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ON THE CR STRUCTURE OF THE TANGENT SPHERE BUNDLE

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We adopt the methods of pseudohermitian geometry (cf. [16]) to study the tangent sphere bundle $U(M)$ over a Riemannian manifold M. If M is an elliptic space form of sectional curvature 1 then $U(M)$ is shown to be globally pseudo-Einstein (in the sense of J.M. Lee, [12]).

1. Introduction.

Let M be a Riemannian manifold and $U(M)$ its tangent sphere bundle. The natural almost complex structure $\widetilde{J}$ of $T(M)$ induces an almost CR structure $\mathcal{H}$ on $U(M)$ (as a real hypersurface of $T(M)$). Although $\widetilde{J}$ is rarely integrable (in fact, only when M is locally Euclidean, cf. P. Dombrowski, [5]) $\mathcal{H}$ may turn out to be a CR structure. For instance, if M is a space form then $\mathcal{H}$ is integrable (cf. Section 3). We establish the following:

Theorem. *Let M be a n-dimensional Riemannian manifold and $\mathcal{H}$ the natural almost CR structure of $U(M)$. The following statements are equivalent:*

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i) $(U(M), \mathcal{H})$ *is a strictly pseudoconvex CR manifold (of CR dimension* $n-1$*) whose Webster connection has a vanishing pseudohermitian torsion.*

ii) M *is an elliptic space form* $M^n(c)$ *of sectional curvature* $c = 1$. *Moreover, the natural pseudohermitian structure of* $U(M^n(1))$ *is globally pseudo-Einstein (i.e. the Ricci tensor of the Webster connection is proportional to the Levi form). In particular* $U(M^n(1))$ *has positive pseudohermitian scalar curvature and the first Chern class of its CR structure* $\mathcal{H}$ *vanishes.*

As a corollary, the first statement in our Theorem yields a short proof of a result by Y. Tashiro, [14] (the contact vector of $U(M)$ is Killing iff $M = M^n(1)$). We employ the methods of pseudohermitian geometry (cf. S. Webster, [16]) rather those of contact (Riemannian) geometry (cf. D.E. Blair, [2], pp. 131-138). The proof of the second statement in the Theorem relies on a result by E.T. Davies & K. Yano, [4].

2. The Webster connection.

Let $(X, T_{1,0}(X))$ be a nondegenerate CR manifold of CR dimension m (where $T_{1,0}(X) \subset T(X) \otimes \boldsymbol{C}$ denotes its CR structure).
Let $H(X) = \mathrm{Re}\{T_{1,0}(X) \oplus T_{0,1}(X)\}$ be the maximally complex distribution of X. Here $T_{0,1}(X) = \overline{T_{1,0}(X)}$ and overbars stand for complex conjugation. Then $H(X)$ carries the complex structure J_X given by $J_X(V+\overline{V}) = i(V-\overline{V})$, for any $V \in T_{1,0}(X)$, where $i = \sqrt{-1}$. Assume X to be orientable and let θ be a contact 1-form on X (i.e. $\theta \wedge (d\theta)^m \neq 0$ everywhere on X) so that $\mathrm{Ker}(\theta) = H(X)$. Let T be the characteristic direction of $d\theta$ (i.e. $T \rfloor d\theta = 0$) normalized so that $\theta(T) = 1$. Then T is transverse to $H(X)$. Let g_θ be the *Webster metric* of (X, θ), given by $g_\theta(V, W) = (d\theta)(V, J_X W)$, $g_\theta(V, T) = 0$ and $g_\theta(T, T) = 1$, for any $V, W \in H(X)$ (cf. also (2.18) in [16], p. 34). By a result of S. Webster, [16], there is a unique linear connection ∇ on X (the *Webster connection* of (X, θ)) determined by the following axioms 1) $H(X)$ is parallel with respect to ∇, 2) $\nabla J_X = 0$, $\nabla g_\theta = 0$, and 3) $\pi_+ T_\nabla(V, W) = 0$ for any $V \in T_{1,0}(X)$, $W \in T(X) \otimes \boldsymbol{C}$, where $\pi_+ : T(X) \otimes \boldsymbol{C} \to T_{1,0}(X)$ is the natural projection (associated with the direct sum decomposition $T(X) \otimes \boldsymbol{C} = T_{1,0}(X) \oplus T_{0,1}(X) \oplus \boldsymbol{C}T$) and T_∇ denotes the torsion tensor field of ∇. Cf. also N. Tanaka, [13]. The *pseudohermitian torsion* τ of ∇ is given by $\tau V = T_\nabla(T, V)$ for any $V \in T(X)$ (cf. also (1.20) in [16], p. 28). Then τ is a trace-less self-adjoint (with respect to g_θ) endomorphism of $T(X)$. The Levi-Civita connection ∇^θ of the semi-Riemannian manifold (X, g_θ) is related to the Webster connection by:

$$\nabla^\theta_V W = \nabla_V W + (\Omega_\theta(V, W) - A(V, W))T + \tag{1}$$
$$+ \tau(V)\theta(W) + \theta(V) J_X W + \theta(W) J_X V$$

for any $V, W \in T(X)$, cf. Theorem 1 in [6]. This corrects the identity (A.6) in [7] (the 1/2 factor should be omitted there). Here $\Omega_\theta(V, W) = g_\theta(V, J_X W)$ and $A(V, W) = g_\theta(V, \tau W)$. Note that $\nabla T = 0$ (by axiom 2). Hence:

$$\nabla_T V = \tau V + \mathcal{L}_T V \tag{2}$$

for any $V \in T(X)$. Here $\mathcal{L}$ denotes the Lie derivative. On the other hand, axiom 3 yields:

$$\tau \circ J_X + J_X \circ \tau = 0\,. \tag{3}$$

Using (2) - (3) we may conduct the calculation:

$$\begin{aligned} 0 = (\nabla_T J_X)V = \nabla_T J_X V - J_X \nabla_T V = \\ = \tau J_X V + \mathcal{L}_T J_X V - J_X \tau V - J_X \mathcal{L}_T V = \\ = -2 J_X \tau V + (\mathcal{L}_T J_X) V \end{aligned}$$

so that to get (as τ is $H(X)$-valued):

$$\tau = -\frac{1}{2} J_X \circ (\mathcal{L}_T J_X)\,. \tag{4}$$

3. The tangent sphere bundle.

Let (M, G) be a n-dimensional Riemannian manifold. Let D be the Levi-Civita connection of (M, G). Then D gives rise to a *nonlinear connection* (in the sense of W. Bartel, [1]) N on $T(M)$. Precisely, there is a distribution $N : v \in T(M) \mapsto N_v \subset T_v(T(M))$ so that:

$$T_v(T(M)) = N_v \oplus \mathrm{Ker}(d_v \Pi) \tag{5}$$

for any $v \in T(M)$. Here $\Pi : T(M) \to M$ is the natural projection. Let (U, x^i) be a local coordinate system on M and $(\Pi^{-1}(U), x^i, y^i)$ the naturally induced local coordinates on $T(M)$. Let $\Gamma^i_{jk}(x)$ be the coefficients of D (with respect to (U, x^i)) and set:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}$$

where:

$$N^i_j(x, y) = \Gamma^i_{jk}(x) y^k\,.$$

Then $\{\delta/\delta x^i\}$ is a local frame of N (on $\Pi^{-1}(U)$). For each $v \in T(M)$ let $\beta_v : T_x(M) \to N_v$ be the *horizontal lift* (i.e. the inverse of $d_v\Pi : N_v \to T_x(M)$) where $x = \Pi(v)$. Locally:

$$\beta \frac{\partial}{\partial x^i} = \frac{\delta}{\delta x^i}.$$

The *vertical lift* $\gamma_v : T_x(M) \to \mathrm{Ker}(d_v\Pi)$ is given by:

$$\gamma_v(w) = \frac{dC}{dt}(0)$$

for any $w \in T_x(M)$. Here $C : (-\varepsilon, \varepsilon) \to T_x(M)$ is the curve given by $C(t) = v + tw$, $|t| < \varepsilon$. Locally:

$$\gamma \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}$$

so that γ is a bundle isomorphism. Let $Q_v : T_v(T(M)) \to \mathrm{Ker}(d_v\Pi)$ be the natural projection (associated with (5)). The *Dombrowski map* $K_v : T_v(T(M)) \to T_x(M)$ is given by $K = \gamma^{-1} \circ Q$ (cf. also [5]). Locally:

$$K \frac{\delta}{\delta x^i} = 0, \quad K \frac{\partial}{\partial y^i} = \frac{\partial}{\partial x^i}$$

Next, we shall need the *Sasaki metric* $\tilde{g}$ on $T(M)$ given by:

$$\tilde{g}(V, W) = G(KV, KW) + G(\Pi_* V, \Pi_* W)$$

for any $V, W \in T(T(M))$. This is a Riemannian metric on $T(M)$ and the distributions N and $\mathrm{Ker}(\Pi_*)$ are orthogonal (with respect to $\tilde{g}$).

Set $U(M)_x = \{v \in T_x(M) : G_x(v, v) = 1\}$. The disjoint union $U(M)$ of $U(M)_x$ for all $x \in M$ is a compact real hypersurface of $T(M)$ (and the total space of a S^{n-1}-bundle $\pi : U(M) \to M$). The portion of $U(M)$ over U is given by the equation:

$$g_{ij}(x) y^i y^j = 1 \tag{6}$$

where g_{ij} are the components of G (with respect to (U, x^i)). Note that:

$$N_v \subset T_v(U(M))$$

$$\mathrm{Ker}(d_v\pi) = T_v(U(M)) \cap \mathrm{Ker}(d_v\Pi)$$

for any $v \in U(M)$. Let $\tilde{J}$ be the natural almost complex structure of $T(M)$ given by $\tilde{J} \circ \beta = \gamma$ and $\tilde{J} \circ \gamma = -\beta$. Locally:

$$\tilde{J}\frac{\delta}{\delta x^i} = \frac{\partial}{\partial y^i}, \quad \tilde{J}\frac{\partial}{\partial y^i} = -\frac{\delta}{\delta x^i}.$$

Set:

$$T_{1,0}(U(M))_v = T^{1,0}(T(M))_v \cap \left[T_v(U(M)) \otimes \boldsymbol{C}\right]$$

where $T^{1,0}(T(M)) \subset T(T(M)) \otimes \boldsymbol{C}$ is the eigenbundle of $\tilde{J}$ corresponding to the eigenvalue i. Then $\mathcal{H} = T_{1,0}(U(M))$ is an almost CR structure on $U(M)$ (i.e. $\mathcal{H} \cap \overline{\mathcal{H}} = 0$). By a result of P. Dombrowski, [5], if D is flat then $(T(M), \tilde{J})$ is a complex manifold so that $\mathcal{H}$ follows to be integrable. For instance $U(\boldsymbol{R}^n)$ is a CR manifold (of CR dimension $n-1$).

Let $P = \beta \circ K$. Then $P_v : \mathrm{Ker}(d_v\Pi) \to N_v$ is a linear isomorphism. We shall need the following:

Lemma 1. *The maximally complex distribution $H(U(M))$ of the almost CR manifold $(U(M), \mathcal{H})$ is given by:*

$$H(U(M))_v = Ker(d_v\pi) \oplus [P_v Ker(d_v\pi)]$$

for any $v \in U(M)$.

Proof. Let $E(\iota) \to U(M)$ be the normal bundle of $\iota : U(M) \subset T(M)$. Set $\nu = y^i\partial/\partial y^i$. Then ν is a (globally defined) unit normal ($\nu_v \in E(\iota)_v$) on $U(M)$. Set $\xi' = -\tilde{J}\nu$. Then ξ' is tangent to $U(M)$. Locally $\xi' = y^i\delta/\delta x^i$. Let η' be the real 1-form on $U(M)$ given by $\eta'(V) = g'(V, \xi')$, for any $V \in T(U(M))$, where $g' = \iota^*\tilde{g}$ is the metric induced on $U(M)$ by the Sasaki metric $\tilde{g}$ of $T(M)$. Note that $H(U(M)) = \mathrm{Ker}(\eta')$. Also $\mathrm{Ker}(\pi_*) \subset \mathrm{Ker}(\eta')$. Set $y_i = g_{ij}y^j$. Then $\eta'(\delta/\delta x^i) = y_i$. At this point Lemma 1 follows from the fact that a vertical tangent vector $X = B^i\partial/\partial y^i$ is tangent to $U(M)$ iff $g_{ij}B^iy^j = 0$ (by taking into account (6)).

Set $\tilde{U} = \{y_n \neq 0\} \subset \pi^{-1}(U)$. The portion of $\mathrm{Ker}(\pi_*)$ over $\tilde{U}$ is the span of $\{Y_\alpha\}$, $1 \leq \alpha \leq n-1$, where:

$$Y_\alpha = \frac{\partial}{\partial y^\alpha} - A_\alpha\frac{\partial}{\partial y^n}, \quad A_\alpha = \frac{y_\alpha}{y_n}.$$

Set $\varphi V = \tan\{\tilde{J}V\}$ for any $V \in T(U(M))$. Here $\tan_v : T_v(T(M)) \to T_v(U(M))$ is the natural projection associated with the decomposition:

$$T_v(T(M)) = T_v(U(M)) \oplus E(\iota)_v$$

for any $v \in U(M)$. The restriction J of φ to $H(U(M))$ and the complex structure $J_{U(M)}$ of $H(U(M))$ (cf. Section 2) actually coincide. Note that $JX_\alpha = Y_\alpha$ where:

$$X_\alpha = \frac{\delta}{\delta x^\alpha} - A_\alpha \frac{\delta}{\delta x^n}.$$

Thus (by Lemma 1) $\mathcal{H} = T_{1,0}(U(M))$ is (locally) the span of $\{T_\alpha\}$ where $T_\alpha = X_\alpha - iY_\alpha$. Let $R^i_{jk\ell}$ be the components of the curvature tensor field R of D (with respect to (U, x^i)). Set:

$$R^i_{k\ell} = R^i_{jk\ell} y^j .$$

Note that:

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = -R^k_{ij} \frac{\partial}{\partial y^k} \tag{7}$$

(so that the Pfaffian system $dy^i + N^i_j(x, y)dx^j = 0$ is integrable iff $R^k_{ij} = 0$). We establish the following:

Lemma 2. *The almost CR structure $\mathcal{H}$ on $U(M)$ is integrable if and only if:*

$$R^i_{\alpha\beta} + A_\alpha R^i_{\beta n} + A_\beta R^i_{n\alpha} = 0 \tag{8}$$

on $\tilde{U}$, for any local coordinate neighbourhood (U, x^i) on M.

Proof. The generators X_α, Y_α satisfy the commutation formulae:

$$[X_\alpha, X_\beta] = (X_\beta(A_\alpha) - X_\alpha(A_\beta)) \frac{\delta}{\delta x^n} - (R^i_{\alpha\beta} + A_\alpha R^i_{\beta n} + A_\beta R^i_{n\alpha}) \frac{\partial}{\partial y^i},$$

$$[Y_\alpha, Y_\beta] = (Y_\beta(A_\alpha) - Y_\alpha(A_\beta)) \frac{\partial}{\partial y^n},$$

$$[X_\alpha, Y_\beta] = Y_\beta \frac{\delta}{\delta x^n} - X_\alpha(A_\beta) \frac{\partial}{\partial y^n} + (Y_\beta(N^i_\alpha) - A_\alpha Y_\beta(N^i_n)) \frac{\partial}{\partial y^i}.$$

These follow from (7) together with the identities:

$$\left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] = \Gamma^k_{ij} \frac{\partial}{\partial y^k}, \left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0.$$

A straightforward calculation shows that:

$$X_\alpha(A_\beta) = X_\beta(A_\alpha), Y_\alpha(A_\beta) = Y_\beta(A_\alpha),$$

$$Y_\alpha(N^i_\beta) - A_\beta Y_\alpha(N^i_n) = Y_\beta(N^i_\alpha) - A_\alpha Y_\beta(N^i_n)$$

so that:

$$[T_\alpha, T_\beta] = -\left(R^i_{\alpha\beta} + A_\alpha R^i_{\beta n} + A_\beta R^i_{n\alpha}\right)\frac{\partial}{\partial y^i}\,. \tag{9}$$

Finally, from (9) and from:

$$T_\alpha = \frac{\delta}{\delta x^\alpha} - i\frac{\partial}{\partial y^\alpha} - A_\alpha\left(\frac{\delta}{\delta x^n} - i\frac{\partial}{\partial y^n}\right)$$

we see that $[T_\alpha, T_\beta] \in \mathcal{H}$ iff (8) holds on $\widetilde{U}$.

Let X be a real $(2m+1)$-dimensional manifold. Let (φ, ξ, η) be an *almost contact structure* on X (in the sense of [2], pp. 19-20). The restriction of φ to $\mathrm{Ker}(\eta)$ is a complex structure. Extend φ to $\mathrm{Ker}(\eta) \otimes \boldsymbol{C}$ by $\boldsymbol{C}$-linearity and set $T_{1,0}(X) = Eigen(i)$. Then $T_{1,0}(X)$ is an almost CR structure (of CR dimension m) on X. If (φ, ξ, η) is *normal* (in the sense of [2], p. 48) then (by a result of [10]) $T_{1,0}(X)$ is integrable. Going back to $X = U(M)$, set $\xi = 2\xi'$ and $\eta = (1/2)\eta'$. Then (φ, ξ, η) is an almost contact structure on $U(M)$. Next, set $g = (1/4)g'$. Then (φ, ξ, η, g) is a contact metric structure on $U(M)$ (in the sense of [2], p. 25). By a result of Y. Tashiro, [14], if $M = M^n(1)$ (i.e. (M, G) has constant sectional curvature 1) then (φ, ξ, η, g) is a Sasakian structure on $U(M)$. In particular it is normal so that, by applying the theorem of S. Ianus cited above, we may conclude that $U(M^n(1))$ is a CR manifold. Our Lemma 2 may be used to indicate examples of Riemannian manifolds M (other than those covered by Tashiro's theorem) for which $U(M)$ is CR. We recall (cf. e.g. [15]) that (M, G) is a Riemannian manifold of *quasi-constant curvature* if its curvature tensor field is given by:

$$R^\ell_{ijk} = c\{\delta^\ell_j g_{ki} - \delta^\ell_k g_{ji}\} + b\{(\delta^\ell_j v_k - \delta^\ell_k v_j)v_i + (v_j g_{ki} - v_k g_{ji})v^\ell\} \tag{10}$$

for some real valued functions $c, b \in C^\infty(M)$ and some unit tangent vector field $V = v^i\partial/\partial x^i$ on M. If this is the case we set $M = M^n_{c,b}(V)$. See also [9]. We obtain the following:

Proposition. *Let M be a n-dimensional Riemannian manifold. Then:*

$$R^\ell_{jk}y_i + R^\ell_{ki}y_j + R^\ell_{ij}y_k = 0 \tag{11}$$

on $U(M)$, is a sufficient condition for the integrability of the natural almost CR structure $\mathcal{H}$ of $U(M)$. If $M = M^n_{c,b}(V)$ then (11) holds if and only if either $n = 2$ or $n \geq 3$ and $b = 0$. In particular $U(M^n(c))$ is CR for any space form $M^n(c)$.

Proof. Clearly (11) yields (8) so that the first statement in the Proposition follows from Lemma 2. Next (by taking into account (10)):

$$R^{\ell}_{jk} y_i + R^{\ell}_{ki} y_j + R^{\ell}_{ij} y_k = bf T^{\ell}_{jik}$$

where:

$$T^{\ell}_{jik} = \delta^{\ell}_j y_i v_k - \delta^{\ell}_i y_i v_k + \delta^{\ell}_i y_k v_j - \delta^{\ell}_j y_k v_i + \delta^{\ell}_k y_i v_i - \delta^{\ell}_k y_i v_j$$

and $f : U(M) \to \boldsymbol{R}$ is $f = v_i y^i$. Note that:

$$T^{\ell}_{jik} + T^{\ell}_{ijk} = T^{\ell}_{jik} + T^{\ell}_{jki} = T^{\ell}_{jik} + T^{\ell}_{kij} = 0 .$$

Clearly $n = 2$ or $b = 0$ yield (11). Viceversa, assume (11) holds. Then:

$$bf T^{j}_{jik} v^i = 0$$

or

$$(n-2) bf (f v_k - y_k) = 0 .$$

If $n \geq 3$ then $b = 0$. Indeed, if $b(x) \neq 0$ for some $x \in M$ then one may choose $u \in U(M)_x$ so that $\{u, V_x\}$ span a 2-plane and $G_x(u, V_x) \neq 0$. Then $f(u) \neq 0$ and $f(u) v^i(x) - u^i \neq 0$, a contradiction.

4. Proof of the theorem.

Let X be a manifold carrying the contact metric structure (φ, ξ, η, g). Set $T = -\xi$ and $\theta = -\eta$. By the contact condition ($d\eta = \Phi$ where $\Phi(V, W) = g(V, \varphi W)$) one has $T \rfloor d\theta = 0$. Assume $T_{1,0}(X) = Eigen(i)$ (cf. section 3) to be a CR structure on X. Again by the contact condition $g_\theta = g$ so that $(X, T_{1,0}(X))$ is a strictly pseudoconvex CR manifold. Let ∇, ∇^θ be respectively the Webster connection and the Levi-Civita connection of (X, g). By axiom 2) one has $\nabla T = 0$ so that (1) leads to:

$$\nabla^\theta \xi = -\varphi - \tau . \tag{12}$$

In particular, the almost contact metric structure (φ, ξ, η, g) of $X = U(M)$ satisfies the contact condition (cf. [2], p. 133) so that the considerations above may be applied to compute the pseudohermitian torsion τ of $(U(M), \eta)$. Precisely, we establish:

Lemma 3. *Assume $\mathcal{H} = T_{1,0}(U(M))$ to be a CR structure. Then:*

$$\tau \frac{\delta}{\delta x^i} = H_i^k \left(\delta_k^\ell - y_k y^\ell\right) \frac{\partial}{\partial y^\ell} , \tag{13}$$

$$\tau \frac{\partial}{\partial y^i} = H_i^k \frac{\delta}{\delta x^k} \tag{14}$$

where:

$$H_i^k = R_{ij}^k y^j + y_i y^k - \delta_i^k$$

Proof. Let us use (4) (with $X = U(M)$). Then (13) - (14) may be gotten from the following identities:

$$\left[\xi, \frac{\delta}{\delta x^j}\right] = 2\left\{N_j^i \frac{\delta}{\delta x^i} - y^i R_{ij}^k \frac{\partial}{\partial y^k}\right\} ,$$

$$\left[\xi, \frac{\partial}{\partial y^j}\right] = 2\left\{ - \frac{\delta}{\delta x^j} + N_j^k \frac{\partial}{\partial y^k}\right\} ,$$

$$[\xi, \nu] = -\xi ,$$

$$\xi(y_j) = 2y^i y^k |jk, i| ,$$

$$\tan\left(\frac{\partial}{\partial y^i}\right) = \left(\delta_i^k - y_i y^k\right) \frac{\partial}{\partial y^k}$$

where $\xi = 2y^i \delta/\delta x^i$ and $\nu = y^i \partial/\partial y^i$.

At this point we may prove the Theorem. Assume ii) holds. Then (11) holds on $U(M)$ so that (by the Proposition) $U(M)$ is a CR manifold. On the other hand $M = M^n(1)$ yields:

$$R_{ij}^k y^j = \delta_i^k - y_i y^k \tag{15}$$

so that (by Lemma 3) $\tau = 0$. Viceversa, assume i) holds. Then Lemma 3 yields (15). Let $x \in M$ and $X, v \in T_x(M)$ two unit tangent vectors so that $G_x(X, v) = 0$. Set $X = X^i \partial/\partial x^i$. Let us apply (15) at v and contract with X^i in the resulting identity. This procedure leads to:

$$R_{\ell ij}^k(x) X^i v^j v^\ell = X^k$$

(as $X^i v_i = 0$) or:

$$R_x(X, v)v = X$$

which (by taking the inner product with X) yields constant sectional curvature 1.

Let us show Tashiro's theorem (cf. [14]) follows from the first statement in our Theorem. Indeed, if $M = M^n(1)$ then (with the arguments above) it makes sense to consider the Webster connection and $\tau = 0$ yields normality by a result in [6]. Thus $U(M)$ is Sasakian (and any Sasakian structure is in particular K-contact). Viceversa, if the contact structure (φ, ξ, η, g) of $U(M)$ is K-contact then $\nabla^\theta \xi = -\varphi$ (by (8) in [2], p.64) which together with (12) yields $\tau = 0$ and the Theorem applies.

Since $U(M^n(1))$ is (by the Theorem) a compact strictly pseudoconvex CR manifold it follows (by a result of L. Boutet de Monvel, [3]) that $U(M^n(1))$ is locally embeddable (as a CR hypersurface of $\mathbf{C}^n$). In view of [12] it is natural to ask whether $U(M^n(1))$ is globally pseudo-Einstein.

Let $\widetilde{K}$ be the Ricci tensor field of the Sasaki metric $\tilde{g}$ on $T(M)$. By a result of E.T. Davies & K. Yano, [4], one has:

$$\widetilde{K}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right) = R_{jk} + \frac{1}{4} y^m \left\{ R^r_{ji} R^i_{krm} + R^r_{ki} R^i_{jrm} \right\}, \tag{16}$$

$$\widetilde{K}\left(\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^k}\right) = \frac{1}{2} y^\ell \left\{ \nabla_i R^i_{k\ell j} + R^i_{k\ell j} \frac{\partial}{\partial x^i} (\log \sqrt{\Delta}) \right\}, \tag{17}$$

$$\widetilde{K}\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right) = \frac{1}{4} R^\ell_{irk} R^i_{\ell js} y^r y^s \tag{18}$$

where R_{jk} denotes the Ricci curvature of (M, G) and $\Delta = \det[g_{ij}]$. See also [8]. We need the Gauss equation (cf. [11], vol. II, p. 23) of $U(M)$ in $(T(M), \tilde{g})$:

$$\begin{aligned} \widetilde{R}(X, Y, Z, W) &= R(X, Y, Z, W) + \\ &+ g'(X, a_\nu Z) g'(Y, a_\nu W) - g'(X, a_\nu W) g'(Y, a_\nu Z) \end{aligned} \tag{19}$$

for any $X, Y, Z, W \in T(U(M))$. Here a_ν is the shape operator of ι. Taking traces in (19) leads to:

$$K^\theta(X, Y) = \widetilde{K}(X, Y) + g'(a_\nu X, a_\nu Y) - g'(X, a_\nu Y)\|\mu\| - \widetilde{R}(X, \nu, Y, \nu) \tag{20}$$

where K^θ is the Ricci curvature of $(U(M), g')$ and μ is the mean curvature vector of ι. At this point, a calculation based on the identities (2) in [2], p. 130, shows that:

$$\widetilde{R}(X, \nu, Y, \nu) = 0$$

for any $X, Y \in T(U(M))$. Next (by taking into account:

$$a_\nu \frac{\delta}{\delta x^i} = 0, \quad a_\nu X = -X$$

for any $X \in \text{Ker}(\pi_*)$, cf. [2], p.132) we obtain:

$$K^\theta\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = \widetilde{K}\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right),$$

$$K^\theta(X, Y) = \widetilde{K}(X, Y) + (1 + \|\mu\|)g'(X, Y), \tag{21}$$

$$K^\theta\left(X, \frac{\delta}{\delta x^j}\right) = \widetilde{K}\left(X, \frac{\delta}{\delta x^j}\right)$$

for any $X, Y \in \text{Ker}(\pi_*)$. Note that $g(X_\alpha, X_\beta) = 2h_{\alpha\bar\beta}$ where $h_{\alpha\bar\beta}$ is the Levi form of $(U(M), \theta)$. If $M = \boldsymbol{R}^n$ then (16) - (18) and (21) lead to:

$$K^\theta_{\alpha\bar\beta} = 2(1 + \|\mu\|)h_{\alpha\bar\beta}$$

where $K^\theta_{\alpha\bar\beta} = K^\theta(T_\alpha, T_{\bar\beta})$. Recall (cf. ii) of Theorem 9 in [6]) that:

$$K^\theta_{\alpha\bar\beta} = K_{\alpha\bar\beta} - \frac{1}{2}h_{\alpha\bar\beta}$$

where $K_{\alpha\bar\beta}$ is the (pseudohermitian) Ricci tensor (of the Webster connection of $(U(M), \theta)$). Thus $U(\boldsymbol{R}^n)$ is globally pseudo-Einstein. Similarly, if $M = M^n(1)$ then (16) - (18) may be written as:

$$\widetilde{K}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right) = \frac{2n-3}{2}g_{jk} - \frac{n-2}{2}y_j y_k,$$

$$\widetilde{K}\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right) = \frac{1}{2}(g_{jk} - y_j y_k),$$

$$\widetilde{K}\left(\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^k}\right) = \frac{1}{2}(y^i g_{jk} - \delta^i_j)\frac{\partial}{\partial x^i}(\log\sqrt{\Delta})$$

(on $U(M^n(1))$ which together with (21) furnish:

$$K^\theta_{\alpha\bar\beta} = 2(n + \|\mu\|)h_{\alpha\bar\beta}.$$

Thus (again by ii) of Theorem 9 in [6]) $R = 2n(n + \|\mu\|) + n/2 > 0$ (where R is given by (2.17) in [16], p. 34). Let $c_1(\mathcal{H}) \in H^2(U(M^n(1)); \boldsymbol{R})$ be the first Chern class of $\mathcal{H}$. As $U(M^n(1))$ is globally pseudo-Einstein we may apply a result of J.M. Lee, [12], to conclude that $c_1(\mathcal{H}) = 0$.

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