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GLOBAL SOLUTION OF REAL VISCOUS COMPRESSIBLE TRIPOLAR HEAT CONDUCTIVE FLUID ON FINITE CHANNEL

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The global existence of a weak solution of viscous compressible real tripolar heat conductive fluid of initial boundary problem is proved.

1. Introduction.

In the classical Navier-Stokes theory, the viscosity of fluids is modelled by the dependence of stress on the first spatial gradients of velocity. It is well-known that the corresponding mathematical theory contains a number of unsolved problems, in particular there is no existence theory for the global solution of compressible fluids for the full system, only for the isentropique case (see [4]), and no uniqueness theory. There are many results of the existence and uniqueness of the classical solution but only under the assumption of smallness of data or for small time (see Matsumura-Nishida [6], [7], Valli [18], Solonnikov, Kazhikov [16], Tani [17], Zajaczkowski [19]).

From experimental results it follows that we can consider a stronger mechanism of dissipation than the one proposed by Stokes. The fundamental postulate assumes that the dependence of stress on higher order gradients of velocity has to occur in flows of viscous fluids. These problems have led Nečas,

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Šilhavý and Novotný to study the physical background and mathematical aspects of the dynamics of multipolar fluids. In the paper [12] by Nečas, Šilhavý an axiomatic theory of viscous multipolar fluids is described. Further works by Nečas, Novotný, Šilhavý, see for example [11], Nečas, Novotný [9] and Nečas [13], are devoted to the study of the problem of adequate existence theory and qualitative properties of equations (regularity, uniqueness, etc.). In compressible motion the pressure p is not an unknown but a known function which can be determined by $p = p(\rho, \theta)$. A change in pressure will promote thermal molecular agitation. From the axioms of thermodynamics we know that

$$p = \rho^2 \frac{\partial (\varepsilon_1 - \theta\eta)}{\partial \rho},$$

where η is the specific entropy, ε_1 is the internal energy and θ is the temperature.

The pressure of ideal fluids is governed by the state equation

$$p = R\rho\theta.$$

For more realistic fluids the state equation has the form

$$p = R\rho\theta + B_1(\rho)\rho^2 + B_2(\rho)\rho^3 + \dots,$$

where B_1, B_2 are virial coefficients which are functions of density. We consider the state equation in the easier form

$$p = R\rho\theta + B_1\rho^2, \tag{1.1}$$

where B_1 is a virial coefficient, ρ is density, θ is a temperature and R is a gas constant and assume that B_1 is a constant. We recall the equations governing the general motion of compressible multipolar fluids

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0, \tag{1.2}$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + p(\rho, \theta)\delta_{ij} - \tau_{ij}^{v}) = \rho F_i, \tag{1.3}$$

$$\frac{\partial}{\partial t}(c_v \rho\theta) + \frac{\partial}{\partial x_j}(c_v \rho\theta v_j) + R\rho\theta \frac{\partial v_j}{\partial x_j} - \lambda \frac{\partial^2 \theta}{\partial x_j \partial x_j} = \tag{1.4}$$

$$= \tau_{ij}^{\nu}(v)e_{ij}(v) + \sum_{m=1}^{k-1} \frac{\partial}{\partial x_j}\left(\tau_{ii_1\ldots i_m j}^{\nu}(v)\frac{\partial^m v_i}{\partial x_{i_1}\ldots\partial x_{i_m}}\right),$$

$$\rho(x,0) = \rho_0,\, v(x,0) = v_0,\, \theta(x,0) = \theta_0, \tag{1.5}$$

$$v = 0 \quad \text{on } \partial\Omega \times I, \tag{1.6}$$

$$\frac{\partial\theta}{\partial\nu} = 0 \quad \text{on } \partial\Omega \times I, \tag{1.7}$$

$$[[v, w]] = 0 \text{ on } \partial\Omega \times I, \text{ for every } w \in V \cap W^{2k,2}(\Omega, \mathbb{R}^N), \tag{1.8}$$

where $\Omega \subset \mathbb{R}^N$, Ω is a sufficiently smooth bounded domain ($N = 2, 3$), c_v (specific heat at constant volume) and λ (heat conductivity) are positive constants and F denotes the density of external forces. We solve this problem in $Q_T = \Omega \times (0, T)$. ($V$, $[[.,.]]$ will be specified later, $k = 3$).

The aim of this paper is to prove the existence and the uniqueness of multipolar fluids for one type of "real" fluids. The article is divided into several parts. Section 2 deals with the formulation of the problem. Section 3 is devoted to a priori estimates. In Sections 4 and 5 we prove the existence and the uniqueness of the weak solution for homogeneous and nonhomogeneous Dirichlet conditions.

2. A formulation of the problem.

We are interested in compressible heat conductive tripolar real fluids. We consider a standard stress tensor τ_{ij} where

$$\tau_{ij} = -p\delta_{ij} + \tau_{ij}^{\nu} \tag{2.1}$$

and higher order stress tensors

$$\tau_{ii_1\ldots i_m j}^{\nu};\ 1 \le m \le k-1,\ k = 3 \quad \text{and} \quad \tau_{ii_1\ldots i_k j} = 0. \tag{2.2}$$

The tensors τ_{ij}^{ν}, $\tau_{ii_1\ldots i_m j}^{\nu}$, $1 \le m \le k-1$, depend linearly on the spatial gradients of velocity up to order $2k-1$; the coefficients of linear dependence are real constants such that τ_{ij}^{ν}, $\frac{\partial}{\partial x_j}\tau_{ii_1j}^{\nu}$ are symmetric and $\tau_{ii_1\dots i_mj}^{\nu}$ are symmetric in $i_1 \dots i_m$. Due to the Clausius-Duhem inequality [12] we suppose that

$$\tau_{ij}^{\nu}(v)e_{ij}(v) + \sum_{m=1}^{k-1} \frac{\partial}{\partial x_j}\left(\tau_{ii_1\dots i_mj}^{\nu}(v)\frac{\partial^m v_i}{\partial x_{i_1}\dots\partial x_{i_m}}\right) \geq 0. \tag{2.3}$$

Let $V = W_0^{1,2}(\Omega, \mathbb{R}^N) \cap W^{k,2}(\Omega, \mathbb{R}^N)$. We assume that there exists a symmetric bilinear V-coercive form $((v, w))$ such that

$$((v, w)) = \int_\Omega \left(\sum_{m=1}^{k} A_{iji_1\dots i_mj_1\dots j_m}^{m} \frac{\partial^m v_i}{\partial x_{i_1}\dots\partial x_{i_m}} \frac{\partial^m w_j}{x_{j_1}\dots\partial x_{j_m}}\right) dx, \tag{2.4}$$

$$(v, v)) \geq \alpha\|v\|^2_{W^{k,2}(\Omega,\mathbb{R}^N)}, \quad \alpha > 0, \tag{2.5}$$

and that for $v, w \in V \cap W^{2k,2}(\Omega, \mathbb{R}^N)$ we have

$$((v, w)) = -\int_\Omega \frac{\partial}{\partial x_j}\left(\tau_{ij}^{\nu}(v)\right) w_i \, dx + [[v, w]], \tag{2.6}$$

where

$$[[v, w]] = \sum_{m=1}^{k-1} \int_{\partial\Omega} \tau_{ii_1\dots i_mj}^{\nu}(v) \frac{\partial^m w_i}{\partial x_{i_1}\dots\partial x_{i_m}} \nu_j \, dS.$$

We define

$$\langle\langle v, w\rangle\rangle = \tau_{ij}^{\nu}(v)e_{ij}(v) + \sum_{m=1}^{k-1} \frac{\partial}{\partial x_j}\left(\tau_{ii_1\dots i_mj}^{\nu}(v)\frac{\partial^m w_i}{\partial x_{i_1}\dots\partial x_{i_m}}\right). \tag{2.7}$$

From (2.3) it follows that the form $\langle\langle v, w\rangle\rangle$ includes derivatives of v and w respectively at most of order k.

The weak formulation of (1.3), (1.4) reads

$$\begin{aligned}\int_{Q_T} \frac{\partial}{\partial t}(\rho v_i)\varphi_i \, dxdt - \int_{Q_T} \rho v_i v_j \varphi_{i,j} \, dxdt + \int_0^T ((v, \varphi)) \, dt \\ - \int_{Q_T} R\rho\theta\varphi_{i,i} \, dxdt - \int_{Q_T} B_1\rho^2\varphi_{i,i} \, dxdt = \int_{Q_T} \rho F_i \, dxdt;\end{aligned} \tag{2.8}$$

$$\begin{aligned}-c_v \int_{Q_T} \rho\theta\frac{\partial\eta}{\partial t} \, dxdt - c_v \int_\Omega \rho_0\theta_0\eta(0) \, dx + \lambda \int_{Q_T} \frac{\partial\theta}{\partial x_j}\frac{\partial\eta}{\partial x_j} \, dxdt \\ -R\int_{Q_T} \rho\theta\frac{\partial v_j}{\partial x_j}\eta \, dxdt - c_v \int_{Q_T} \rho\theta v_j \frac{\partial\eta}{\partial x_j} \, dxdt = \int_{Q_T} \langle\langle v, v\rangle\rangle\eta \, dxdt\end{aligned} \tag{2.9}$$

for every $\eta \in C^\infty(Q_T)$, $\eta(T) = 0$, $\varphi \in C^\infty(Q_T)$, $\varphi(T) = 0$, $\varphi(t) \in W^{3,2}(\Omega)$, $\eta(t) \in W^{1,2}(\Omega)$, $t \in I$.

3. Modified Galerkin method.

The orthogonal basis $\{w^r\}_{r=1}^{+\infty}$ in V is given by solving the following eigenvalue problem

$$((v, w^r)) = \lambda_r \int_\Omega v_i w_i^r \, dx \tag{3.1}$$

for every $v \in V (0 < \lambda_1 \le \lambda_2 \le \dots)$. From the regularity of the elliptic systems (see [10]) we obtain

$$w^r \in C^\infty(\overline{Q}, \mathbb{R}^N). \tag{3.2}$$

Let $\gamma = (\gamma_1, \dots, \gamma_n) \in C^1(\overline{I}, \mathbb{R}^N)$ $(n = 1, 2, \dots)$ and put

$$v^n(x, t) = \sum_{r=1}^{n} \gamma_r(t) w^r. \tag{3.2$'$}$$

Let

$$\rho_0 \in C^d(\overline{\Omega}), \ \rho_0 > 0 \quad \text{in} \quad \Omega, d = 1, 2, \dots , \tag{3.3}$$

$$v_0 \in C(\overline{Q}_T), \tag{3.4}$$

$$\theta_0 \in L^2(\Omega), \quad \theta > 0 \quad \text{a.e. in} \quad \Omega, \tag{3.5}$$

$$F \in L^\infty(Q_T, \mathbb{R}^N). \tag{3.6}$$

We refer to [9] that there exists a solution

$$\rho^n \in C^1(\overline{Q}_T), \quad \rho^n > 0 \quad \text{in } Q_T, \ v^n = \sum_{r=1}^{n} \gamma_r(t) w^r, \ \gamma = (\gamma_1, \dots, \gamma_n),$$

$\gamma \in C^1(\overline{I}, \mathbb{R}^N)$, $\theta^n \in C^1(\overline{Q}_T) \cap C(I, C^2(\overline{Q}))$, $\theta^n \ge 0$ in $\overline{Q}_T$, satisfying the equations

$$\frac{\partial \rho^n}{\partial t} + \frac{\partial}{\partial x_i}(\rho^n v_i^n) = 0 \, ; \tag{3.7}$$

$$\frac{\partial}{\partial t}(c_v \rho^n \theta^n) + \frac{\partial}{\partial x_j}(c_v \rho^n \theta^n v_j^n) - \lambda \frac{\partial^2 \theta^n}{\partial x_j \partial x_j} = \tag{3.8}$$

$$= -R \rho^n \theta^n \frac{\partial v_j^n}{\partial x_j} + \langle\langle v^n, v^n \rangle\rangle \, ;$$

(3.8) can be written in the equivalent form

$$(3.8)' \quad \frac{\partial}{\partial t}\left(c_v \rho^n \theta^n + \frac{1}{2}\rho^n \mid v^n \mid^2\right) + \frac{\partial}{\partial x_j}\left(c_p \rho^n \theta^n v_j^n + \frac{1}{2}\rho^n \mid v^n \mid^2 v_j^n\right)$$

$$+ \frac{\partial}{\partial x_i}(B_1 \rho_n^2) v_i - \frac{\partial}{\partial x_j}\left(\tau_{ij}^{\nu}(v^n) v_i^n + \sum_{m=1}^{k} \tau_{ii_1\dots i_m j}^{\nu}(v^n) \frac{\partial^m v_i^n}{\partial x_{i_1}\dots\partial x_{i_m}}\right)$$

$$= \lambda \frac{\partial^2 \theta^n}{\partial x_j \partial x_j} + \rho^n F_i v_i^n ;$$

and

$$(3.9) \quad \int_\Omega \left(\rho^n \frac{\partial v_i^n}{\partial t} + \rho^n v_j^n \frac{\partial v_i^n}{\partial x_j} + \frac{\partial}{\partial x_j}\left\{R\rho^n\theta^n + B_1\rho_n^2\right\}\delta_{ij} - \rho^n F_i\right) w_i^r \, dx =$$

$$= -((v^n, w^r)), r = 1, \dots, n.$$

We have the following initial conditions

$$(3.10) \qquad \rho^n(0) = \rho_0 ,$$

$$(3.11) \qquad \theta^n(0) = \theta_0^n , \quad \frac{\partial \theta^n}{\partial \nu} = 0 \text{ on } \partial\Omega \times I,$$

where $\theta_0^n \in C^2(\overline{Q})$, $\theta_0^n > 0$ in $\overline{\Omega}$, $\partial\theta_0^n/\partial\nu = 0$ on $\partial\Omega, \theta_0^n \to \theta_0$ strongly in $L^2(\Omega)$,

$$(3.12) \qquad \int_\Omega v_i^n(0) w_i^r \, dx = \int_0 v_{0i} w_i^r \, dx.$$

We solve the continuity equation by the method of characteristic to have

$$(3.13) \qquad \rho^n(t, x) = \rho_0(x) \exp\left(-\int_0^t \frac{\partial}{\partial x_j} v_j^n(\tau, x^n(\tau))\, d\tau\right),$$

where $x = x^n(0)$, $y = x^n(t)$; the characteristic $x^n(\tau)$ are solutions to the problem

$$(3.14) \qquad \dot{x}^n(t) = v^n\left(t, x^n(\tau)\right), \quad \tau \in (0, T), x^n(0) = x \in \overline{\Omega}.$$

$x \to x^n(\tau)$ is a diffeomorphism from $\overline{\Omega}$ onto $\overline{\Omega}$ (for more details see [9]). Now, integrating (3.7) over Q_T, using (1.6) we get

$$\int_{\Omega_t} \rho^n \, dx = \int_{\Omega_0} \rho_0 \, dx. \tag{3.15}$$

Multiplying (3.7) by ρ_n we obtain

$$\int_{\Omega} (\rho^n)_t^2 = -\int_{\Omega} (\rho^n)^2 \nabla v \, . \tag{3.15$'$}$$

(i) We multiply (3.9) by γ_r, add from $r = 1$ to n and integrate over $I = (0, T)$.
(ii) We integrate (3.8) over Q_T.
Adding (i), (ii), using (3.15)′ it follows that

$$\frac{1}{2}\int_{\Omega_t} \rho^n |v^n|^2 \, dx - \frac{1}{2}\int_{\Omega_0} \rho_0 |v_0|^2 \, dx + \int_0^T ((v^n, v^n)) \, dt + \tag{3.16}$$

$$+ B_1 \int_{\Omega_t} \rho_n^2 \, dx - B_1 \int_{\Omega_0} \rho_0^2 \, dx - \int_{Q_T} (R\rho^n\theta^n) \frac{\partial v_i^n}{\partial x_i} \, dxdt = \int_{Q_T} \rho^n F_i v_i^n \, dxdt.$$

Integrating (3.8)′ over Q_T, using (3.15)′, we have

$$\int_{Q_t} c_v \rho^n \theta^n - \int_{\Omega_0} c_v \rho_0^n \theta_0^n + B_1 \int_{\Omega_t} \rho_n^2 - B_1 \int_{\Omega_0} \rho_0^2 + \int_{\Omega_t} \frac{1}{2} \rho^n |v^n|^2 - \tag{3.17}$$

$$- \int_{\Omega_0} \rho^n |v^n|^2 = \int_{Q_t} \rho^n F_i v_i^n.$$

From (3.15), (3.16), (3.17) we get the following estimates

$$\|\rho^n \mid v^n \mid^2 \|_{L^\infty(I, L^1(\Omega))} \leq K_2, \quad K_2 > 0, \tag{3.18}$$

$$\|\rho^n \theta^n\|_{L^\infty(I, L^1(\Omega))} \leq K_1, \quad K_1 > 0, \tag{3.19}$$

$$\|\rho^n\|_{L^\infty(I, L^1(\Omega))} \leq K_3, \quad K_3 > 0, \tag{3.20}$$

$$\|v^n\|_{L^2(I, W^{k,2}(\Omega))} \leq K_2, \tag{3.21}$$

$$\|\rho^n\|_{L^\infty(I, L^2(\Omega))} \leq K_1. \tag{3.22}$$

Similarly as in [9] we obtain

Lemma 3.2. *Let* $k = 3$, $1 \le q < +\infty$ $(N = 2)$, $1 \le q \le 6$ $(N = 3)$, $0 < \alpha < 1/2$ *and let* (3.3) - (3.6) *be satisfied. Then*

$$\|\rho^n\|_{L^\infty(I,W^{1,q}(\Omega))} \le K_5, \tag{3.23}$$

$$\left\|\frac{\partial \rho^n}{\partial t}\right\|_{L^2(I,L^q(\Omega))} \le K_5. \tag{3.24}$$

$$\|\rho^n\|_{C^{0,\alpha}(\bar I,W^{1,q}(\Omega))} \le K_5,\ 0 < \alpha < 1/2. \tag{3.24$'$}$$

Also,

$$\theta^n \ge 0 \quad \text{a.e. in} \quad Q_T \tag{3.25}$$

holds.

Proof. See [9].

Now we assume that $v_0 \in W^{k,2}(\Omega, \mathbb{R}^N) \cap W_0^{1,2}(\Omega, \mathbb{R}^N)$. Using $\varphi = \partial v^n/\partial t$ as the test function in (2.8) we obtain

$$\int_{Q_t} \frac{\partial}{\partial t}(\rho^n v_i^n)\frac{\partial v_i^n}{\partial t} + \frac{\partial}{\partial x_j}(\rho^n v_i^n v_j^n + p(\rho^n, \theta^n)\delta_{ij} - \tau_{ij}^{\nu}(v^n))\frac{\partial v_i^n}{\partial t} = \tag{3.26}$$

$$= \int_{Q_T} \rho^n F_i \frac{\partial v_i^n}{\partial t}.$$

From (3.26) it follows that

$$\int_{Q_T} \rho^n \frac{\partial v_i^{n^2}}{\partial t} + \rho^n v_j^n \frac{\partial v_i^n}{\partial t}\frac{\partial v_i^n}{\partial x_j} + \frac{\partial}{\partial x_i}(R\rho^n\theta^n)\frac{\partial v_i^n}{\partial t} + \frac{\partial}{\partial x_i}(B_1\rho_n^2)\frac{\partial v_i^n}{\partial t} \tag{3.27}$$

$$+ \int_0^T \left(\left(v^n, \frac{\partial v^n}{\partial t}\right)\right) = \int_{Q_T} \rho^n F_i \frac{\partial v_i^n}{\partial t},$$

$$\int_\Omega \frac{\partial}{\partial x_i}(R\rho^n\theta^n)\frac{\partial v_i^n}{\partial t} = R\int_\Omega \rho^n \frac{\partial \theta^n}{\partial x_i}\frac{\partial v_i^n}{\partial t} + \int_\Omega R\frac{\partial \rho^n}{\partial x_i}\theta^n \frac{\partial v_i^n}{\partial t},$$

$$\left|\int_\Omega \frac{\partial \rho^n}{\partial x_i}\theta^n \frac{\partial v_i^n}{\partial t}\right| \le \frac{1}{\varepsilon}\left(\int_\Omega \left|\frac{\partial \rho^n}{\partial x_i}\right|^6\right)^{1/6}\left(\int_\Omega |\theta^n|^3\right)^{1/3} + \varepsilon\int_\Omega \left|\frac{\partial v_i^n}{\partial t}\right|^2,$$

$$\left|\int_\Omega \rho^n \frac{\partial \theta^n}{\partial x_i}\frac{\partial v_i^n}{\partial t}\right| \le \frac{c}{\varepsilon_1}\|\theta^n\|^2_{W^{1,2}(\Omega)} + c\varepsilon_1 \left\|\frac{\partial v^n}{\partial t}\right\|^2_{L_2(\Omega)},$$

$$\left|\int_\Omega 2\rho^n \frac{\partial \rho^n}{\partial x_i}\frac{\partial v_i^n}{\partial t}\right| \le c\left\|\frac{\partial v^n}{\partial t}\right\|_{L^2(\Omega)} \|\rho^n\|_{L^\infty(\Omega)}\|\rho^n\|_{W^{1,2}(\Omega)},$$

$$\left|\int_\Omega \rho^n v_j^n \frac{\partial v_i^n}{\partial t}\frac{\partial v_i^n}{\partial x_j}\right| \le |\rho^n|_{L^\infty(\Omega)}\|v^n\|_{W^{k,2}(\Omega)}\left\|\frac{\partial v^n}{\partial t}\right\|_{L^2(\Omega)}\|v^n\|_{W^{1,2}(\Omega)}.$$

And thus we have

$$\left\|\frac{\partial v^n}{\partial t}\right\|^2_{L^2(Q_T,\mathbb{R}^N)} + \sup_{t\in[0,T]}((v^n(t), v^n(t))) \le K_5(1 + \|\theta^n\|^2_{L^2(I,W^{1,2}(\Omega))}). \tag{3.28}$$

From the regularity of the elliptic system we get

$$\|v^n\|_{L^2(I,W^{2k,2}(\Omega))} \le K_5\big(1 + \|\theta^n\|_{L^2(I,W^{1,2}(\Omega))}\big). \tag{3.29}$$

Remark. This estimate plays an essential role in the proof of the uniqueness.

Lemma 3.3.

$$\|\theta^n\|^2_{L^2(I,W^{1,2}(\Omega))} + \|\theta^n\|_{L^\infty(I,L^2(\Omega))} \le K_6,\ K_6 \ge 0. \tag{3.30}$$

Proof. We multiply (3.8) by θ^n and integrate over Ω.

$$\int_\Omega \frac{\partial}{\partial t}(c_v\rho^n\theta^n)\theta^n + \frac{\partial}{\partial x_j}(c_v\rho^n\theta^n v_j^n)\theta^n - \int_\Omega \lambda\frac{\partial^2\theta^n}{\partial x_j\partial x_j}\theta^n \tag{3.31}$$

$$= \int_\Omega R\rho^n(\theta^n)^2\frac{\partial v_j^n}{\partial x_j} + \int_\Omega ((v^n, v^n))\theta^n.$$

We rewrite the left-hand side of (3.31). We obtain

$$\int_\Omega \left(c_v\frac{\partial \rho^n}{\partial t}(\theta^n)^2 + c_v\rho^n\frac{\partial\theta^n}{\partial t}\theta^n + c_v\frac{\partial(\rho^n v_j^n)}{\partial x_j}(\theta^n)^2 + c_v\frac{\partial\theta^n}{\partial x_j}\theta^n\rho^n v_j^n\right)dx$$

$$+ \int_\Omega \lambda\frac{\partial\theta^n}{\partial x_j}\frac{\partial\theta^n}{\partial x_j}dx \overset{(3.7)}{=} \frac{1}{2}\int_\Omega\left(c_v\rho^n\frac{\partial}{\partial t}(\theta^n)^2 + c_v\rho^n v_j^n\frac{\partial}{\partial x_j}(\theta^n)^2\right)dx$$

$$+ \int_\Omega \lambda\frac{\partial\theta^n}{\partial x_j}\frac{\partial\theta^n}{\partial x_j}dx = \frac{d}{dt}\int_\Omega \frac{c_v}{2}\rho^n(\theta^n)^2dx + \int_\Omega \lambda\frac{\partial\theta^n}{\partial x_j}\frac{\partial\theta^n}{\partial x_j}dx.$$

Thus we obtain

$$\frac{d}{dt}\left(\frac{1}{2}c_v\int_\Omega \rho^n(\theta^n)^2dx\right)+\lambda\int_\Omega \frac{\partial\theta^n}{\partial x_j}\frac{\partial\theta^n}{\partial x_j}\,dx \tag{3.32}$$

$$= R\int_\Omega \rho^n\frac{\partial v_j^n}{\partial x_j}(\theta^n)^2dx+\int_\Omega \langle\langle v^n, v^n\rangle\rangle\theta^n dx\,.$$

We known that the first term on the right-hand side is bounded by $\|\theta^n\|^2_{L^2(\Omega)}$. The term $\int_\Omega\langle\langle v^n, v^n\rangle\rangle\theta^n dx$ is more complicated. We use the relation

$$\|v^n\|_{C^k(\overline{\Omega},\mathbb{R}^N)} \leq \varepsilon_1\|v^n\|_{W^{2k,2}(\Omega,\mathbb{R}^N)}+\overline{K}_1(\varepsilon_1)\|v^n\|_{L^2(\Omega,\mathbb{R}^N)} \tag{3.33}$$

for every $\varepsilon_1>0(\overline{K}_1(\varepsilon_1)>0)$, see [3].

$$\left(\int_{\Omega_t}\langle\langle v^n, v^n\rangle\rangle^2dx\right)^{1/2}\|\theta^n\|_{L^2(\Omega)} \tag{3.34}$$

$$\leq c_2\varepsilon_1\|v^n(t)\|_{W^{k,2}}\|v^n(t)\|_{W^{2k,2}}\|\theta^n\|_{L^2}+\overline{K}_2(\varepsilon_1)\|\theta^n\|_{L^2},\ \ c_2,\overline{K}_2(\varepsilon_1)>0.$$

Hence, due to (3.29) and the Young inequality, (3.34) on the right-hand side is bounded by

$$K(\varepsilon_1)\|v^n(t)\|^2_{W^{k,2}}\|\theta^n\|^2_{L^2}+\varepsilon_1\|\theta^n\|^2_{W^{1,2}}+K(\varepsilon_1),\ \ K(\varepsilon_1)>0. \tag{3.34$'$}$$

Now we use Gronwall's lemma and obtain (3.30).

Thus we have

Lemma 3.4. *Let* $v_0\in W^{k,2}(\Omega,\mathbb{R}^N)\cap W_0^{1,2}(\Omega,\mathbb{R}^N)$. *Then*

$$\left\|\frac{\partial v^n}{\partial t}\right\|^2_{L^2(Q_t,\mathbb{R}^N)}+\sup_{[0,T]}((v^n, v^n))\leq K_{16} \tag{3.35}$$

$$\|v^n\|_{L^2(I,W^{2k,2}(\Omega,\mathbb{R}^N))}\leq K_{16},\quad K_{16}>0. \tag{3.36}$$

Lemma 3.5.

$$\left\|\frac{\partial}{\partial t}(\rho^n\theta^n)\right\|_{L^2(I,W^{-1,2}(\Omega,\mathbb{R}^N))}\leq K_{17}, \tag{3.37}$$

$$\|\rho^n\theta^n\|_{L^2(I,W^{1,2}(\Omega,\mathbb{R}^N)}\leq K_{17},\quad K_{17}>0. \tag{3.38}$$

Proof. The estimate (3.38) is a consequence of Lemma 3.3 and (3.23). (3.37) follows from (3.8).

Lemma 3.6. *Let $\theta_0 \in W^{1,2}(\Omega)$. Then*

$$\left\| \frac{\partial \theta^n}{\partial t} \right\|_{L^2(Q_T)} + \|\theta^n\|^2_{L^\infty(I, W^{1,2}(\Omega))} \leq K_{18}, \quad K_{18} > 0, \tag{3.39}$$

$$\|\theta^n\|_{L^2(I, W^{2,2}(\Omega))} \leq K_{18}. \tag{3.40}$$

Proof. Let us choose a sequence $\theta_0^n \in C^2(\overline{\Omega})$, $\theta_0^n > 0$, $\theta_0^n \to \theta_0$ strongly in $W^{1,2}(\Omega)$, $\partial\theta_0^n/\partial\nu = 0$ on $\partial\Omega$. Multiplying (3.8) by $\partial\theta^n/\partial t$ and integrating over $Q_t = (0, t) \times \Omega$, $t \in I$, we get

$$c_v \int_{Q_T} \rho^n \left(\frac{\partial \theta^n}{\partial t}\right)^2 dxdt + \frac{\lambda}{2} \int_\Omega \frac{\partial \theta^n}{\partial x_j} \frac{\partial \theta n}{\partial x_j} dx = \tag{3.41}$$

$$-c_v \int_{Q_T} \rho^n v_j^n \frac{\partial \theta^n}{\partial x_j} \frac{\partial \theta^n}{\partial t} dxdt - \int_{\Omega_T} R\rho^n\theta^n \frac{\partial v_j^n}{\partial x_j} \frac{\partial \theta^n}{\partial t} dxdt +$$

$$+ \int_{Q_T} \langle\langle v^n, v^n \rangle\rangle \frac{\partial \theta^n}{\partial t} dxdt + \frac{\lambda}{2} \int_{\Omega_0} \frac{\partial \theta_0^n}{\partial x_j} \frac{\partial \theta_0^n}{\partial x_j} dx.$$

The right-hand side of (3.41) is bounded by $K_{20}(1 + \left\| \frac{\partial \theta^n}{\partial t} \right\|_{L^2(Q_T)})$, $K_{20} > 0$. We use the same technique for the estimate of the term $\langle\langle v^n, v^n \rangle\rangle \frac{\partial \theta^n}{\partial t}$ as in Lemma 3.3. Applying Young's inequality we verify (3.39). (3.40) is a consequence of the regularity properties to elliptic equations.

4.1. The global existence of a weak solution.

Lemma 4.1. *Let $\widetilde{B}_0$, $\widetilde{B}$, $\widetilde{B}_1$ be Banach spaces, $\widetilde{B}_0$, $\widetilde{B}_1$ reflexive such that $\widetilde{B}_0 \subset\subset \widetilde{B} \subset \widetilde{B}_1$ ($\subset\subset$ is a compact imbedding), let $1 < p_0, p_1 < +\infty$. Then $L^{p_0}(I, \widetilde{B}) \subset\subset \{g; g \in L^{p_0}(I, \widetilde{B}_0), \partial g/\partial t \in L^{p_1}(I, \widetilde{B}_1)\}$.*

Proof. See [3].

Lemma 4.2. *Let the assumptions* (3.2)-(3.6), $v_0 \in W^{k,2}(\Omega, \mathbb{R}^N) \cap W^{1,2}(\Omega, \mathbb{R}^N)$ *be satisfied. Then one can choose a subsequences of* $\{(\rho^n, v^n, \theta^n)\}_{n=1}^{\infty}$ *(denoted* $\{(\rho^n, v^n, \theta^n)\}$ *again) such that*

(i) $\rho^n \to \rho$ *strongly in* $L^p(Q_T)$, $1 < p < +\infty$, $\rho > \varepsilon > 0$ *a.e. in* Q_T*;*

(ii) $v^n \to v$ *strongly in* $L^2(I, W^{2k-1,2}(\Omega, \mathbb{R}^N))$,
$v^n \to v$ *strongly in* $L^p(I, W^{k-1,2}(\Omega, \mathbb{R}^N))$,
$D^i v^n \rightharpoonup D^i v$ *weakly in* $L^2(Q_T, \mathbb{R}^N)$ $(i = 1, \ldots, 2k)$*;*

(iii) $D^i\theta^n \rightharpoonup D^i\theta$ *weakly in* $L^2(Q_T)$*;*

(iv) $\theta \geq 0$ *a.e. in* Q_T,
$\rho^n\theta^n \to \rho\theta$ *strongly in* $L^2(Q_T)$*;*

(v) $\dfrac{\partial \rho^n}{\partial t} \rightharpoonup \dfrac{\partial \rho}{\partial t}$ *weakly in* $L^2(Q_T)$;

(vi) $\dfrac{\partial v^n}{\partial t} \rightharpoonup \dfrac{\partial v}{\partial t}$ *weakly in* $L^2(Q_T)$*;*

(vii) $\rho^n v^n \to \rho v$ *strongly in* $L^2(Q_T)$*;*

(viii) $\rho^n\theta^n \dfrac{\partial v_j^n}{\partial x_j} \rightharpoonup \rho^n\theta^n \dfrac{\partial v_j}{\partial x_j}$ *weakly in* $L^2(Q_T)$;

(ix) $\rho^n v_i^n v_j^n \rightharpoonup \rho v_i v_j$ *weakly in* $L^2(Q_T)$*;*

(x) $\displaystyle\int_{Q_T} ((v^n, v^n))\phi\, dx d\tau \to \int_{Q_T} ((v, v))\phi\, dx d\tau$, $\phi \in C^1(\overline{Q}_T)$.

Proof.

(i) this assertion follows from Lemma 4.1, where we put $\widetilde{B}_0 = W^{1,6}$, $\widetilde{B} = L^p$, $\widetilde{B}_1 = L^q$, $1 < p < +\infty$, $p_0 = p$, $p_1 = 2$, $1 \leq q \leq 6$ $(N = 3)$, $1 \leq q < +\infty$ $(N = 2)$;

(ii) the first two assertions follow from Lemma 4.1 with $\widetilde{B}_0 = W^{2k,2}$, $\widetilde{B} = W^{2k-1,2}$, $\widetilde{B}_1 = L^2$, $p_0 = p_1 = 2$ and $\widetilde{B}_0 = W^{k,2}$, $\widetilde{B} = W^{k-1,2}$, $\widetilde{B}_1 = L^2$, $\rho_0 = p$, $p_1 = 2$ respectively; the last assertion of (ii) follows from the boundedness of $\{v^n\}_{n=1}^{\infty}$ in $L^2(I, W^{2k,2}(\Omega, \mathbb{R}^N))$;

(iii) is a consequence of Lemma 3.3;

(iv) we obtain that $\rho^n\theta^n \to a$ (we use Lemma 4.1 with $\widetilde{B} = L^2$, $\widetilde{B}_1 = W^{-1,2}$, $\widetilde{B}_0 = W^{1,2}$, $p_0 = p_1 = 2$) and that $a = \rho\theta$ we get form (i), (iii), first assertion of (iv) holds due to (3.25);

(v) is a consequence of (3.24);

(vi) is a consequence of (3.35);

(vii) is a consequence of (i) and (ii);

(viii) is a consequence of (ii) and (iv);

(ix) follows from (ii), (i), (vii);

(x) follows from (ii).

Due to Lemma 4.1 we can pass to the limit in (3.7), (3.8), (3.9) to obtain the following theorems

Theorem 4.1 (weak solution). *Let* $\rho_0 \in C^d(\overline{Q})$, $d = 1, 2, \dots, \rho_0 > 0$ *in* $\overline{Q}, \theta_0 \in L^2(\Omega)$, $\theta_0 > 0$ *a.e. in* Ω, $v_0 \in W^{k,2}(\Omega, \mathbb{R}^N) \cap W_0^{1,2}(\Omega, \mathbb{R}^N)$. *Let* $1 \le q < +\infty$ $(N = 2)$ *or* $1 \le q \le 6$ $(N = 3)$, $0 < \alpha < 1/2$ *then there exists* (ρ, v, θ) *such that*
(4.1) $\rho \in L^\infty(I, W^{1,q}(\Omega)) \cap C^{0,\alpha}(\overline{I}, L^q(\Omega))$, $\rho > \varepsilon$ *a.e. in* Q_T *for some* $\varepsilon > 0$,
(4.2) $\dfrac{\partial \rho}{\partial t} \in L^\infty(I, L^q(\Omega))$,
(4.3) $v \in L^\infty(I, W^{k,2}(\Omega, \mathbb{R}^N) \cap W_0^{1,2}(\Omega, \mathbb{R}^N)) \cap L^2(I, W^{2k,2}(\Omega, \mathbb{R}^N))$,
(4.4) $\dfrac{\partial v}{\partial t} \in L^2(Q_T, \mathbb{R}^N)$,
(4.5) $\theta \in L^\infty(I, L^2(\Omega)) \cap L^2(I, W^{1,2}(\Omega))$, $\theta \ge 0$ *a.e. in* Q_T,
such that (1.2),(1.3) *holds a.e. in* Q_T *and* (2.12) *is fulfilled; if* $d > 1$ *then*
(4.6) $\rho \in L^\infty(I, W^{p,q}(\Omega)) \cap C^{0,\alpha}(\overline{I}, W^{p-1,q}(\Omega))$,
(4.7) $\dfrac{\partial \rho}{\partial t} \in L^2(I, W^{p-1,q}(\Omega))$,
where $p = \min(d, 4)$.

Theorem 4.2 (strong solution). *Let the assumptions of Theorem 4.1 be satisfied and let* $\theta \in W^{1,2}(\Omega)$. *Then there exists* (ρ, v, θ) *satisfying* (4.1)-(4.5) *and*
(4.8) $\theta \in L^2(I, W^{2,2}(\Omega)) \cap L^\infty(I, W^{1,2}(\Omega))$,
(4.9) $\dfrac{\partial \theta}{\partial t} \in L^2(Q_T)$,
such that the equations (1.2)-(1.4) *are satisfied a.e. in* Q_T.

4.2. Uniqueness.

Theorem 4.3. *Let the assumptions of Theorem 4.1 with* $d \ge 2$ *be satisfied. Then in the class* (4.1)-(4.9) *there exists at most one solution* (ρ, v, θ) *satisfying* (1.2)-(1.4) *a.e. in* Q_T.

Proof. Let (ρ, v, θ), $(\overline{\rho}, \overline{v}, \overline{\theta})$ be two solutions with the same initial conditions. Then $(\xi, w, \eta) = (\rho - \overline{\rho}, v - \overline{v}, \theta - \overline{\theta})$ satisfies

$$\frac{\partial}{\partial t}\xi + \frac{\partial}{\partial x_j}(\xi v_j) + \frac{\partial}{\partial x_j}(\overline{\rho} w_j) = 0 \quad \text{a.e. in} \quad Q_T, \tag{4.10}$$

$$\overline{\rho}\frac{\partial w_i}{\partial t} + \xi v_j \frac{\partial v_i}{\partial x_j} + \overline{\rho}\overline{v}_j \frac{\partial w_i}{\partial x_j} + \overline{\rho} w_j \frac{\partial v_i}{\partial x_j} - \frac{\partial}{\partial x_j}((\tau_{i,j}^v(w)) + \tag{4.11}$$

$$R\frac{\partial}{\partial x_i}(\xi\theta) + R\frac{\partial}{\partial x_i}(\overline{\rho}\eta) + B_1\frac{\partial}{\partial x_i}(\rho^2 - \overline{\rho}^2) + \xi\frac{\partial v_i}{\partial t} = \xi F_i \quad \text{a.e. in} \quad Q_T,$$

(4.12)
$$c_v\overline{\rho}\frac{\partial \eta}{\partial t} + c_v\xi v_j\frac{\partial \theta}{\partial x_j} + c_v\overline{\rho}v_j\frac{\partial \eta}{\partial x_j} + c_v\overline{\rho}\frac{\partial \overline{\theta}}{\partial x_j}w_j -$$
$$\lambda\frac{\partial^2 \eta}{\partial x_j \partial x_j} + R\overline{\rho}\frac{\partial v_j}{\partial x_j}\eta + R\overline{\rho}\overline{\theta}\frac{\partial w_j}{\partial x_j} + R\theta\frac{\partial v_j}{\partial x_j}\xi +$$
$$+ c_v\xi\frac{\partial \theta}{\partial t} = \langle\langle v, v\rangle\rangle - \langle\langle \overline{v}, \overline{v}\rangle\rangle \quad \text{a.e. in} \quad Q_T.$$

From (4.10) one obtains the estimates

(4.13)
$$\frac{\partial}{\partial t}\|\xi(t)\|^2_{W^{1,2}(\Omega)} \leq \overline{K}(\varepsilon_1)a_1(t)\|\xi(t)\|^2_{W^{1,2}(\Omega)} + \varepsilon_1\|w(t)\|^2_{W^{k,2}(\Omega,\mathbb{R}^N)},$$

$K_4(\varepsilon_1) > 0$, where

$$a_1 = (1 + \|w\|_{W^{k,2}(\Omega,\mathbb{R}^N)} + \|\rho\|_{W^{2,6}(\Omega)})^2$$

for every $\varepsilon_1 > 0$, $a_1 \in L^\infty(I)$. From (4.11) we obtain

(4.14)
$$\frac{\partial}{\partial t}\left(\int_\Omega \tilde{\rho} \mid w \mid^2 dx\right) + \|w(t)\|^2_{W^{k,2}(\Omega,\mathbb{R}^N)} \leq$$
$$\leq K(\varepsilon)a(t)\left(\|\xi(t)\|^2_{W^{1,2}(\Omega)} + \|w(t)\|^2_{L^2(\Omega,\mathbb{R}^N)} + \|\eta(t)^2_{L^2(\Omega)}\right) + \varepsilon_1\|w(t)\|^2_{W^{k,2}(\Omega,\mathbb{R}^N)}$$

($\overline{K}(\varepsilon_1) > 0$) for a.e. $t \in I$ and every $\varepsilon_1 > 0$, where

$$a = \left(1 + \left\|\frac{\partial \overline{\rho}}{\partial t}\right\|_{L^6(\Omega)} + \|\overline{\rho}\|_{W^{1,6}(\Omega)} + \|\rho\|_{W^{1,6}(\Omega)}\right)^2 \cdot$$
$$\cdot\left(1 + \|\overline{v}\|_{W^{2k,2}(\omega,\mathbb{R}^N)} + \|v\|_{W^{2k,2}(\Omega,\mathbb{R}^N)} + \left\|\frac{\partial v}{\partial t}\right\|_{L^2(\Omega,\mathbb{R}^N)} + \right.$$
$$\left. + \left\|\frac{\partial \theta}{\partial t}\right\|_{L^2(\Omega,\mathbb{R}^N)}\right)^2\left(1 + \|\overline{\theta}\|_{W^{1,2}(\Omega)} + \|\theta\|_{W^{1,2}(\Omega)}\right)^2,$$

hence $a \in L^1(I)$. From (4.12) after some computation we get the estimate

(4.15)
$$\frac{\partial}{\partial t}\left(\int_{\Omega_t} \rho\eta^2 dx\right) + \|\nabla\eta(t)\|^2_{L^2} \leq$$
$$\overline{K}_2(\varepsilon_1)a(t)(\|\xi(t)\|^2_{W^{1,2}(\Omega)} + \|w(t)\|^2_{L^2(\Omega,\mathbb{R}^N)} + \|\eta(t)\|^2_{L^2}) +$$
$$+ \varepsilon_1(\|w(t)\|^2_{W^{k,2}(\Omega,\mathbb{R}^N)} + \|\nabla\eta(t)\|^2_{L^2})$$

for a.e. $t \in I$ and every $\varepsilon_1 > 0$. From (4.13)-(4.15), after applying Gronwall's lemma, we obtain $\xi = 0$, $w = 0$, $\eta = 0$ a.e. in I.

5. Non-homogeneous condition.

Now we are interested in the case of non-homogeneous boundary conditions. We consider that Ω is a finite channel, where $\partial\Omega = \Gamma_{\text{inp}} \cup \Gamma_{\text{out}} \cup \Gamma_r$. Besides the initial conditions

$$\rho(0) = \rho_0,\ v(0) = v_0,\ \theta(0) = \theta_0 \tag{5.1}$$

we consider the following boundary conditions:

$$\begin{aligned} v &= v_0 \text{ on } \Gamma_{\text{inp}} \cup \Gamma_{\text{out}}, \\ v &= 0 \text{ on } \Gamma_r, \\ &\int_{\Gamma_{\text{inp}}} v\nu > 0, \\ &\int_{\Gamma_{\text{out}}} v\nu < 0, \end{aligned} \tag{5.2}$$

where ν is the outer normal,

$$\begin{aligned} \rho &= \rho_0 \text{ on } \Gamma_{\text{inp}}, \\ \rho &> 0. \end{aligned}$$

Further, we consider that $\theta_0 > 0$; and that there exist constants m and M such that

$$0 \le m \le \theta(x,t) \le M \quad \text{on} \quad \partial\Omega \times I \quad \text{(see [5])}. \tag{5.2$'$}$$

We consider that v_0 is such a function that there exists the extension to the entire cylinder. The function is denoted again v_0. So we can write $v = v_0 + z$. (This means $z = 0$ on $\partial\Omega \times I$) Moreover, we assume the boundary conditions

$$[[z, w]] = 0 \quad \text{on} \quad \partial\Omega \times I, \quad \text{for every } w \in V \cap W^{2k,2}(\Omega, \mathbb{R}^N). \tag{5.3}$$

5.1. The formulation of the problem.

We consider the system (1.2)-(1.4). We state the weak formulation of the problem as follows

$$\int_{Q_T} \rho v_i \frac{\partial \varphi}{\partial t} + \int_{\Omega} \rho_0 v_{0i} \varphi(0) + \int_{Q_T} p(\rho, \theta) \frac{\partial \varphi}{\partial x_i} + \tag{5.5}$$

$$+ \int_0^T ((v, \varphi)) = \int_{Q_T} \rho F_i \varphi ,$$

$$\varphi(T) = 0, \ \varphi \in C^\infty(Q_T), \ \varphi = 0 \text{ on } \partial\Omega \times I, \ \varphi(t) \in W^{3,2}(\Omega);$$

$$\int_{Q_T} c_v \rho \theta \frac{\partial \eta}{\partial t} - \int_{Q_T} c_v \rho_0 \theta_0 \eta(0) + \int_0^T \int_{\partial\Omega} c_v \rho \theta v_j \eta \nu_j \tag{5.6}$$

$$- \int_{Q_T} c_v \theta \rho v_j \frac{\partial \eta}{\partial x_j} + \int_{Q_T} R \rho \theta \frac{\partial v_j}{\partial x_j} \eta + \int_{Q_T} \lambda \frac{\partial \theta}{\partial x_j} \frac{\partial \eta}{\partial x_j} = \int_{Q_T} \langle\langle v, w \rangle\rangle \eta ,$$

$$\eta(T) = 0, \eta \in C^1(Q_T), \eta(t) \in W^{1,2}(\Omega).$$

We apply the Galerkin method, we take the same orthogonal basis as in Section 3 and we solve the eigenvalue problem

$$((z, w^r)) = \lambda_r \int_{\Omega} v_i w_i^r \, dx \tag{5.7}$$

for every $z \in V = W^{k,2} \cap W_0^{1,2} (0 < \lambda_1 \leq \lambda_2 \leq \ldots)$. From the regularity of the elliptic system (see [10]) we obtain

$$w^r \in C^\infty(\overline{\Omega}, \mathbb{R}^N). \tag{5.7'}$$

Let

$$c = (c_1, \ldots, c_n) \in C^1(\overline{I}, \mathbb{R}^N) \quad (n = 1, 2, \ldots) \tag{5.8}$$

and we put

$$v^n(x, t) = \sum_{r=1}^{n} c_r(t) w^r(x) + v^0 \tag{5.9}$$

and let

$$v^0 \in C^5(\overline{Q}_T). \tag{5.9)'$$

Let (3.4)-(3.6) be satisfied and

$$\rho_0^n \in C(\overline{Q}),\ \rho_0^n > 0. \tag{5.9)''$$

Then there exists a solution satisfying (5.8), (5.9), $\rho^n \in C(\overline{Q}_T) \cap W^{1,\infty}(Q_T)$, $\rho^n > 0$ in Q_T, $\theta^n \in C(\overline{Q}_T) \cap C(I, C^2(\overline{\Omega}))$, $\theta^n \geq 0$ in Q_T satisfying equations (3.7)-(3.12). We solve the continuity equation by the method of characteristics and obtain

$$\rho^n(t,x) = \rho_0\left(t - \tilde{t}, x^n(\tilde{t})\right) \exp\left(-\int_0^{\tilde{t}} \frac{\partial}{\partial x_j} v_j^n\left(t - \tau, x^n(\tau)\right) d\tau\right), \tag{5.10}$$

where $x = x^n(0)$, $y = x^n(\tilde{t})$; the characteristics $x^n(\tau)$ are solution to the problem

$$x^n(t) = -v^n(t - \tau, x^n(\tau)),\ \tau \in (0, T),\ x^n(0) = x \in \overline{\Omega}. \tag{5.10)'$$

($\tilde{t}$ denotes the time which we achieve when going from the point $[x, t]$ to the point which lies in Ω_0 or Γ_{inp}). For every $\tau \in I_{\tilde{t}}$, $I_{\tilde{t}} = (0, \tilde{t})$, $\tilde{t} > 0$, $x \to x^n(\tau)$ is a local diffeomorphism of $\overline{\Omega}$ onto $\overline{\Omega}$. More precisely see [8]. We integrate (3.7) over Q_T to have

$$\int_{\Omega_t} \rho^n \leq \int_{\Omega_0} \rho^n - \int_0^T \int_{\Gamma_{\text{inp}}} \rho_0^n v_j^n \nu_j. \tag{5.11}$$

From (3.8)′ after integrating over Q_T and using the continuity equation we get

$$\int_{Q_T} c_v \rho^n \theta^n + \frac{1}{2} \int_{\Omega_T} \rho^n |v^n|^2 - \int_{\Omega_T} B_1 \rho_n^2 \leq \tag{5.12}$$

$$\leq -\int_0^T \int_{\Gamma_{\text{inp}}} \left(c_p \rho_0^n \theta_0^n v_0^n \nu_j + \frac{1}{2} \rho^n |v^n|^2 v_j^n \nu_j \right) + \int_0^T \int_{\Gamma_{\text{out}}} B_1 \rho_n^2 v_{0i}^n \nu_i$$

$$+ \int_{Q_T} \rho^n F_i v_i^n + \int_{Q_T} \langle\langle v_0, v_0 \rangle\rangle.$$

Now we put $\varphi = z^n$ in (5.5). So we obtain

$$\frac{1}{2}\int_{Q_T} \rho^n |z^n|^2 + \int_{Q_T} \left(\rho^n \frac{\partial v_{0i}}{\partial t} z_j^n + \rho^n v_j^n z_i^n \frac{\partial v_{0i}}{\partial x_j} \right) \tag{5.13}$$

$$+ \int_0^T \int_{\partial\Omega} B_1 \rho^2 v_i v_j + \int_{Q_T} \rho_n^2 - \int_{\Omega_0} \rho_n^2 + \int_{Q_T} B_1 \rho_n^2 \frac{\partial v_i^0}{\partial x_i} - \int_{Q_T} \rho^n F_i v_i^n$$
$$= -\int_0^T ((v^n, z^n)).$$

Thus we obtain

$$\|v^n\|_{L^2(I, W^{k,2}(\Omega))} \le k_1, \tag{5.14}$$

$$\|\rho^n\|_{L^\infty(I, L^2(\Omega))} \le k_2. \tag{5.15}$$

From (5.14) and (5.10) we obtain

$$\|\rho^n\|_{L^\infty(I, W^{p,q}(\Omega))} \le k_3, \tag{5.16}$$

$(N = 2,\ 1 \le q \le +\infty,\ N = 3,\ 1 \le q \le 6,\ p = k - 2)$
and

$$\left\| \frac{\partial \rho^n}{\partial t} \right\|_{L^2(I, W^{p-1,q}(\Omega))} \le k_4. \tag{5.17}$$

Now we use $\partial v^n / \partial t$ as the test function in (5.5) and we obtain (3.28), (3.29). Analogously as in Lemma 3.3 we obtain (3.30), (3.37), (3.38).

Theorem 5.1. *Let the assumptions* (3.2) - (3.6), (5.9)′, (5.9)″,

$$v_0 \in W^{k,2}(\Omega, \mathbb{R}^N) \cap W^{1,2}(\Omega, \mathbb{R}^N)$$

be satisfied. Then one can choose a subsequence of $\{(\rho^n, v^n, \theta^n)\}_{n=1}^{+\infty}$ *(denoted* $\{(\rho^n, v^n, \theta^n)\}$ *again) such that*

(i) $\rho^n \to \rho$ *strongly in* $L^p(Q_T)$, $1 < p < +\infty$, $\rho > \varepsilon > 0$ *in* Q_T*;*

(ii) $v^n \to v$ *strongly in* $L^2(I, W^{2k-1,2}(\Omega, \mathbb{R}^N))$,
$v^n \to v$ *strongly in* $L^p(I, W^{k-1,2}(\Omega, \mathbb{R}^N))$,
$D^i v^n \to D^i v$ *weakly in* $L^2(Q_T, \mathbb{R}^N)$ $(i = 1, \ldots, 2k)$*;*

(iii) $D^i \theta^n \to D^i \theta$ *weakly in* $L^2(Q_T)$*;*

(iv) $\theta \geq 0$ *a.e. in* Q_T,
$\rho^n\theta^n \to \rho\theta$ *strongly in* $L^2(Q_T)$;

(v) $\dfrac{\partial \rho^n}{\partial t} \to \dfrac{\partial \rho}{\partial t}$ *weakly in* $L^2(Q_T)$;

(vi) $\dfrac{\partial v^n}{\partial t} \to \dfrac{\partial v}{\partial t}$ *weakly in* $L^2(Q_T)$;

(vii) $\rho^n v^n \to \rho v$ *strongly in* $L^2(Q_T)$;

(viii) $\rho^n\theta^n \dfrac{\partial v_j^n}{\partial x_j} \to \rho\theta \dfrac{\partial v_j}{\partial x_j}$ *weakly in* $L^2(Q_T)$;

(ix) $\rho^n v_i^n v_j^n \to \rho v_i v_j$ *weakly in* $L^2(Q_T)$;

(x) $\displaystyle\int_{Q_T} \langle\langle v^n, v^n\rangle\rangle \phi \, dxdt \to \int_{Q_T} \langle\langle v, v\rangle\rangle \phi \, dxdt, \ \phi \in C^1(\overline{Q}_T)$.

Due to Theorem 5.1 we can pass to the limit in (3.7), (3.8), (3.9). We obtain the following theorem

Theorem 5.2. *Let the assumptions of Theorem 5.1 be satisfied and let* $1 \leq q < +\infty$ $(N = 2)$ *or* $1 \leq q \leq 6$ $(N = 3)$. *Then there exists* (ρ, v, θ) *such that*
(5.18) $\rho \in L^\infty(I, W^{1,q}(\Omega))$, $\rho > \varepsilon$ *a.e. in* Q_T *for some* $\varepsilon > 0$,
(5.19) $\dfrac{\partial \rho}{\partial t} \in L^\infty(I, L^q(\Omega))$,
(5.20) $v \in L^\infty(I, W^{k,2}(\Omega, \mathbb{R}^N) \cap W_0^{1,2}(\Omega, \mathbb{R}^N)) \cap L^2(I, W^{2k,2}(\Omega, \mathbb{R}^N))$,
(5.21) $\dfrac{\partial v}{\partial t} \in L^2(Q_T), \mathbb{R}^N)$,
(5.22) $\theta \in L^\infty(I, L^2(\Omega)) \cap L^2(I, W^{1,2}(\Omega))$, $\theta \geq 0$ *a.e. in* Q_T,
such that (1.2), (1.3) *holds a.e. in* Q_T *and* (2.12) *is fulfilled.*

Remark. When we repeat all steps of Section 4 we obtain uniqueness as well.

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