag{5.6}
\int_{Q_T} c_v \rho \theta \frac{\partial \eta}{\partial t} - \int_{Q_T} c_\theta \rho_0 \theta_0 \eta(0) + \int_{0}^{T} \int_{\partial \Omega} c_v \rho \theta v_j \eta v_j \\
- \int_{Q_T} c_v \rho \theta v_j \frac{\partial \eta}{\partial x_j} + \int_{Q_T} R \rho \theta \frac{\partial v_j}{\partial x_j} \eta + \int_{Q_T} \lambda \frac{\partial \theta}{\partial x_j} \frac{\partial v_j}{\partial x_j} = \int_{Q_T} ((v, w)) \eta,
\end{equation}

\[ \eta(T) = 0, \ \eta \in C^1(Q_T), \ \eta(t) \in W^{1,2}(\Omega). \]

We apply the Galerkin method, we take the same orthogonal basis as in Section 3 and we solve the eigenvalue problem

\begin{equation}
\tag{5.7}
(((z, w')) = \lambda_r \int_{\Omega} v_i w'_i \, dx
\end{equation}

for every \( z \in V = W^{k,2} \cap W^{1,2}_0(0 < \lambda_1 \leq \lambda_2 \leq \ldots) \). From the regularity of the elliptic system (see [10]) we obtain

\begin{equation}
\tag{5.7}'
w' \in C^\infty(\overline{\Omega}, \mathbb{R}^N).
\end{equation}

Let

\begin{equation}
\tag{5.8}
c = (c_1, \ldots, c_n) \in C^1(\overline{I}, \mathbb{R}^N) \quad (n = 1, 2, \ldots)
\end{equation}

and we put

\begin{equation}
\tag{5.9}
v^n(x, t) = \sum_{r=1}^{n} c_r(t) w'_r(x) + v^0
\end{equation}
and let

\[(5.9)\]

\[v^0 \in C^5(\overline{Q}_T).\]

Let (3.4)-(3.6) be satisfied and

\[(5.9)''\]

\[\rho_0^n \in C(\overline{Q}), \rho_0^n > 0.\]

Then there exists a solution satisfying (5.8), (5.9), \(\rho^n \in C(\overline{Q}_T) \cap W^{1,\infty}(Q_T), \rho^n > 0 \text{ in } Q_T, \theta^n \in C(\overline{Q}_T) \cap C(I, C^2(\overline{\Omega})), \theta^n \geq 0 \text{ in } Q_T \text{ satisfying equations } (3.7)-(3.12).\] We solve the continuity equation by the method of characteristics and obtain

\[(5.10)\]

\[\rho^n(t, x) = \rho_0(t - \tilde{\tau}, x^n(\tilde{\tau})) \exp \left( -\int_0^{\tilde{\tau}} \frac{\partial}{\partial x_j} v^n_j(t - \tau, x^n(\tau)) d\tau \right),\]

where \(x = x^n(0), y = x^n(\tilde{\tau})\); the characteristics \(x^n(\tau)\) are solution to the problem

\[(5.10)'

\[x^n(t) = -v^n(t - \tau, x^n(\tau)), \tau \in (0, T), x^n(0) = x \in \overline{\Omega}.\]

(\(\tilde{\tau}\) denotes the time which we achieve when going from the point \([x, t]\) to the point which lies in \(\Omega_0\) or \(\Gamma_{\text{inp}}\)). For every \(\tau \in I_{\tilde{\tau}}, I_{\tilde{\tau}} = (0, \tilde{\tau}), \tilde{\tau} > 0, x \to x^n(\tau)\)

is a local diffeomorphism of \(\overline{\Omega}\) onto \(\overline{\Omega}\). More precisely see [8]. We integrate (3.7) over \(Q_T\) to have

\[(5.11)\]

\[\int_{\Omega_t} \rho^n \leq \int_{\Omega_0} \rho^n - \int_0^T \int_{\Gamma_{\text{inp}}} \rho_0^n v^n_j v_j.\]

From (3.8)' after integrating over \(Q_T\) and using the continuity equation we get

\[(5.12)\]

\[\int_{Q_T} c_p \rho^n \theta^n + \frac{1}{2} \int_{\Omega_T} \rho^n |v^n|^2 - \int_{\Omega_T} B_1 \rho^n_2 \leq \]

\[\leq - \int_0^T \int_{\Gamma_{\text{inp}}} \left( c_p \rho_0^n \theta_0^n v^n_0 v_j + \frac{1}{2} \rho^n |v^n|^2 v^n_j v_j \right) + \int_0^T \int_{\Gamma_{\text{out}}} B_1 \rho^n_2 v^n_0 v_i \]

\[+ \int_{Q_T} \rho^n F_i v^n_i + \int_{Q_T} \langle (v_0, v_0) \rangle.\]
Now we put $\varphi = z^n$ in (5.5). So we obtain

\begin{equation}
\frac{1}{2} \int_{Q_T} \rho^n |z^n|^2 + \int_{Q_T} \left( \rho^n \frac{\partial v^i}{\partial t} z^n_j + \rho^n v^i_j z^n_i \frac{\partial v^0}{\partial x_i} \right) \\
+ \int_0^T \int_{\partial \Omega} B_1 \rho^n v^i v_j + \int_{Q_T} \rho^n - \int_{\Omega_0} \rho^n + \int_{Q_T} B_1 \rho^n \frac{\partial v^0}{\partial x_i} - \int_{Q_T} \rho^n F_i v^i \\
= - \int_0^T \langle (v^n, z^n) \rangle.
\end{equation}

Thus we obtain

\begin{equation}
\|v^n\|_{L^2(I, W^{1,2}(\Omega))} \leq k_1,
\end{equation}

\begin{equation}
\|\rho^n\|_{L^\infty(I, L^2(\Omega))} \leq k_2.
\end{equation}

From (5.14) and (5.10) we obtain

\begin{equation}
\|\rho^n\|_{L^\infty(I, W^{p-4}(\Omega))} \leq k_3,
\end{equation}

\((N = 2, 1 \leq q \leq +\infty, N = 3, 1 \leq q \leq 6, p = k - 2)\)

and

\begin{equation}
\left\| \frac{\partial \rho^n}{\partial t} \right\|_{L^2(I, W^{p-4}(\Omega))} \leq k_4.
\end{equation}

Now we use $\partial v^n/\partial t$ as the test function in (5.5) and we obtain (3.28), (3.29). Analogously as in Lemma 3.3 we obtain (3.30), (3.37), (3.38).

**Theorem 5.1.** Let the assumptions (3.2) - (3.6), (5.9)', (5.9)''

\[ v_0 \in W^{k,2}(\Omega, \mathbb{R}^N) \cap W^{1,2}(\Omega, \mathbb{R}^N) \]

be satisfied. Then one can choose a subsequence of \{$(\rho^n, v^n, \theta^n)$\} again (denoted \{$(\rho^n, v^n, \theta^n)$\} again) such that

(i) $\rho^n \to \rho$ strongly in $L^p(Q_T)$, $1 < p < +\infty$, $\rho \geq \varepsilon > 0$ in $Q_T$;
(ii) $v^n \to v$ strongly in $L^2(I, W^{2k-1,2}(\Omega, \mathbb{R}^N))$,
\[ v^n \to v \text{ strongly in } L^p(I, W^{k-1,2}(\Omega, \mathbb{R}^N)), \]
\[ D^i v^n \to D^i v \text{ weakly in } L^2(Q_T, \mathbb{R}^N) \ (i = 1, \ldots, 2k); \]
(iii) $D^i \theta^n \to D^i \theta$ weakly in $L^2(Q_T)$;
(iv) \( \theta \geq 0 \) a.e. in \( Q_T \),
\[ \rho^n \theta^n \to \rho \theta \text{ strongly in } L^2(Q_T); \]
(v) \[ \frac{\partial \rho^n}{\partial t} \to \frac{\partial \rho}{\partial t} \text{ weakly in } L^2(Q_T); \]
(vi) \[ \frac{\partial v^n}{\partial t} \to \frac{\partial v}{\partial t} \text{ weakly in } L^2(Q_T); \]
(vii) \( \rho^n v^n \to \rho v \text{ strongly in } L^2(Q_T); \)
(viii) \[ \rho^n \theta^n \frac{\partial v^n_j}{\partial x_j} \to \rho \theta \frac{\partial v_j}{\partial x_j} \text{ weakly in } L^2(Q_T); \]
(ix) \[ \rho^n v^n_i v^n_j \to \rho v_i v_j \text{ weakly in } L^2(Q_T); \]
(x) \[ \int_{Q_T} \langle (v^n, v^n) \rangle \phi \, dx \, dt \to \int_{Q_T} \langle (v, v) \rangle \phi \, dx \, dt, \phi \in C^1(\overline{Q}_T). \]

Due to Theorem 5.1 we can pass to the limit in (3.7), (3.8), (3.9). We obtain the following theorem

**Theorem 5.2.** Let the assumptions of Theorem 5.1 be satisfied and let \( 1 \leq q < +\infty \) (\( N = 2 \)) or \( 1 \leq q \leq 6 \) (\( N = 3 \)). Then there exists \((\rho, v, \theta)\) such that
\[(5.18) \quad \rho \in L^\infty(I, W^{1,q}(\Omega)), \quad \rho > \varepsilon \text{ a.e. in } Q_T \text{ for some } \varepsilon > 0, \]
\[(5.19) \quad \frac{\partial \rho}{\partial t} \in L^\infty(I, L^q(\Omega)), \]
\[(5.20) \quad v \in L^\infty(I, W^{k,2}(\Omega, \mathbb{R}^N) \cap W_0^{1,2}(\Omega, \mathbb{R}^N)) \cap L^2(I, W^{2k,2}(\Omega, \mathbb{R}^N)), \]
\[(5.21) \quad \frac{\partial v}{\partial t} \in L^2(Q_T), \mathbb{R}^N), \]
\[(5.22) \quad \theta \in L^\infty(I, L^2(\Omega)) \cap L^2(I, W^{1,2}(\Omega)), \quad \theta \geq 0 \text{ a.e. in } Q_T, \]
\[ \text{such that (1.2), (1.3) holds a.e. in } Q_T \text{ and (2.12) is fulfilled.} \]

**Remark.** When we repeat all steps of Section 4 we obtain uniqueness as well.

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