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CUMULATIVE HIERARCHIES AND COMPUTABILITY OVER UNIVERSES OF SETS

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Various metamathematical investigations, beginning with Fraenkel's historical proof of the independence of the axiom of choice, called for suitable definitions of hierarchical universes of sets. This led to the discovery of such important cumulative structures as the one singled out by von Neumann (generally taken as the universe of *all* sets) and Gödel's universe of the so-called *constructibles*. Variants of those are exploited occasionally in studies concerning the foundations of analysis (according to Abraham Robinson's approach), or concerning non-well-founded sets. We hence offer a systematic presentation of these many structures, partly motivated by their relevance and pervasiveness in mathematics. As we report, numerous properties of hierarchy-related notions such as *rank*, have been verified with the assistance of the ÆtnaNova proof-checker. Through SETL and Maple implementations of procedures which effectively handle the Ackermann's hereditarily finite sets, we illustrate a particularly significant case among those in which the entities which form a universe of sets can be algorithmically constructed and manipulated; hereby, the fruitful bearing on pure mathematics of cumulative set hierarchies ramifies into the realms of theoretical computer science and algorithmics.

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Introduction

Axiomatic set theory had already evolved (by the essential contributions of Zermelo [47], Fraenkel and Skolem) into today's version *ZF*, when von Neumann modeled it by his renowned *cumulative hierarchy* [33]. Having been conceived downstream, after decades-long investigations on the foundations of mathematics, that structure perhaps does not fully deserve the relevance of *intended model*: suffice it to say that ZF admits, besides that model (having as its own domain of support a proper class[1] and encompassing sets whose cardinality exceeds the countable), also models of countable cardinality.[2] Actually, Gödel later proposed a model alternative to the one due to von Neumann, the universe of *constructibles* [21], which comes to coincide with von Neumann's model only when a specific axiom so caters.

Anyway, the cumulative hierarchy is, for anybody who undertakes an advanced study on sets, a precious conceptual tool some preliminary exposure to which—if only at an intuitive, but nonetheless rigorous, level—will pay off when one arrives at the axiomatic formalization, traditionally based on first-order predicate calculus.

One of the benefits that can ensue from placing the semantics before the formal-logical description is the gradualness made possible by such approach. Preparatory to the cumulative hierarchy in which, as proposed by von Neumann, one level corresponds to every ordinal (even to each *transfinite* ordinal), this paper will introduce similar but less demanding hierarchies. One of these structures, the one of the *hereditarily finite* sets, may be taken as a reference model for an axiomatic theory close in spirit to the one due to Zermelo but focused exclusively on *finite* sets [46]; other hierarchies, the so-called *superstructures* play an important rôle in setting up the ground for non-standard analysis [14]. The study of these scaled-down versions of the von Neumann's hierarchy brings to light algorithmic manipulations which make sense insofar as one deals with relatively simple sets only, but which bear a lot of significance for those whose interests are more deeply oriented towards computer science than towards foundational issues.[3]

Various hierarchies similar to the von Neumann's one held a historically crucial rôle in the investigations on axiomatic theories antithetic to ZF. Models deserving attention in this more speculative framework were proposed for the study on the independence of the axiom of choice and of the continuous hypoth-

[1] As we will belabor in Sections 11 and 13, a proper class differs from a set inasmuch as it does not belong as an element to any other class.

[2] This constitutes the so called *Skolem paradox*.

[3] The importance of algorithms for the manipulation of nested sets is witnessed by the availability of sets (with their correlated associative *maps*) in several programming languages: SETL, Maple, Python, etc..

esis: a "cumulative proto-hierarchy" proposed by Fraenkel in the far 1922 [20] and the already mentioned class of Gödel's constructible sets. Moreover, as we will examine on a "small scale" instance (referring to *finite hypersets* only), it is easy to obtain models of theories of non well-founded sets [2, 5] from standard hierarchies.

For the above-mentioned reasons, we deem it useful to revisit cumulative hierarchies (as well as other structures akin to them) emerging in various fields of mathematics and theoretical computer science. We do not intend to write an erudition piece of work, but to set up the subject in useful terms, on the one hand

- for anybody who wants to undertake the development of *software* libraries for the management of sets (or maps, multi-sets, hypersets, etc.), with full awareness of the principles lying behind applications (for instance—as regards hypersets—applications to π-*calculus* [32, 44]); on the other hand,

- for anybody who wants to develop with the appropriate degree of formal rigor, in the form of reusable *proofware*, all basic proofs regarding such hierarchies.

In order to move towards the former of these goals, we have implemented several Maple procedures which handle (as we report in the appendix Sec. 18) the hereditarily finite sets; to move towards the latter, we have developed various 'theories'[4] within our automatic proof-verification system Referee/ÆtnaNova [9, 11, 12, 35, 38, 39].

1. The simplest of all cumulative hierarchies

> *By analyzing mathematical arguments, logicians became convinced that the notion of "set" is the most fundamental concept of mathematics. This is not meant to detract from the fundamental character of the integers. Indeed a very reasonable position would be to accept the integers as primitive entities and then use sets to form higher entities. However, it can be shown that even the notion of an integer can be derived from the abstract notion of a set, and this is the approach we shall take.* [13, p. 50]

[4] Here the word *theory* has a very technical meaning, referring to the THEORY construct of the verifier Referee.

We take for granted a certain familiarity with sets, that includes the notions of *empty* set, of formation of *singletons*, of *union* of (two or of a plurality of) sets, of formation of the *power* set of a set.

We give here the sequence of definitions of

$$\begin{array}{lcl}
\textbf{numbers}: & \text{and} & \textbf{levels}: \\
0 =_{\text{Def}} \emptyset\,, & & \mathscr{V}_0 =_{\text{Def}} \emptyset\,; \\
1 =_{\text{Def}} \{0\}\,, & & \mathscr{V}_1 =_{\text{Def}} \mathcal{P}(\mathscr{V}_0)\,; \\
2 =_{\text{Def}} 1 \cup \{1\}\,, & & \mathscr{V}_2 =_{\text{Def}} \mathcal{P}(\mathscr{V}_1)\,; \\
3 =_{\text{Def}} 2 \cup \{2\}\,, & & \mathscr{V}_3 =_{\text{Def}} \mathcal{P}(\mathscr{V}_2)\,; \\
\quad\ldots & & \quad\ldots
\end{array}$$

expanding which we can effectively determine the levels as follows:

$$\begin{aligned}
&\mathscr{V}_1 = \{0\} = 1 = \{\mathscr{V}_0\},\\
&\mathscr{V}_2 = \{0,1\} = 2 = \{0,\ \mathscr{V}_1\},\\
&\mathscr{V}_3 = \big\{0,1,\{1\},2\big\} = \big\{0,1,\{1\},\mathscr{V}_2\big\},\\
&\mathscr{V}_4 = \Big\{0,1,\{1\},2,\big\{\{1\}\big\},\big\{2\big\},\big\{0,\{1\}\big\},\big\{0,2\big\},\big\{1,\{1\}\big\},\big\{1,2\big\},\\
&\qquad\qquad \big\{0,1,\{1\}\big\},3,\big\{\{1\},2\big\},\big\{0,\{1\},2\big\},\big\{1,\{1\},2\big\},\mathscr{V}_3\Big\},
\end{aligned}$$

etc.

Note that (natural) numbers, intended as above, are just peculiar sets. Likewise, all levels $\mathscr{V}_i$, as well as the elements of each of these levels, are sets.

Every one of the levels $\mathscr{V}_i$ has *finitely many* elements—by precise assessment of this number of elements, one discovers that it is (if $i > 0$) the hyperexponential amount

$$2^{\cdot^{\cdot^{2}}}\Big\}\ i-1 \text{ times}\,.$$

Moreover, each $\mathscr{V}_i$ belongs to the next level $\mathscr{V}_{i+1}$ and "*boxes*" (as subsets, besides having them as elements) the $\mathscr{V}_j$s of lower index; moreover the inclusion of each $\mathscr{V}_i$ in $\mathscr{V}_{i+1}$ is *strict* (i.e., $\mathscr{V}_i \subsetneq \mathscr{V}_{i+1}$), since each number i belongs to the LAYER $\mathscr{V}_{i+1} \setminus \mathscr{V}_i$. Starting from $\mathscr{V}_3$, to which the set $\{1\}$ belongs, the $\mathscr{V}_i$s own elements which are not numbers.

If x belongs to $\mathscr{V}_i$, both x and any of its elements is finite, and so are all the elements of elements of x, etc.. In view of that, x is said to be HEREDITARILY FINITE.

The CUMULATIVE HIERARCHY $\mathscr{V}_\omega$ of all PURE *hereditarily finite* sets (here 'pure'is in the sense that their formation involves exclusively sets, all ultimately founded over $\emptyset$) consists of all those sets that enter as elements into $\mathscr{V}_i$ after a finite number of steps of the preceding construction:

$$\begin{array}{llll} \omega =_{\mathrm{Def}} \mathbb{N} & =_{\mathrm{Def}} & \{0,1,2,\ldots\} & \\ \mathcal{V}_\omega & =_{\mathrm{Def}} & \mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \cup \cdots & \end{array} \quad (\textit{ad infinitum})$$ [5].

For every x in $\mathcal{V}_\omega$, we call RANK of x the first value i for which x belongs to $\mathcal{V}_{i+1}$ (that is, x is included in $\mathcal{V}_i$). Examples: every number i has rank i; the rank of $\{0,2\}$ is 3. One can see the rank of a set as a measure of how deeply nested $\emptyset$ occurs within it when the set is written in primitive notation—for instance $\{0,2\}$ must be written as $\{\emptyset,\{\emptyset,\{\emptyset\}\}\}$. The following recursive formula enables us to determine the rank of any set X in $\mathcal{V}_\omega$:

$$\mathsf{rk}(X) \; = \; \bigcup\Big\{\mathsf{rk}(y) \cup \{\mathsf{rk}(y)\} : y \in X\Big\}.$$

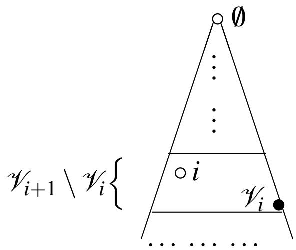

Figure 1: Cumulative hierarchy founded on the void

2. A few remarks about natural numbers

The notion of natural numbers seen above is due the Hungarian-American mathematician John (János) von Neumann (1903-1957). One might object that the intuitive notion of number comes *before* sets can be conceived and is in some sense presupposed in the construction of $\mathcal{V}_\omega$ seen above. A counter-objection is that "being able to count" comes even earlier than the notion of number and this is the only presupposed ability for the construction of $\mathcal{V}_\omega$.

However if, on the side of the numbers, "counting" means reiterating the operation

$$X \overset{+1}{\longmapsto} X \cup \{X\}$$

of unitary increment, on the side of sets it is the much more elaborate operation

$$X \longmapsto \mathcal{P}(X)$$

[5] It is almost superfluous to say that the infinity we are referring to is the one consisting of natural numbers only.

of power set formation that gets repeated.

Notice that for numbers intended *à la* von Neumann the ordering relation, traditionally designated by the symbols $<$ and $\leqslant$, mixes up with the ones of membership and inclusion among sets, of which it comes to be a special case. Actually, when i and j are numbers, we have

$$\begin{array}{lcl} i<j & \text{iff} & i \in j, \\ i<j & \text{iff} & i \subsetneq j, \\ i \leqslant j & \text{iff} & i \subseteq j, \\ \max(i,j) & = & i \cup j, \\ \min(i,j) & = & i \cap j. \end{array}$$

3. Transitive sets and lexicographic orderings

Besides numbers, other sets exist for which the distinction between membership and inclusion is also tenuous. Indeed, innumerable other sets S satisfy the condition[6]

$$\text{when } x \in S \text{ then } x \subset S,$$

or, equivalently, the condition of being TRANSITIVE according to the definition

$$\mathsf{Trans}(S) \ \leftrightarrow_{\text{Def}} \ S \subset \mathcal{P}(S)\,,$$

or, yet equivalently,[7] the condition that

$$\bigcup S \ \subseteq \ S\,.$$

Just to see an example, each one of the levels $\mathscr{V}_i$ is transitive.

Suppose that a transitive family of sets S (even one which is not encompassed by the limited universe $\mathscr{V}_\omega$ seen so far) is endowed with a total ordering $\lhd$. If we consider an element x of S, the fact that x is also a subset of S allows us to put its elements in the order induced by $\lhd$ and therefore to single out, in case x is finite and non-null, the maximum among them: $\max_{\lhd}(x)$. We will say that the order $\lhd$ is LEXICOGRAPHIC if whenever $x \lhd y$, with x and y finite elements of S, one of the following three situations hold:

[6]In this condition, as well as in the subsequent definition of $\mathsf{Trans}(\_)$, it is immaterial whether $\subset$ is intended as $\subseteq$ or as $\subsetneq$, if one excludes—as it normally happens—that any set S may satisfy self-membership $S \in S$. Without relying on the assumption that $S \notin S$, the seminal paper [47] whence axiomatic set theory draws its origin provided an agile proof of the fact that $\mathcal{P}(S) \nsubseteq S$: the key idea was that $\{x \in S \mid x \notin x\} \in \mathcal{P}(S) \setminus S$.

[7]If $S \subset \mathcal{P}(S)$, then any $y \in \bigcup S$ also satisfies $y \in S$, because $y \in x$ and $x \subseteq S$ hold for some $x \in S$. Conversely, if $\bigcup S \subseteq S$ and $x \in S$, then $x \subseteq \bigcup S$, and thus $x \subseteq S$: therefore, $x \in \mathcal{P}(S)$.

- $x = \emptyset$; or
- $x \neq \emptyset$, $y \neq \emptyset$ and $\max_{\lhd}(x) \lhd \max_{\lhd}(y)$; or
- $x \neq \emptyset$, $y \neq \emptyset$, $\max_{\lhd}(x) = \max_{\lhd}(y)$ and $x \setminus \{\max_{\lhd}(x)\} \lhd y \setminus \{\max_{\lhd}(x)\}$.

Put otherwise, the ordering $\lhd$ is *lexicographic* if, for every pair x, y of finite sets in S, it turns out that $x \lhd y$ when either

- $x \subsetneq y$; or
- $x \not\subseteq y$, $y \not\subseteq x$ and, assuming that $x = \{x_0, \ldots, x_n\}$, $y = \{y_0, \ldots, y_m\}$, $x_0 \lhd \cdots \lhd x_n$, and $y_0 \lhd \cdots \lhd y_m$, it is found that $x_h \lhd y_{m-(n-h)}$ holds in the last position h for which $x_h \neq y_{m-(n-h)}$.

At first one might suppose that there is just one lexicographic ordering over a transitive set S; this is indeed the case (as we will examine closely in Sec. 7) as long as $S \in \mathscr{V}_\omega$ or even $S = \mathscr{V}_\omega$, but the situation changes when S owns some infinite elements, because no constraint is imposed by the lexicographic condition regarding the comparison with such sets.

Perhaps it would be more appropriate to call *anti*lexicographic an ordering $\lhd$ which meets the property stated above, as the comparison between two finite sets does not proceed from the "left" but from the "right"; we prefer the shortest word, though, also for historical reasons. Concerning the reason for proceeding from the right, we observe that if we proceeded in the opposite direction, we would be forced to place, for example, $\{0, 2\}$ *before* $\{1\}$, although $\{1\}$ appears at a lower level of the cumulative hierarchy seen above; on the contrary, if we carry out the lexicographic comparison from right to left, this will ensure that the sets belonging to one layer precede those of any subsequent layer.

4. Superstructures

> *However, starting with nothing but the empty set is rather tedious and unnatural, especially when you want to apply set theory outside mathematics. We want to be able to have sets of sticks, stones, and broken bones, as well as more abstract objects like numbers and whatnot.* [5, p. 21]

We are about to introduce an obvious generalization of the cumulative hierarchy seen above. Rather than from the empty basis $\mathscr{V}_0 = \emptyset$, this time we start from a basis of INDIVIDUALS intended as "ur-elements" (or "atoms") plainly distinguishable from sets (and hence, in particular, distinct from $\emptyset$), usable as members in the formation of sets. Denoting by B such a basis, we must retouch the definition of the $\mathscr{V}_i$s if we want to preserve the boxing of each level of the hierarchy in the subsequent ones:

$$\begin{aligned}
\mathscr{V}_0^B &=_{\text{Def}} B;\\
\mathscr{V}_1^B &=_{\text{Def}} \mathcal{P}(\mathscr{V}_0^B) \cup B;\\
\mathscr{V}_2^B &=_{\text{Def}} \mathcal{P}(\mathscr{V}_1^B) \cup B;\\
&\text{etc.}
\end{aligned}$$

The infinite family

$$\mathscr{V}_\omega^B =_{\text{Def}} \mathscr{V}_0^B \cup \mathscr{V}_1^B \cup \mathscr{V}_2^B \cup \mathscr{V}_3^B \cup \cdots \qquad (\textit{ad inf.})$$

of sets and individuals is sometimes called SUPERSTRUCTURE generated by B. Obviously we have

$$\mathscr{V}_i^B = \mathscr{V}_i \text{ when } B = \emptyset,$$

as can be proved by a simple inductive argument. When $B \supseteq B'$, then the inclusion $\mathscr{V}_i^B \supseteq \mathscr{V}_i^{B'} \cup B$ holds: in particular

$$\mathscr{V}_i^B \supsetneq \mathscr{V}_i \text{ when } B \neq \emptyset.$$

Consequently—still assuming that $B \neq \emptyset$—, we will have $\mathscr{V}_\omega^B \supsetneq \mathscr{V}_\omega$.

There is no guarantee anymore, with this modified cumulative hierarchy, that all the sets that enter into it (as members of any level $\mathscr{V}_i^B$) are finite. Actually, in case B is infinite, the level $\mathscr{V}_1^B$ already includes an infinity of sets, every one of which is endowed with an infinity of elements: the following among others

$$B,\ B \setminus \{b_0\},\ B \setminus \{b_0, b_1\},\ B \setminus \{b_0, b_1, b_2\}, \dots,$$

if we suppose the b_js to be pairwise distinct elements of B. If we want to limit ourselves to HEREDITARILY FINITE sets, as we have done in the hierarchy seen at the beginning, we must alter the construction as follows. Instead of the constructor $\mathcal{P}(X)$, that produces the family of *all* subsets of X, we use $\mathcal{F}(X)$, that produces the family of the *finite* subsets; thus we obtain

$$\begin{aligned}
\mathscr{H}_0^B &=_{\text{Def}} B;\\
\mathscr{H}_1^B &=_{\text{Def}} \mathcal{F}(\mathscr{H}_0^B) \cup B;\\
\mathscr{H}_2^B &=_{\text{Def}} \mathcal{F}(\mathscr{H}_1^B) \cup B;\\
&\text{etc.,}
\end{aligned}$$

to conclude with

$$\mathscr{H}_\omega^B =_{\text{Def}} \mathscr{H}_0^B \cup \mathscr{H}_1^B \cup \mathscr{H}_2^B \cup \mathscr{H}_3^B \cup \cdots \qquad (\textit{ad inf.}).$$

It is obvious that

$$\mathscr{H}_i^B = \mathscr{V}_i^B \text{ when the basis } B \text{ is finite.}$$

Let us recall here from [14, pp. 12-15], where they are inventoried and proved in detail, various closure properties of the superstructures. First of all, in a superstructure each level is "*transitive*", in the sense that

$$\mathscr{V}_i^B \setminus B \subsetneq \mathcal{P}(\mathscr{V}_i^B) \subsetneq \mathscr{V}_\omega^B \setminus B \subsetneq \mathcal{P}(\mathscr{V}_\omega^B) \quad \text{for } i \in \omega\,;$$

moreover

when $x \in \mathscr{V}_\omega^B \setminus B$ then $\mathcal{P}(x) \in \mathscr{V}_\omega^B \setminus B$ and $\bigcup(x \setminus B) \in \mathscr{V}_\omega^B \setminus B$,
when $x_1, \ldots, x_k \in \mathscr{V}_\omega^B$ then $\{x_1, \ldots, x_k\} \in \mathscr{V}_\omega^B \setminus B$,
when $x_1, \ldots, x_k \in \mathscr{V}_\omega^B \setminus B$ then $x_1 \cup \cdots \cup x_k \in \mathscr{V}_\omega^B \setminus B$,
when $x_1, \ldots, x_k \in \mathscr{V}_\omega^B$ and $k \geqslant 2$ then $\langle x_1, \ldots, x_k \rangle \in \mathscr{V}_\omega^B \setminus B$,
when $x, y \in \mathscr{V}_\omega^B \setminus B$ then $x \times y \in \mathscr{V}_\omega^B \setminus B$,

if we intend the ORDERED PAIR *à la* Kuratowski,[8] or—as we prefer to do [19]—as

$$\langle X, Y \rangle \quad =_{\text{Def}} \; \big\{\{X, Y\}, \{Y\} \setminus \{X\}\big\}$$

(so that $U \times V \quad =_{\text{Def}} \; \{\langle x, y \rangle : x \in U, y \in V\}$), and recursively we put

$$\langle x_1, \ldots, x_{k+1} \rangle \quad =_{\text{Def}} \; \big\langle x_1, \langle x_2, \ldots, x_{k+1} \rangle \big\rangle, \;\text{ for } k \geqslant 2\,.$$

Thanks to the hierarchical constitution of $\mathscr{V}_\omega^B$, *no* descending infinite chain

$$\cdots \in x_2 \in x_1 \in x_0$$

exists originating from any $x_0 \in \mathscr{V}_\omega^B \setminus B$. Therefore it is possible to define an operation

$$x \longmapsto \mathsf{arb}(x)$$

in such a way that

- when $x \in \mathscr{V}_\omega^B \setminus (B \cup \{\emptyset\})$, then $\mathsf{arb}(x) \in x$ and

$$\mathsf{arb}(x) \notin B \;\; \rightarrow \;\; \mathsf{arb}(x) \cap x = \emptyset\,;$$

- when $x \in B \cup \{\emptyset\}$ then $\mathsf{arb}(x) = x$.

[8] Kazimierz Kuratowski (1896–1980), who in 1922 defined the ordered pair as $\langle X, Y \rangle \quad =_{\text{Def}} \; \big\{\{X\}, \{X, Y\}\big\}$.

Thus we have, obviously, that

$$\mathsf{arb}(x) \in \mathscr{V}_\omega^B \text{ for every } x \in \mathscr{V}_\omega^B.$$

We can, finally, define multi-images and single images of any set f (neglecting possible elements of f that are not ordered pairs), by putting:

$$\begin{array}{lll} f \upharpoonright^{\!\!\rightarrow} v & =_{\text{Def}} & \{y : \langle x,y\rangle \in f \mid x \in v\}\,, \\ f \upharpoonright x & =_{\text{Def}} & \mathsf{arb}(f \upharpoonright^{\!\!\rightarrow} \{x\})\,. \end{array}$$

Even for these APPLICATION constructs, obvious closure properties hold:

$$\begin{array}{l} \text{when } f, v \in \mathscr{V}_\omega^B \setminus B \text{ then } f \upharpoonright^{\!\!\rightarrow} v \in \mathscr{V}_\omega^B \setminus B\,, \\ \text{when } f \in \mathscr{V}_\omega^B \setminus B \text{ and } x \in \mathscr{V}_\omega^B \text{ then } f \upharpoonright x \in \mathscr{V}_\omega^B\,. \end{array}$$

5. Embedding of $\mathscr{H}_\omega^B$ in a Herbrand universe $\mathscr{H}$

We could develop the construction of $\mathscr{H}_\omega^B$ in another way [17], by means of a binary constructor $w(X,Y)$ with which to represent the operation $X \cup \{Y\}$. At first we construct the family $\mathscr{H}$ of all the TERMS over the signature

$$a_{/0} \quad \text{with } a \text{ in } B, \qquad \emptyset_{/0}, \qquad w_{/2}\,,$$

on the basis of the following rules:

- $\emptyset$ constitutes a freestanding term;
- each a in B constitutes a freestanding term;
- if s,t are terms, then $w(s,t)$ is a term;
- only those expressions that can be recognized to be terms on the basis of the preceding three clauses are terms.[9]

To capture the properties of sets, we establish the rewriting rules

$$w(w(X,Y),Y) \;\rightsquigarrow\; w(X,Y)$$

and

$$w(w(X,Y),Z) \;\rightsquigarrow\; w(w(X,Z),Y).$$

[9] The set of all terms constructed over a signature made up, like this one, of constants (at least one) and function symbols, is often called a HERBRAND UNIVERSE (see [28]), from the name of the French mathematician Jacques Herbrand (1908–1931).

The former of these says that inserting twice or more than twice the same element in a set amounts to the same as inserting it only once. Moreover, as implied by the latter rewriting rule, when several elements are inserted, the order of insertions is immaterial for the result. Without these identities, the universe of terms would be *free*; but it now comes to be split into equivalence classes, so that two terms will be regarded as being "the same set" when they are equivalent up to rearrangement of the elements and elimination of duplicates. Thus, the "universe" turns out to be richer than the $\mathscr{H}_\omega^B$ seen earlier, unless one bans from the domain of interest any term whose COLOR differs from $\emptyset$, where the following definition applies:

$$\begin{array}{lll} \mathrm{color}(\emptyset) & =_{\mathrm{Def}} \emptyset; & \\ \mathrm{color}(a) & =_{\mathrm{Def}} a, & \text{for every } a \text{ in } B; \\ \mathrm{color}(w(s,t)) & =_{\mathrm{Def}} \mathrm{color}(s), & \text{for every pair } s,t \text{ of terms.} \end{array}$$

6. The von Neumann universe $\mathscr{V}$

Thus in our system, all objects are sets. We do not postulate the existence of any more primitive objects. To guide his intuition, the reader should think of our universe as all sets which can be built up by successive collecting processes, starting from the empty set.

[13, p. 50]

In proposing his cumulative hierarchy, rather than adding individuals to it, von Neumann extended its construction to *all* ORDINALS, including the transfinite ones.[10]

Finite ordinals are the same as natural numbers; the first infinite ordinal is the ω already seen—namely the set $\mathbb{N}$ of all natural numbers—, immediately followed by the *successor* $\omega+1$, followed in its turn by $(\omega+1)+1$, etc. Very much like ω is not the immediate successor of another ordinal, and in this sense it is a LIMIT ordinal, at the end of the infinite sequence of successors stemming from ω we will find the second limit ordinal $\omega+\omega$, etc. To escape the *impasse* of an unsteady intuition about infinities so remote from experience, we can define the notion $\mathscr{O}(\_)$ of ordinal in an autonomous way, in the context of an *axiomatic* set theory that we do not care to expound here:

$$\mathscr{O}(\alpha) \;\leftrightarrow_{\mathrm{Def}}\; \mathsf{Trans}(\alpha) \;\&\; \langle \forall x \in \alpha, y \in \alpha \,|\, x \in y \vee x = y \vee y \in x \rangle\,;$$

[10]Von Neumann catered thereby for the *infinity postulate*, by which Zermelo [47] required the existence of a set having $\emptyset$ among its elements and closed with respect to singleton formation $X \longmapsto \{X\}$.

that is: we mean by ORDINAL a transitive set α within which either the relation $x \in y$ or the relation $y \in x$ holds between any of its two distinct elements x, y. Assuming that the membership relation is, as von Neumann conceived it, *well founded* (i.e., devoid of cycles and, more generally, of descending chains

$$\cdots \in x_2 \in x_1 \in x_0$$

of infinite length), this definition imposes that, internally, each ordinal is well ordered by the membership relation.[11] We can define *d'emblée*

$$\mathscr{V}_\alpha \quad =_{\text{Def}} \bigcup \Big\{ \mathcal{P}(\mathscr{V}_\beta) : \beta \in \alpha \Big\}, \text{ for every ordinal } \alpha\,.$$

This recursive definition would make sense for any *set* α (thanks to the assumed well-foundedness of $\in$), but is devoid of interest unless α is an ordinal—briefly $\mathscr{O}(\alpha)$. When α is a SUCCESSOR ordinal, that is, when there is a β such that $\alpha = \beta + 1$ ($=_{\text{Def}} \beta \cup \{\beta\}$), one has by virtue of the boxing property (which continues to hold) that

$$\mathscr{V}_\alpha \quad = \quad \mathcal{P}(\mathscr{V}_\beta)\,;$$

when α *does not have* an immediate predecessor instead, we will have

$$\mathscr{V}_\alpha = \emptyset \quad \text{or} \quad \mathscr{V}_\alpha = \bigcup_{\beta < \alpha} \mathcal{P}(\mathscr{V}_\beta)$$

according as whether α is 0 or it is a *limit* ordinal. The latter expression is again the definition given at the outset, where we are stressing by use of the symbol $<$ that $\in$ acts, between ordinals, as an ordering relation.

Now the characterization of RANK, provided in Sec. 1 as the formal counterpart of an intuitive notion, becomes—within an axiomatic set theory—a definition:

$$\mathsf{rk}(X) \quad =_{\text{Def}} \bigcup \big\{ \mathsf{rk}(y) + 1 : y \in X \big\}.$$

This notion tries to capture the notion of "nesting degree" of any set—even of an infinite set. We can in fact organize the class of all sets as the following stratified hierarchy, known under the name of VON NEUMANN UNIVERSE:

$$\mathscr{V} \quad = \quad \bigcup_{\mathscr{O}(\alpha)} \mathscr{V}_\alpha\,,$$

where the union ranges over the whole *proper class* $\mathscr{O}$ of ordinals. It remains true, as it held for the pure hereditarily finite sets, that the rank of any set X

[11] In fact, just to make it independent of the *postulate of foundation*, several authors (see [25]) define an ordinal just as a transitive set well ordered by $\in$. Another definition available in literature is the following: an ORDINAL is a transitive set whose elements are transitive (cf. [30]).

is the immediate predecessor ρ of the first ordinal α for which $X \in \mathscr{V}_\alpha$ (thus $\alpha = \rho + 1$); said in another way, $\mathsf{rk}(X)$ is the first ordinal ρ for which $X \subseteq \mathscr{V}_\rho$.

One easily proves the following laws:

$$\begin{array}{rcl}
\mathscr{O}\big(\mathsf{rk}(X)\big) & \& & \Big(\mathscr{O}(Y) \leftrightarrow Y = \mathsf{rk}(Y)\Big), \\
X \in Y & \rightarrow & \mathsf{rk}(X) < \mathsf{rk}(Y), \\
X \subseteq Y & \rightarrow & \mathsf{rk}(X) \leqslant \mathsf{rk}(Y), \\
\mathsf{rk}(X) = 0 & \leftrightarrow & X = \emptyset, \\
\mathsf{rk}(X \cup Y) & = & \mathsf{rk}(X) \cup \mathsf{rk}(Y), \\
\mathsf{rk}\big(\mathcal{P}(X)\big) \;=\; \mathsf{rk}\big(\{X\}\big) & = & \mathsf{rk}(X) + 1\,, \\
\mathsf{rk}\big(\{X_0, X_1, \ldots, X_n\}\big) & = & \mathsf{rk}(\{X_0\}) \cup \mathsf{rk}(\{X_1\}) \cup \cdots \cup \mathsf{rk}(\{X_n\})\,, \\
\mathsf{rk}\big(\bigcup X\big) & = & \bigcup\big\{\mathsf{rk}(y) : y \in X\big\}, \\
\mathsf{rk}\big(\langle X, Y\rangle\big) & = & \mathsf{rk}\big(\{X, Y\}\big) + 1\,, \\
1 < \mathsf{rk}(X) & \leftrightarrow & \mathsf{rk}(X) = \mathsf{rk}\big((X \setminus 1) \cup (1 \setminus X)\big)\,.
\end{array}$$

Moreover, if we recursively define the transitive closure of a set X to be

$$\mathsf{trCl}(X) =_{\mathrm{Def}} X \cup \bigcup\{\, \mathsf{trCl}(y) : y \in X\,\}\,,$$

and denote by $\mathsf{Finite}(X)$ the *finiteness* property that a set X meets if and only if its overall number of elements is finite, then we have the following alternative characterization of hereditarily finite sets:

$$\mathsf{rk}(X) \in \omega \quad \leftrightarrow \quad \mathsf{Finite}(\mathsf{trCl}(X))\,.$$

7. Ackermann's lexicographic ordering

Let us consider a set $\mathscr{D}$ with a total and strict (i.e., antireflexive) ordering over it, and let us define the operation

$$\partial_\prec \,:\, \mathcal{P}(\mathscr{D}) \times \mathcal{P}(\mathscr{D}) \longrightarrow \mathcal{P}(\mathscr{D})$$

by putting

$$P \,\partial_\prec\, Q \quad =_{\mathrm{Def}} \quad \big\{v \in P \mid Q \supseteq \{w \in P \mid v \prec w\}\big\} \setminus Q\,.$$

Thus $P \,\partial_\prec\, Q$ is the set of those elements v of $P \setminus Q$ whose strict upper bounds inside P are all included in Q. Since $\prec$ is total, the set $P \,\partial_\prec\, Q$ has at most one element. Such an element, if it exists, is the maximum of $P \setminus Q$. Globally, the operation $\partial_\prec$ produces as result $\emptyset$ (obtainable by taking $P \subseteq Q$) and the singletons $\{v\}$ with v in $\mathscr{D}$ (obtainable, for instance, by taking $Q \subseteq \mathscr{D} \setminus \{v\}$ and $P = Q \cup \{v\}$).

We observe that when P is finite, $P\,\partial_{\prec}\,Q=\emptyset$ cannot hold, unless $P\subseteq Q$, since $P\setminus Q$ has a maximum when $P\not\subseteq Q$; things may go differently when P is infinite, as for instance is the case when P is an infinite ascending chain and Q is any set disjoint from P.

It is convenient to introduce a "symmetrization" $\delta_{\prec}$ of $\partial_{\prec}$, which gives as result—when defined for the operands X,Y—the maximum element of the symmetric difference $X\triangle Y =_{\text{Def}} (X\cup Y)\setminus(X\cap Y)$. For any subsets X,Y of $\mathscr{D}$, we put

$$X\,\delta_{\prec}\,Y =_{\text{Def}} (X\cup Y)\,\partial_{\prec}\,(X\cap Y)\,.$$

At this point we could easily introduce the LEXICOGRAPHIC ordering $\prec\!\cdot$ of the *finite* subsets $\{X\subseteq\mathscr{D}\mid \mathsf{Finite}(X)\}$ of $\mathscr{D}$ by putting

$$X \prec\!\cdot\, Y \leftrightarrow_{\text{Def}} (X\,\delta_{\prec}\,Y)\cap Y\neq\emptyset\,.$$

It is easy to see that we are dealing with an ordering which is total and strict as well, and which will also be a well-ordering when $\prec$ is such.

Having oriented $\prec\!\cdot$ in the way we did, rather than in the opposite way, yields the relation $\prec\subseteq\prec\!\cdot$ in the particular case when $\mathscr{D}=\mathscr{V}_{\alpha}$ is a level of the cumulative hierarchy and, accordingly, $\mathsf{Trans}(\mathscr{D})$ holds. By reiterating the construction of the induced ordering over all ordinals, without worrying about assigning a distinct name to each of the orderings, we come up to define *d'emblée*

$$\begin{array}{lll} P\,\partial\,Q & =_{\text{Def}} & \{v\in P\mid Q\supseteq\{w\in P\mid v\lhd w\}\}\setminus Q\,,\\ X\lhd Y & \leftrightarrow_{\text{Def}} & \big((X\cup Y)\,\partial\,(X\cap Y)\big)\cap Y\neq\emptyset\,. \end{array}$$

However cryptic this formulation may look, it is easy to realize that it is not affected by a vicious circularity; on the contrary, it relies on an acceptable form of recursion. Indeed, when we expand the $\lhd$ appearing in the first *definiens* according to the second line of the definition,

$$P\,\partial\,Q \;=_{\text{Def}}\; \{v\in P\mid Q\supseteq\{w\in P\mid \big((v\cup w)\,\partial\,(v\cap w)\big)\cap w\neq\emptyset\}\}\setminus Q\,,$$

we note that the operand $v\cup w$ of the ∂ occurring on the right has a rank lower than the rank of the P in the *definiendum.* Due to the well foundedness of $\in$, the rank of the left operand of ∂ may decrease a finite number of times only, till it reduces to zero; but then, as it immediately follows from the definition of ∂, we have that

$$\emptyset\,\partial\,\_ \;=\; \emptyset\,.$$

To grasp, in the limited context of *hereditarily finite* sets P,Q,X,Y, the meaning of the definitions of ∂ and $\lhd$, suppose we know that the relation $X\lhd Y$ establishes a *total ordering* as long as X,Y range over a certain level $\mathscr{V}_i$ of the hierarchy $\mathscr{V}_{\omega}$ (which is obvious when $i<2$). Then consider a pair X',Y' of sets of level $i+1$, one of them at least having rank i, and determine $(X'\cup Y')\partial(X'\cap Y')$. The latter is obtained from the family of all sets that, in the union $X'\cup Y'$, are bounded from above only by elements of the intersection $X'\cap Y'$: if none among them lies outside the intersection, then $X'=Y'$ and

$$\emptyset=(X'\cup Y')\partial(X'\cap Y')=\big((X'\cup Y')\partial(X'\cap Y')\big)\cap Y'=\big((X'\cup Y')\partial(X'\cap Y')\big)\cap X'$$

otherwise there will be exactly one of them, the placement of which, either in Y' or in X', establishes whether $X'\lhd Y'$ or vice versa.

All this can be summarized by saying that $\lhd$, restricted to any level $\mathscr{V}_\beta$ with $\beta\leqslant\omega$, is a LEXICOGRAPHIC ORDERING (see Sec. 3).

The ordering of $\mathscr{V}_\omega$ just proposed differs from the familiar lexicographic ordering of *strings over an alphabet* because of the following properties:

- The "alphabet" is not fixed from the beginning; it acquires instead new "characters" at each layer (i.e. all the sets which are added at each level).
- Unlike characters in strings, which may be repeated and occur in an arbitrary order, set elements may appear only once and must occupy the positions imposed by $\lhd$ itself.
- The "scanning" of the elements to determine which set comes lexicographically before the other, is not performed "from left to right", but proceeds from the big towards the small elements.

It is because of these properties that we could specify the ordering $\lhd$ of $\mathscr{V}_\omega$ *in a declarative way* (such specification abstracts, for instance, from the concept of scanning). Rather than recursively describing the lexicographic ordering "from the top", it is easy—as we will see in Sec. 17—to integrate its construction in a technique for generating iteratively the hierarchy "from the bottom" by levels.

Unlike its behaviour in $\mathscr{V}_\omega$, the relation $\lhd$ is no longer well founded already in $\mathscr{V}_{\omega+1}$, where one can find, for instance, the following infinite descending chain:

$$\cdots\lhd\mathscr{V}_\omega\setminus n\lhd\cdots\lhd\mathscr{V}_\omega\setminus 2\lhd\mathscr{V}_\omega\setminus 1\lhd\mathscr{V}_\omega\qquad(\text{with } n\in\omega).$$

In order to extend the restriction of $\lhd$ to $\mathscr{V}_\omega$ to the whole universe $\mathscr{V}$, in such way it remains a lexicographic well ordering, we should operate again from the bottom (see Sec. 11 below). In other words, we should order the sets by increasing ranks, keeping as a *requirement* to be satisfied for the lexicographicity the pair of conditions by means of which we have introduced, simultaneously, ∂ and $\lhd$: such conditions should hold only for sets of *finite* cardinality (even though possibly of an infinite rank).

We mean that a well ordering $\prec$ of $\mathscr{V}$ shall be regarded as LEXICOGRAPHIC if and only if it satisfies the condition

$$\big(\mathsf{Finite}(X)\,\&\,\mathsf{Finite}(Y)\big) \rightarrow \big((X \cup Y)\,\partial_{\prec}\,(X \cap Y)\big) \cap Y \neq \emptyset\,,$$

where $\partial_{\prec}$ is tied to $\prec$ as said at the beginning of the section.

8. Arithmetics of set theoretic operations

The Ackermann's[12] ordering

$$\emptyset,\ \{\emptyset\},\ \{\{\emptyset\}\},\ \{\emptyset,\{\emptyset\}\},\ \Big\{\{\{\emptyset\}\}\Big\},\ \Big\{\emptyset,\{\{\emptyset\}\}\Big\},\ \Big\{\{\emptyset\},\{\{\emptyset\}\}\Big\},\ldots$$

introduced above enables us to establish a bijective correspondence

$$p \ \longmapsto\ \widehat{p}$$

between natural numbers and pure hereditarily finite sets (cf. [1, 27]). For example, since 0, 1, 2, 3, and 4 are the *positions* belonging to the sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, $\{\emptyset,\{\emptyset\}\}$ and $\Big\{\{\{\emptyset\}\}\Big\}$, respectively, we have that

$$\widehat{0} = \emptyset,\ \widehat{1} = \{\emptyset\},\ \widehat{2} = \{\{\emptyset\}\},\ \widehat{3} = \{\emptyset,\{\emptyset\}\},\ \widehat{4} = \Big\{\{\{\emptyset\}\}\Big\},$$

and so on.

It is interesting to notice that if we express the position numbers in binary notation, this can be elegantly interpreted as a membership function, in the sense that the i^{th} bit of p indicates whether the set $\widehat{i}$ is an element of $\widehat{p}$ or not. Thus we can very naturally represent an hereditarily finite set by a binary string of finite length or, more generally, by an infinite binary string containing only a finite number of the digit 1.

Having established these correspondences, it is interesting to find out which relations and operations between numbers and strings correspond to the basic relations and set theoretical operations. The answer is immediate in the case of binary strings, or *bitmaps*: intersection, union, and symmetric difference just correspond to the bitwise Boolean connectives of conjunction, disjunction, and exclusive disjunction.

Concerning the *numerical* representation, the answer must be more elaborate: for instance, a partial list of operations and relations can be found in the following table.

[12] Wilhelm Ackermann, 1896–1962.

$\widehat{q} \lhd \widehat{p}$	$q < p$
$\widehat{q} \in \widehat{p}$	$\lfloor p/2^q \rfloor \mod 2 = 1$
$\emptyset$	0
$\{\widehat{p}\}$	2^p
$\widehat{p} \cup \widehat{q}$	$p+q$ when $\widehat{p} \cap \widehat{q} = \emptyset$
$\widehat{p} \setminus \widehat{q}$	$p-q$ when $\widehat{p} \subseteq \widehat{q}$
$\max\ \widehat{p}$	$\lfloor \log_2 p \rfloor$
$\langle \widehat{p}, \widehat{q} \rangle$	$(1+2^{2^p}) \cdot (\mathsf{if}\ \ p = q\ \ \mathsf{then}\ \ 1\ \ \mathsf{else}\ \ 2^{2^q}\ \ \mathsf{fi})$

Despite leading to slightly less natural specifications, this representation is better suited for a high level language like Maple, in which we have implemented the operations treated here. Thus we will insist on it in what follows.

We observe that the membership test looks quite elaborate, but in essence it just selects the q^{th} bit of p and verifies whether its value is 1; analogously, singleton formation amounts to flipping one bit in the string constituted by all zeros to 1. The union of two sets (very natural when viewed as a bitmap operation) is correctly emulated by the arithmetic sum only when the operands represent disjoint sets, and so on. For the sake of brevity, we denote by $\mathtt{hd}(\_)$ and $\mathtt{tl}(\_)$ the "head" and "tail" selectors of a nonempty set, respectively:

$$\mathtt{hd}(p) \quad =_{\mathrm{Def}} \quad \lfloor \log_2 p \rfloor , \qquad\qquad \mathtt{tl}(p) \quad =_{\mathrm{Def}} \quad p - 2^{\lfloor \log_2 p \rfloor} .$$

In view of the considerations we are carrying out, these selectors represent the operations $\max\ \widehat{p}$ and $\widehat{p} \setminus \{\max\ \widehat{p}\}$, unless $\widehat{p}$ is empty.

Going beyond such introductory stage requires, in some points, relatively sophisticated algorithms. Which positions belong, for instance, to the number n (intended *à la* von Neumann) and to the level $\mathscr{V}_n$? The answer is given by the following recursive specifications:

$$\begin{aligned} \mathtt{num}(n) \quad &=_{\mathrm{Def}} \quad \mathsf{if}\ \ n = 0\ \ \mathsf{then}\ \ 0\ \ \mathsf{else}\ \ \mathtt{num}(n-1) + 2^{\mathtt{num}(n-1)}\ \ \mathsf{fi}\,; \\ \mathtt{cum}(n) \quad &=_{\mathrm{Def}} \quad \mathsf{if}\ \ n = 0\ \ \mathsf{then}\ \ 0\ \ \mathsf{else}\ \ 2^{\mathtt{cum}(n-1)+1} - 1\ \ \mathsf{fi}\,. \end{aligned}$$

Notice that the latter specification is equivalent to the observation that in $\lhd$ the immediate successor of $\mathscr{V}_n$, that is the first set of rank $n+1$ (with $n \neq 0$), is the singleton

$$\underbrace{\{\cdots\{}_{n+1 \text{ times}}\ \emptyset\ \}\cdots\}$$

having as element the immediate successor of $\mathscr{V}_{n-1}$.

To translate the Boolean operations $\triangle, \cap$ of symmetric difference and intersection (by means of which the definitions of $\cup$ and of $\setminus$ are immediate), a selector arb conforming to the specifications given in Sec. 4, and conjugated projections $\_^{[1]}, \_^{[2]}$ relative to the pairing operation $\langle \_, \_ \rangle$, we can proceed as follows:

$$\begin{aligned}
\mathtt{sy}(p,q) \quad &=_{\text{Def}} \quad \textsf{if } p=0 \textsf{ then } q \textsf{ elseif } q=0 \textsf{ then } p \\
&\qquad \textsf{elseif } \mathtt{hd}(p) < \mathtt{hd}(q) \textsf{ then } \mathtt{sy}(p,\mathtt{tl}(q)) + 2^{\mathtt{hd}(q)} \\
&\qquad \textsf{elseif } \mathtt{hd}(q) < \mathtt{hd}(p) \textsf{ then } \mathtt{sy}(q,\mathtt{tl}(p)) + 2^{\mathtt{hd}(p)} \\
&\qquad \textsf{else } \mathtt{sy}(\mathtt{tl}(p),\mathtt{tl}(q)) \textsf{ fi};
\end{aligned}$$

$$\begin{aligned}
\mathtt{nt}(p,q) \quad &=_{\text{Def}} \quad \textsf{if } p=0 \textsf{ or } q=0 \textsf{ then } 0 \\
&\qquad \textsf{elseif } \mathtt{hd}(p) < \mathtt{hd}(q) \textsf{ then } \mathtt{nt}(p,\mathtt{tl}(q)) \\
&\qquad \textsf{elseif } \mathtt{hd}(q) < \mathtt{hd}(p) \textsf{ then } \mathtt{nt}(q,\mathtt{tl}(p)) \\
&\qquad \textsf{else } \mathtt{nt}(\mathtt{tl}(p),\mathtt{tl}(q)) + 2^{\mathtt{hd}(p)} \textsf{ fi};
\end{aligned}$$

$$\begin{aligned}
\mathtt{arb}(p) \quad &=_{\text{Def}} \quad \textsf{if } p=0 \textsf{ then } 0 \textsf{ else } \min_{q\in\mathbb{N}}\left(\widehat{q}\in\widehat{p}\right) \textsf{ fi}; \\
\mathtt{sn}(p) \quad &=_{\text{Def}} \quad \mathtt{hd}\Big(\mathtt{hd}(p) \setminus \mathtt{hd}\big(\mathtt{tl}(p)\big)\Big); \\
\mathtt{dx}(p) \quad &=_{\text{Def}} \quad \textsf{if } p \mod 2 = 1 \textsf{ then } \mathtt{hd}\big(\mathtt{hd}(p)\big) \textsf{ else } \mathtt{hd}\Big(\mathtt{hd}\big(\mathtt{tl}(p)\big)\Big) \textsf{ fi}.
\end{aligned}$$

Then we translate the monadic operations $\bigcup$, $\mathcal{P}$ into the following $\mathtt{U}(\_), \mathtt{P}(\_)$:

$$\begin{aligned}
\mathtt{U}(p) \quad &=_{\text{Def}} \quad \textsf{if } p=0 \textsf{ then } 0 \textsf{ else } \mathtt{un}\Big(\mathtt{U}\big(\mathtt{tl}(p)\big),\mathtt{hd}(p)\Big) \textsf{ fi}, \\
\mathtt{S}(r,k,q) \quad &=_{\text{Def}} \quad \textsf{if } r=0 \textsf{ then } k \textsf{ else } \mathtt{S}\big(\mathtt{tl}(r),k,q\big) + 2^{\mathtt{hd}(r)+2^q} \textsf{ fi}, \\
\mathtt{P}(p) \quad &=_{\text{Def}} \quad \textsf{if } p=0 \textsf{ then } 1 \textsf{ else } \mathtt{S}\Big(\mathtt{P}\big(\mathtt{tl}(p)\big),\mathtt{P}\big(\mathtt{tl}(p)\big),\mathtt{hd}(p)\Big) \textsf{ fi}.
\end{aligned}$$

We can finally introduce the operation of *choice*

$$\mathtt{ch}(p) \quad =_{\text{Def}} \quad \textsf{if } p \leqslant 1 \textsf{ then } 0 \textsf{ else } \mathtt{ch}\big(\mathtt{tl}(p)\big) + 2^{\mathtt{hd}(\mathtt{hd}(p))} \textsf{ fi},$$

which, *in case* $\widehat{p}$ is a PARTITION, that is a set of pairwise disjoint nonempty "*blocks*", returns the position of a set whose intersection with each block is a singleton.

9. Intensionally specified subsets

When we draw from a set x the elements y that enjoy a certain property $\varphi(y)$, to form the subset

$$\{\, y \in x \mid \varphi(y) \,\},$$

the latter is said to be obtained by SEPARATION.

Which forms of specification should be allowed for $\varphi(\_)$? When, as is customary nowadays, set theory is formalized as an axiomatic theory in a first order predicate language $\mathscr{L}_{\in,=}$, it is generally admitted that $\varphi(y)$ may be *any* formula of $\mathscr{L}_{\in,=}$, within which y usually occurs as a free variable. This liberal criterion was proposed by the Norwegian mathematician Skolem.[13] Before him, in [20], Fraenkel[14] proposed for $\varphi(y)$ two acceptable forms only:

$$\sigma(y) \notin \tau(y), \qquad \sigma(y) \in \tau(y),$$

where $\sigma(\_)$ and $\tau(\_)$ are called functions and are determined out of descriptions of set constructions recursively defined in Sec. 15.

In the hierarchy of hierarchies of descriptions of sets represented in Sec. 15, the first one of the two forms, $\sigma(y) \notin \tau(y)$, is the most important one since it allows to construct new sets by separation. For instance the subset $X = \{a_2, a_3, \ldots\}$ (of class 1 and level 1 in the hierarchy) of the set $A = \{a_0, a_1, \ldots\}$ is obtained for the first time by a "negative" separation, namely $X = \{x \in A | x \notin \{a_0, a_1\}\}$. X can also be represented by a (more complicated) description of level 2 by applying a "positive" separation, such as $X = \{x \in A | x \in \{x \in A | x \notin \{a_0, a_1\}\}\}$.

Since it trivially holds that

$$\{y \in x \mid \sigma(y) \in \tau(y)\} \;=\; \{y \in x \mid y \in \{y \in x \mid \sigma(y) \notin \tau(y)\}\},$$

if we accept Fraenkel's criterion, it is important to have the "negative" separation, as we can build the other one. The negative separation can be implemented, for pure hereditarily finite sets, as follows

$$\begin{array}{lcl} \mathtt{sp}\big(p, q \mapsto f(q), q \mapsto g(q)\big) & =_{\mathrm{Def}} & \mathsf{if}\ \ p = 0\ \ \mathsf{then}\ \ 0\ \ \mathsf{else} \\ \mathtt{sp}\big(\mathtt{tl}(p), q \mapsto f(q), q \mapsto g(q)\big) & + & \mathtt{if}\ \mathtt{nin}\Big(f\big(\mathtt{hd}(p)\big), g\big(\mathtt{hd}(p)\big)\Big) \\ & & \mathtt{then}\ \ 2^{\mathtt{hd}(p)}\ \ \mathsf{else}\ \ 0\ \ \mathsf{fi}\ \ \mathsf{fi} \end{array}$$

(where $\mathtt{nin}(p,q) \;=_{\mathrm{Def}}\; (\lfloor p/2^q \rfloor \mod 2 = 0)$); the positive one can be implemented indirectly, as

$$\mathtt{sp}\Big(p, q \mapsto q, q \mapsto \mathtt{sp}\big(p, q \mapsto f(q), q \mapsto g(q)\big)\Big).$$

A similar argument holds for the REPLACEMENT schema, which is more general than separation and enables one to obtain a set

$$\{\,\vartheta(y) : y \in x \mid \varphi(y)\,\}$$

[13] Thoralf Albert Skolem, 1887–1963.

[14] Adolf Abraham Halevi Fraenkel, 1891–1965.

from a set x whose elements are first filtered according to a condition $\varphi(\_)$ and then transformed according to a set theoretical expression $\vartheta(\_)$. Allowing the same Fraenkel's format seen above to specify the condition $\varphi(\_)$, we can again confine ourselves to a replacement of "negative" kind, because

$$\{\vartheta(y) : y \in x \mid \sigma(y) \in \tau(y)\} \;=\; \{\vartheta(y) : y \in x \mid y \notin \{y : y \in x \mid \sigma(y) \notin \tau(y)\}\}.$$

The negative replacement will be implemented directly as

$$\begin{array}{lll}
\mathtt{rp}\big(p,\, q \mapsto f(q),\, q \mapsto g(q),\, q \mapsto h(q)\big) & =_{\text{Def}} & \mathsf{if}\;\; p = 0 \;\; \mathsf{then}\;\; 0 \;\; \mathsf{elseif} \\
\mathtt{in}\Big(f\big(\mathtt{hd}(p)\big), g\big(\mathtt{hd}(p)\big)\Big) \vee & & \\
\mathtt{in}\Big(\mathtt{hd}(p), \mathtt{rp}\big(\mathtt{tl}(p),\, q \mapsto f(q),\, q \mapsto g(q),\, q \mapsto h(q)\big)\Big) & & \\
 & & \mathtt{then} \\
\mathtt{rp}\big(\mathtt{tl}(p),\, q \mapsto f(q),\, q \mapsto g(q),\, q \mapsto h(q)\big) & & \mathtt{else} \\
\mathtt{rp}\big(\mathtt{tl}(p),\, q \mapsto f(q),\, q \mapsto g(q),\, q \mapsto h(q)\big) & + & 2^{\mathtt{hd}(p)} \;\; \mathsf{fi}
\end{array}$$

(where $\mathtt{in}(p,q) \;=_{\text{Def}}\; (\lfloor p/2^q \rfloor \mod 2 = 1)$); the positive one indirectly, as

$$\mathtt{rp}\Big(p,\, q \mapsto q,\, q \mapsto \mathtt{rp}\big(p,\, q \mapsto f(q),\, q \mapsto g(q),\, q \mapsto q\big),\, q \mapsto h(q)\Big).$$

Starting with replacement, we could implement more briefly the constructs introduced from Sec. 8 on: directly in terms of negative replacement, `sp`; in terms of the same (and using `U` and pair formation to realize `un`), `sy`; in terms of positive replacement, `nt` and, by means of a self-contained recursion (i.e. one which avoids the `S` defined *ad hoc* above), `P`; in terms of positive replacement, and exploiting `arb`, `ch`.

To show that replacement is more powerful than separation, let us consider the infinite set

$$Z_0 \;=_{\text{Def}}\; \Big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \Big\{\{\{\emptyset\}\}\Big\}, \dots\Big\}.$$

By means of separation, we could form a countable infinity of subsets of Z_0, but not a set so akin to it as

$$\Big\{2, \{2\}, \{\{2\}\}, \Big\{\{\{2\}\}\Big\}, \dots\Big\},$$

which is easily obtainable by replacement as

$$\{\vartheta(x) : x \in Z_0 \mid x \notin \emptyset\},$$

by taking

$$\vartheta(X) \;=\; \mathsf{if}\;\; X = \{\mathsf{arb}(X)\} \;\; \mathsf{then}\;\; \{\vartheta(\mathsf{arb}(X))\} \;\; \mathsf{else}\;\; 2 \;\; \mathsf{fi}.$$

10. Emulation of superstructures in $\mathscr{V}$

> *We now give an informal discussion of the axioms although our results are precise and could be formalized. If the Axiom of Extensionality is dropped, the resulting system may contain atoms, i.e., sets x such that $\forall y(\sim y \in x)$ yet the sets x are different. Indeed, one possible view is that the integers* are *atoms and should not be viewed as sets. Even in this case, one might still wish to prevent the existence of unrestricted atoms. In any case, for the "genuine" sets, Extensionality holds and the other sets are merely harmless curiosities. Fraenkel and Mostowski have used atoms to obtain results about the Axiom of Choice* $[\cdots]$
>
> [13, p. 54]

We will now see how it is possible, starting with any "genuine" set B (i.e., a set drawn from $\mathscr{V}$), to form a cumulative hierarchy "functionally equivalent" to the superstructure $\mathscr{V}_\omega^B$ introduced in Sec. 4, masking the elements of B as individuals. Since the elements x of B are what they are (that is, they are sets of unrestrained structure), we replace each of them with its image $\bar{x} = \big\{\{x, B \cup \mathbb{N}\}\big\}$, where the map

$$x \longmapsto \bar{x}$$

is clearly bijective. Chosen in this way, all sets corresponding to the elements x of B have equal rank (since the rank ρ of $B \cup \mathbb{N}$ always predominates over the one of x); we remark that such rank $(\rho+1)+1$ exceeds the rank of any element of the pure hierarchy $\mathscr{V}_\omega$ $(\subseteq \mathscr{V}_\omega^B)$. If in constructing the cumulative hierarchy we use, instead of the elements x of B, their corresponding sets $\bar{x}$ of high rank, the hierarchy will come to include sets of "low" rank (that is lower than ω), that we can take as being "pure", on one side, and sets of "high" rank (that is, higher than $\rho+1$), on the other. Thus, it will never happen that one of such sets be member of any $\bar{x}$; in this sense, the $\bar{x}$s behave (relative to the rest of the superstructure) like individuals.

Obviously we do not intend in the least, through the technique just proposed for eliminating individuals, to close the debate about the usefulness of having a reservoir of individuals in Set Theory. The quotations of Cohen and Barwise-Moss which appear, respectively, at the top of this section and of Sec. 4, clearly witness that such question is controversial; speaking more generally, it is controversial whether it is convenient to intermix the classical heterogeneous form of set aggregation with rigid forms of typing more or less strictly related to the ones involved in programming.

Taking into account only the technical aspects of the question, we cannot even be sure that satisfiability tests conceived for several decidable fragments of set theory devoid of individuals (see, for instance, [8]) reflect into analogous decision algorithms for set theory with individuals, although some case studies (e.g. [15]) may encourage this expectation.

11. A typical exploitation of the notion of rank

The rank notion may turn out useful when one wants to extend a relation R, initially defined over a set S, i.e.

$$R \subseteq S \times S,$$

to the whole class of sets, preserving some of its properties. As an example, we will see below how to globalize a "toggle" function.

We first draw a distinction of some importance when reasoning about Set Theory: the one between set and *class*. All sets are classes, but it may happen that one has to consider classes "too wide"—a hint will do—in order to consider them sets: the so-called PROPER CLASSES. Two proper classes which have already occurred in this paper are $\mathscr{O}$, that is the class of all ordinals, and $\mathscr{V}$, the class of all sets: indeed, recognizing the *status* of set to similar classes would quickly lead to logical antinomies.

The words *relation* and *function* have the same basic ambiguity: if we think of relations as of *sets* of ordered pairs, do we recognize the *status* of function to an operation like the $X \mapsto \mathcal{P}(X)$, that has $\mathscr{V}$ as its domain? Evidently, the latter is not a set but a *class* of ordered pairs, and since its operand ranges over all sets, we can say that it is a GLOBAL FUNCTION. (On the other hand we will call LOCAL a function f having as its domain a set and consequently being itself a set). On many occasions, to be more explicit, we will indicate by $f \upharpoonright x$ the image through a *local* function f of an element x of its domain, and by $\varphi(x)$ the image through a global function φ of any set x.

Let us now consider a T enjoying the following properties:

- T is a function, understood as a *set* of ordered pairs, such that when $p \in T$ and $q \in T$ have the same first component, $p^{[1]} = q^{[1]}$, then the second one is the same as well, $p^{[2]} = q^{[2]}$; in short, $p = q$;
- T is *self-inverse*: $\{\langle p^{[2]}, p^{[1]}\rangle \,:\, p \in T\} = T$;
- T is *anti-diagonal*: $\{p \in T \,|\, p^{[1]} = p^{[2]}\} = \emptyset$.

The second and the third property can be expressed by asserting that, for every x in the domain of T,

$$T \upharpoonright x \neq x \;\;\&\;\; T \upharpoonright (T \upharpoonright x) = x\,.$$

We call a TOGGLE such a function. Clearly, over a finite domain, we can define a toggle provided that the number of elements is even, whereas over an infinite set a toggle is *always* definable.

Let us suppose we want to extend to the whole von Neumann universe the domain of a given toggle T. In other words, we want to find a global function $\mathsf{tog}(\_)$ such that

$$\mathsf{tog}(x) = T \restriction x \text{ for every } x \text{ in the domain of } T\,,$$

also satisfying the condition

$$\mathsf{tog}(x) \neq x \;\;\&\;\; \mathsf{tog}\big(\mathsf{tog}(x)\big) = x$$

even when x lies outside such a domain.

For such purpose we can proceed this way:

- Let us consider the infinitely many singletons

$$\{T\},\ \{\{T\}\},\ \{\{\{T\}\}\},\cdots$$

  of increasing ranks, all exceeding $\mathsf{rk}(T)$.

- We observe that the union λ of such ranks, being the rank of a set, is an ordinal...

- ...and that $\mathscr{V}_\lambda \setminus \mathsf{domain}(T)$, being an infinite set, has a toggle $\bar{T}$.

- For every set x, we stipulate that

$$\mathsf{tog}(x) \;=\; \begin{cases} T \restriction x & \text{if } x \text{ is in the domain of } T, \\ \bar{T} \restriction x & \text{if } x \text{ is in the domain of } \bar{T}, \\ (x \setminus 1) \cup (1 \setminus x) & \text{otherwise.} \end{cases}$$

This example has analogous, much more powerful, constructions. Suppose, for example, we want to impose a well ordering to the universe $\mathscr{V}$. We can vacuously order $\mathscr{V}_0$, then showing how the ordering $<_\beta$ of a level $\mathscr{V}_\beta$ of the hierarchy can be extended to an ordering $<_{\beta+1}$ of the subsequent level $\mathscr{V}_{\beta+1}$[15] and how, finally, when λ is a limit ordinal, it is possible to combine all the orderings $<_\beta$, with β preceding λ, into a well ordering $<_\lambda$ of $\mathscr{V}_\lambda$, by simply posing $<_\lambda = \bigcup\{<_\beta : \beta \in \lambda\}$. At this point, it only remains to notice how the orderings

[15]Here the well known consequence of the axiom of choice comes into play that any set can be well ordered. For instance, the order of $<_{\omega+1}$ could be obtained as union of the lexicographic order $<_\omega$ with an ordering $\lessdot$ of $\mathscr{V}_{\omega+1} \setminus \mathscr{V}_\omega$ of which the axiom of choice guarantees the existence and with the relation $\{\,\langle u,v\rangle : u \in \mathscr{V}_\omega, v \in \mathscr{V}_{\omega+1} \setminus \mathscr{V}_\omega\,\}$.

so imposed at the different levels of the hierarchy are naturally amalgamated into a single global ordering of $\mathscr{V}$, which furthermore agrees with rank comparison. By proceeding suitably [8, pp. 57–61], it is easy to make such an ordering *lexicographic* (see Sec. 7) over finite sets and to enforce that, within each rank, finite sets precede infinite sets.

The construction just outlined enables us to roughly assess the distance between the postulates of *local* and *universal* choice (cf. [18, 26]), formalizable (among many other possibilities) as follows:

local choice: $\forall x \exists f \forall y \big(y \in x \rightarrow (y = \emptyset \vee f \restriction y \in y)\big)$;

universal choice: $\forall y \big(y = \emptyset \vee \mathsf{arb}(y) \in y\big)$.

On the one hand we have made use of the former of these (even though in another, equivalent version) to impose a well ordering on $\mathscr{V}$; on the other hand, once this ordering has been constructed, we can interpret the global selector arb as the function which extracts the minimum element from each nonempty set. Incidentally, since we have managed to make the global ordering compliant with rank comparison, we can exploit arb also as a witness of the well-foundedness of membership; in fact, for free we can strenghten the constraint imposed on arb by the additional requirement that

$$\forall x \big(x \cap \mathsf{arb}(x) = \emptyset\big) \ \& \ \mathsf{arb}(\emptyset) = \emptyset\,.$$

Notice that we developed this construction at a semantical level *external* to Zermelo-Fraenkel. The latter gets usually formulated as an axiomatic theory within first-order predicate logic; but with its variant implemented inside the Referee system (see Sec. 16), to which the THEORY construct confers a touch of second-order logic, our construction of a well ordering of $\mathscr{V}$ is certainly reproducible inside the formal system.

12. The Gödel universe of constructibles

> *Perhaps the shortest description of L is that it is the smallest transitive model of the axioms of L_1Set which contains all the ordinals. But the working definition of L, from which the name "constructible universe" is derived, is rather different.*
>
> [29, p. 149]

Let us consider the following global operations:

$$
\begin{array}{lll}
F_1(X,Y) & =_{\text{Def}} & \{X,Y\}, \\
F_2(X,Y) & =_{\text{Def}} & X \setminus Y, \\
F_3(X,Y) & =_{\text{Def}} & X \times Y, \\
F_4(X) & =_{\text{Def}} & \{u : \langle u,v \rangle \in X\}, \\
F_5(X) & =_{\text{Def}} & \{\langle u,v \rangle : u \in X, v \in X \mid u \in v\}, \\
F_6(X) & =_{\text{Def}} & \{\langle\langle u,v \rangle, w\rangle : \langle\langle v,w \rangle, u\rangle \in X\}, \\
F_7(X) & =_{\text{Def}} & \{\langle\langle u,v \rangle, w\rangle : \langle\langle w,v \rangle, u\rangle \in X\}, \\
F_8(X) & =_{\text{Def}} & \{\langle\langle u,v \rangle, w\rangle : \langle\langle u,w \rangle, v\rangle \in X\}.
\end{array}
$$

It is possible to prove that every set S of the von Neumann's hierarchy $\mathscr{V}$ has a superset which is closed with respect to these eight operations; so, it will have a superset $\mathfrak{I}(S)$ equally closed and minimal with respect to inclusion. We can therefore exploit a construction similar to the one of $\mathscr{V}$, but with a "slow growth", in order to obtain the following HIERARCHY OF CONSTRUCTIBLES:

$$
\begin{array}{lll}
\mathscr{L}_\alpha & =_{\text{Def}} & \bigcup\left\{\mathcal{P}(\mathscr{L}_\beta) \cap \mathfrak{I}(\mathscr{L}_\beta \cup \{\mathscr{L}_\beta\}) : \beta \in \alpha\right\} \quad \text{for every ordinal } \alpha, \\
\mathscr{L} & =_{\text{Def}} & \bigcup_{\mathscr{O}(\alpha)} \mathscr{L}_\alpha .
\end{array}
$$

This develops exactly like the von Neumann's hierarchy up to the first infinite ordinal ω, but starts suffering the cardinality limitation

$$|\mathscr{L}_\alpha| = |\alpha| \quad \text{when } \omega < \alpha$$

(i.e., $\mathscr{L}_\alpha \subsetneq \mathscr{V}_\alpha$) beyond such threshold.

Growth limitation apart, $\mathscr{L}$ has several common traits with $\mathscr{V}$; first of all, it is indeed a hierarchy, in the sense that every $\mathscr{L}_\alpha$ is transitive and is strictly included in each $\mathscr{L}_\gamma$ for $\alpha < \gamma$. It is easy to observe that, in general, $\mathscr{L}_\alpha \subseteq \mathscr{V}_\alpha$ (the inclusion being strict when α is beyond ω) and that $\mathscr{L}$, which includes all ordinals as its own elements, is a proper class. Concerning the question whether

$$\mathscr{L} = \mathscr{V} \ ?$$

the axiomatic theory of Zermelo-Fraenkel does not pronounce either affirmatively or negatively; however, it is possible to *impose* an affirmative answer by laying down a specific CONSTRUCTIBILITY POSTULATE [31], which as a side benefit makes the so-called GENERALIZED CONTINUUM HYPOTHESIS [13] provable, regarding whose truth the usual axioms have no power to decide.

13. Automatic synthesis of universes of sets

We consider a structure $\mathscr{U}, \mathbf{\emptyset}, \in, \oplus, \ominus$ formed by five components so interrelated:

- $\mathscr{U}$ is the *domain of discourse* and $\mathbf{\emptyset}$ belongs to such a domain;
- $\in$ is a binary relation over $\mathscr{U}$;
- $\oplus$, $\ominus$ are binary operations over $\mathscr{U}$, namely for each pair of operands x, y drawn from $\mathscr{U}$ the results $x \oplus y$ and $x \ominus y$ belong to $\mathscr{U}$.

Intuitively speaking, we want that the structure in question satisfies those minimal requirements that anybody expects that must hold for nested sets. From this point of view, $\mathscr{U}$ represents the totality of the so-called "sets", $\in$ (which does not need to be the usual $\in$) plays the rôle of "membership" relation among such entities, $\mathbf{\emptyset}$ acts as "empty set", $\oplus$ and $\ominus$ act as the operations of addition / removal of a single element to / from a set. For simplicity, we leave ur-elements outside our considerations.

The formal "minimal" properties that we want to be satisfied by our structure are the following ones (these, by definition, will make our quintuple a UNIVERSE OF SETS). For any X, Y in $\mathscr{U}$:

1. X does not precede $\mathbf{\emptyset}$ in $\in$.
2. If X and Y are distinct, they cannot have the same immediate predecessors in $\in$; that is to say, some v in $\mathscr{U}$ will witness their difference satisfying one but not the other of the two relations $v \in X$, $v \in Y$.
3. The entities v in $\mathscr{U}$ such that $v \in (X \oplus Y)$ are the same for which $v \in X$ holds, *plus* (possibly, if $Y \in X$ is false) the $v = Y$.
4. The entities v in $\mathscr{U}$ such that $v \in (X \ominus Y)$ are the same for which $v \in X$ holds, *with the exception* (if $Y \in X$) of $v = Y$.

Such a universe of sets will then be said COMPUTABLE if

- there is a Herbrand universe $\widehat{\mathscr{U}}$ including $\mathscr{U}$ such that for each element v of $\widehat{\mathscr{U}}$ one can algorithmically establish whether v belongs to $\mathscr{U}$ or not;
- operations $\oplus$, $\ominus$ are computable;
- there is a computable binary operation η over $\mathscr{U}$ such that for X, Y in $\mathscr{U}$:
  - if $Y \neq \mathbf{\emptyset}$, then at least one of the relations $(X \,\eta\, Y) \in X$, $(Y \,\eta\, X) \in Y$ holds;

– if both relations $(X\,\eta\,Y) \in X$ and $(X\,\eta\,Y) \in Y$ hold, then $X = Y$.

A UNIVERSE OF CLASSES (computable or not) is defined much in the same way; but its domain of discourse is partitioned as

$$\mathscr{U} = \mathscr{U}_0 \cup \mathscr{U}_1,$$

where $\mathscr{U}_0$ and $\mathscr{U}_1$ are disjunct domains and where the operations $X \oplus Y$ and $X \ominus Y$ are allowed only when Y belongs to $\mathscr{U}_0$. The idea is that $\mathscr{U}_0$ collects the "sets" whereas $\mathscr{U}_1$ contains the "proper classes": thus, the limitation imposed to $\oplus$ (and, less significantly, to $\ominus$) reflects the intuitive idea that "proper classes are too big to belong as elements to other classes (or, in particular, to sets)".

Example. We take as domain of discourse $\mathscr{U}$ the collection $\mathscr{V}_\omega$ of pure hereditarily finite sets extended with all their complements in $\mathscr{V}_\omega$ itself, namely

$$\mathscr{U} \;=\; \mathscr{V}_\omega \cup \{\mathscr{V}_\omega \setminus x : x \in \mathscr{V}_\omega\}.$$

Further, we choose to take as $\boldsymbol{\emptyset}$, $\in$, $\oplus$, $\ominus$ the standard $\emptyset$, $\in$, $(X,Y) \mapsto X \cup \{Y\}$, $(X,Y) \mapsto X \setminus \{Y\}$, and we regard the cofinites $\mathscr{V}_\omega \setminus x$ as the proper *classes* (which are therefore *not* allowed to appear as second argument in the operations of singleton addition and removal).

To generate the universe $\widehat{\mathscr{U}}$ we will use the constructs of the signature

$$\emptyset_{/0}, \qquad \infty_{/0}, \qquad w_{/2}, \qquad \ell_{/2}$$

(where ∞ represents $\mathscr{V}_\omega$ and w, ℓ represent the operations $\oplus$, $\ominus$), and among the terms of such a universe we will select canonical designations for the finite sets $\{x_1,\dots,x_k\}$ and for the cofinites $\mathscr{V}_\omega \setminus \{x_1,\dots,x_k\}$, for example by insisting on the fact that the x_is are in lexicographic ordering (see Sections 3 e 7).

It is not problematic to implement a binary operation η satisfying the requirements indicated above.

Notice that the universe $\mathscr{U}$ under consideration is not closed with respect to separation: if in some cases, as the ones

$$\{x \in \mathscr{V}_\omega \mid x \notin x\}\,(= \mathscr{V}_\omega) \text{ and } \{x \in \mathscr{V}_\omega \mid x \in x\}\,(= \emptyset),$$

the aggregate obtained by separation from a proper class is in turn a class, the same is not true in infinite other cases, as the ones

$$\{x \in \mathscr{V}_\omega \mid \emptyset \notin x\} \text{ and } \{x \in \mathscr{V}_\omega \mid \emptyset \in x\},$$

where the finiteness-or-cofiniteness requirement fails. $\square$

[36] provides the specification of a synthesis algorithm that generates a computable universe $\mathscr{M} = (\mathscr{U}, \boldsymbol{\emptyset}, \in, w, \ell)$ of sets starting from an input sentence of the form

$$\exists y_1 \cdots \exists y_n \forall x\, \varphi(y_1, \ldots, y_n, x),$$

where the formula $\varphi \equiv \varphi(y_1, \ldots, y_n, x)$ involves only the variables $y_1, \ldots, y_n, x$, the relators $\in, =$, the propositional connectives, and no quantifiers. Within the computable universe $\mathscr{M}$, one can also compute any total function that can be specified in the form

$$\varepsilon x\, \psi(y_1, \ldots, y_m, x)$$

(where the formula ψ has a similar structure to the one of φ) given any input assignment

$$y_1 \mapsto v_1, \ldots, y_m \mapsto v_m$$

of $\mathscr{M}$-sets v_j to the formal parameters y_j.

Before generating the universe $\mathscr{M}$, the synthesis algorithm [36] must check that the constraint $\exists y_1 \cdots \exists y_n \forall x\, \varphi$ is consistent with the properties 1.–4. listed above characterizing a universe of sets; to this purpose it incorporates a decision algorithm. Examples of unsatisfiable constraints are $\exists y \forall x\, (x \in y \leftrightarrow x \notin x)$ and $\exists y \forall x\, y \notin x$. The following satisfiable constraints are paradigmatic of the power of the synthesis algorithm:

$$\begin{array}{l}
\exists y_0 \cdots \exists y_n \forall x \Big(\big(\bigwedge_{0 \leqslant i < j \leqslant n} y_i \neq y_j \big) \,\&\, \bigwedge_{i=0}^{n} (x \in y_i \leftrightarrow x = y_i) \Big), \\
\exists y \forall x\, y \notin x, \\
\exists y_0 \cdots \exists y_h\, \exists y_{h+1} \cdots \exists y_n \forall x \Big(\big(\bigvee_{i=0}^{h} x \in y_i \big) \;\leftrightarrow\; \bigwedge_{j=h+1}^{n} x \neq y_j \Big), \\
\exists y_0\, \exists y_1\, \exists y_2 \forall x\, \big(x \notin y_0 \,\&\, (x \in y_1 \leftrightarrow y_0 \in x) \,\&\, (x \in y_2 \leftrightarrow y_0 \notin x) \big).
\end{array}$$

Respectively, these express: the existence of at least $n+1$ autosingletons; the existence of a set of all the sets; the existence of $h+1$ sets that cover almost completely the universe of sets; the existence of a set formed by all sets containing the empty set, and of its complement.

14. Finite rational hypersets

> *Ordinary computability theory can be thought of as the theory of recursion over the set HF^0. It seems that to include things like computable streams and computable binary trees, there should be an analogous corecursion theory. The natural setting for this theory would be either $HF^{1/2}$ or HF^1. The existence of computable streams like $\langle 0, \langle 1, \langle 2, \ldots \rangle\rangle\rangle$ that are not in $HF^{1/2}$ suggests this set is too small.* [5, p. 328]

We resume here the construction outlined in [10, pp. 96–97] of a computable universe of sets (see Sec. 13); for more detailed explanations the reader is referred to [37]. Briefly speaking, the construction that we trace here is based on the one provided by Aczel [2] to obtain from the universe of all sets the—even bigger—universe of *hypersets*.[16] On the formal side, Aczel's hypersets enjoy all the properties of ordinary sets with one exception: while the membership relation among traditional sets is well founded, on the wider hypersets universe it is forced to contravene well foundedness in all possible ways, for instance by forming cycles of any length. Trying to be clearer even at this intuitive level of exposition, we associate to each set/hyperset ξ the graph $trans(\xi)$ having

- as nodes, all ζs such that there is a chain of memberships $\zeta \in \cdots \in \xi$ (of length 0,1 or more) leading from ζ to ξ;
- as edges, all pairs $\langle \zeta_0, \zeta_1 \rangle$ of such nodes satisfying the relation $\zeta_0 \in \zeta_1$.

Moreover we classify ξ as a set when $trans(\xi)$ has no paths of infinite length, and as a hyperset in a strict sense in the contrary case.

The universe of sets (of von Neumann) is rich enough so that, given a graph G with no infinite paths and with a *sink* $v_\star$ (namely, a node reachable in G from any other node), it is possible to find a set ξ and an isomorphism χ between $trans(\xi)$ and G such that $\chi(\xi) = v_\star$, *provided that* G contains no distinct nodes with the same immediate predecessors. The latter requirement reflects the *extensionality postulate*, according to which there can be no distinct sets with the same elements.

The universe of hypersets (of Aczel) satisfies a similar "richness"principle – but not limited by the requirement of paths finiteness –, and it is conceived in such a way as to reflect a cautious variant of the extensionality principle so as to assure, for instance, that when ξ_b is the only member of ξ_b, for $b = 0, 1$, then ξ_0 and ξ_1 coincide. In order to be more precise on this point, we will need to introduce the notion of *bisimulation*.[17] Bisimulations will allow us to easily express a (strengthened) extensionality condition that a pointed graph $\langle G, v_\star \rangle$ must satisfy in order that there is a hyperset ξ such that $trans(\xi)$ is isomorphic to G (that implies, incidentally, that $v_\star$ is a sink): there can be no bisimulation between any two closed subgraphs $\langle G_0, v_\star^0 \rangle$, $\langle G_1, v_\star^1 \rangle$ of G whose sinks $v_\star^0, v_\star^1$ are distinct.

[16] The word 'hyperset', drawn from [4], substitutes here the locution 'non-well-founded set' of Aczel [2].

[17] The concept of bisimulation—and the name itself—originated from studies carried out by Milner and Park (cf. [41]) on the semantics of concurrent processes. Bisimulations have been introduced independently also by other authors, more or less at the same time: among them, we mention Johan van Benthem [6, 7], who called them *pi-morphisms*.

Here we only deal with *hereditarily finite and rational* hypersets, namely hypersets ξ such that both their cardinality and the height of their associated graphs $trans(\xi)$ are finite.[18] Such *finiteness assumptions* allow us to considering hypersets as an algorithmic data-structure.

In extreme synthesis, the construction of our universe $\overline{\mathscr{V}}_\omega$ of hypersets proceeds in the following way:

1. We consider the ordered pairs $\langle h,k\rangle$ in $\mathscr{V}_\omega$ and, among them, the binary relation
$$G =_{\text{Def}} \{ \langle\langle h,k_0\rangle, \langle h,k_1\rangle\rangle : \langle k_0,k_1\rangle \in h \} .$$
Then we establish that the relation $\langle h_0,k_0\rangle \overset{G}{\sim} \langle h_1,k_1\rangle$ holds iff there is a BISIMULATION B over G such that $\langle h_0,k_0\rangle B\langle h_1,k_1\rangle$. This means that B is a symmetric relation among nodes satisfying the condition
$$\big\langle \forall u,v,w \mid uBvGw \rightarrow \langle \exists z \mid zBw \,\&\, uGz\rangle \big\rangle.$$
It can easily be verified that the very relation $\overset{G}{\sim}$ is a bisimulation, hence the largest bisimulation over G with respect to $\subseteq$; furthermore, $\overset{G}{\sim}$ is an equivalence relation.

2. Then we consider the relation $\overline{G}$ over the quotient $\mathscr{V}_\omega \times \mathscr{V}_\omega \,/ \overset{G}{\sim}$ consisting of all pairs $\langle [h,k_0]_{\underset{\sim}{G}}, [h,k_1]_{\underset{\sim}{G}} \rangle$ such that $\langle k_0,k_1\rangle \in h$, where
$$[h_0,k_0]_\sim =_{\text{Def}} \{\langle h_1,k_1\rangle : h_1 \in \mathscr{V}_\omega, k_1 \in \mathscr{V}_\omega, \langle h_0,k_0\rangle \overset{G}{\sim} \langle h_1,k_1\rangle\},$$
for $h_0,k_0 \in \mathscr{V}_\omega$.

3. The blocks of $\mathscr{V}_\omega \times \mathscr{V}_\omega \,/ \overset{G}{\sim}$ form the domain of discourse of our universe $\overline{\mathscr{V}}_\omega$ of HYPERSETS, over which $\overline{G}$ acts as inverse $\ni$ of the membership relation.

4. $\overline{\mathscr{V}}_\omega$ includes entities that can be identified with ordinary sets. More precisely, these are the blocks which can be written in the form $[h,q]_{\underset{\sim}{G}}$, where h is the collection of the pairs $\langle k_1,k_0\rangle$ for which there is in $\mathscr{V}_\omega$ a chain $k_0 \in k_1 \in k_2 \in \cdots \in k_{n+1} = q$ (with n arbitrarily large, although (obviously) finite). Notice, indeed, that when $[h_0,q_0]_{\underset{\sim}{G}}$ and $[h_1,q_1]_{\underset{\sim}{G}}$ are blocks of this kind, then $[h_0,q_0]_{\underset{\sim}{G}} \in [h_1,q_1]_{\underset{\sim}{G}}$ holds in $\overline{\mathscr{V}}_\omega$ iff $q_0 \in q_1$ holds in the ordinary sense, i.e. it holds in $\mathscr{V}_\omega$.

[18] By *height* of a graph we intend the maximal length of any path in the graph in which no node is allowed to appear more than once. Rational hypersets form the collection that in the quotation reported on the top of this section is named $\mathrm{HF}^{1/2}$; those, among them, whose *trans* has no internal cycle, form HF^0, namely our $\mathscr{V}_\omega$.

5. (We leave to the reader the definition of $\mathbf{\emptyset}, \oplus, \ominus$ suitable for $\mathscr{V}_\omega$).

(Notice that the main question occurring in the hyperset notion, that is to determine the pairs μ, ν of nodes of a given graph G such that the relation $\mu \overset{G}{\sim} \nu$ holds, can be framed in the *relational coarsest partition problem* dealt with in [24]. A $\mathscr{O}(|E| \cdot \ell og\,|N|)$-time and $\mathscr{O}(|E| + |N|)$-space algorithm for such a problem, where E and N are respectively the set of edges and the set of nodes of G, was proposed in [40]).

An appealing thesis put forward by Alberto Policriti is that the numerous satisfiability tests devised for decidable fragments of classical set theory reflect in analogous decision algorithms for hyperset theory. As for the case of hereditarily finite sets, this thesis has gained some evidence through [3, 16, 37].

15. Fraenkel's "proto" cumulative hierarchy

In [20], Fraenkel proved the independence of the axiom of choice from the remaining axioms of set theory, referring his analysis to the axiomatic system which had been put forward by Zermelo in [47].

Fraenkel's proof is based on the construction of a universe of objects, $\mathfrak{B}$, that, as we illustrate below, can be described naturally as a cumulative hierarchy, more specifically as a hierarchy of hierarchies of sets.[19]

The universe $\mathfrak{B}$ is built by starting with the primitive objects

- the null set $\emptyset$;
- a countable infinity of individuals $a_1, \bar{a}_1, a_2, \bar{a}_2, \ldots$;
- the set $Z_0 = \Big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\Big\}$; and
- the set $A = \{\{a_1, \bar{a}_1\}, \{a_2, \bar{a}_2\}, \ldots\}$

and then by adding all the sets that can be obtained from the primitive objects by a finite number of applications of Axiom II (of the elementary sets), Axiom III (of separation), Axiom IV (of the power set) and Axiom V (of the union).

It is then proved (see the *Fundamental Theorem* in [20]) that every set M of $\mathfrak{B}$ is *symmetric* with respect to all but a finite number of elements $\{a_k, \bar{a}_k\}$ of the set A, in the sense that if $\overline{M}^k$ is the result of simultaneously substituting a_k

[19] Since it is constituted by syntactical descriptions, $\mathfrak{B}$ is plainly countable and therefore it forms a small model of Zermelo's axioms. In [47], instead, the domain of discourse is described in rather generic terms as being formed by objects (which may or may not be sets) obeying some fundamental conditions (regarding the membership relation) and subject to Zermelo's axioms, with no explicit references to any hierarchy of sets.

with $\bar{a}_k$ and $\bar{a}_k$ with a_k in M, then $\overline{M}^k = M$ holds. Therefore $\mathfrak{B}$ cannot contain any choice set for A, which shows that the axiom of choice cannot hold for $\mathfrak{B}$.

Axioms II, IV, and V are stated as in [47], whereas concerning Axiom III, Fraenkel felt the need of making Zermelo's formulation more precise by introducing the notion of *function* (see Sec. 9 for a discussion on Fraenkel's version of Axiom III and comparisons with other, more recent, versions).

Informally speaking, functions are determined from *descriptions* of set constructions (like for instance $\bigcup\bigcup A$, which is a description of the null set) by substituting a constant object with a variable. For example, $\varphi(x) = \{\{x\}\}$ is a function corresponding to the description $\varphi = \{\{\emptyset\}\}$ of the singleton of the singleton of the null set, where $\emptyset$ is substituted with x. It may also happen that x substitutes a constant not occurring in φ. In this case we have $\varphi(x) = \{\{\emptyset\}\}$.

Axiom III ([20]). If a set M is given, as well as, in a definite order, two functions $\varphi(x)$ and $\psi(x)$, then M possesses a subset M' (resp. M'') containing as elements all the elements $m \in M$ for which $\varphi(m)$ is an element of $\psi(m)$ (resp. $\varphi(m)$ is not an element of $\psi(m)$); and no others.

Descriptions can be recursively defined as follows.

- $Z_0, A, \emptyset, a_i, \bar{a}_i$ (with $i \in \mathbb{N}$) are descriptions of class 0;
- if φ, ψ are descriptions of class at most p, then the descriptions $\{\varphi, \psi\}$ (Axiom II), $\mathcal{P}(\varphi)$ (Axiom IV), and $\bigcup \varphi$ (Axiom V) are of class p,
- if χ, φ, ψ are descriptions of class at most p, then the description $\{x \in \chi \mid \varphi(x) \in \psi(x)\}$ (resp. $\{x \in \chi \mid \varphi(x) \notin \psi(x)\}$), where $\varphi(x)$ is a function obtained from φ by substituting a constant object (primitive, if p=0) with x, and $\psi(x)$ is a function obtained from ψ by substituting a constant object (primitive, if p=0) with x, is of class $p+1$.[20]

From the above definition of description, it follows that every set in $\mathfrak{B}$ may have several different descriptions (for instance, $\mathcal{P}(\emptyset)$ and $\{\emptyset\}$ are both descriptions of the singleton of the null set). By identifying all sets in $\mathfrak{B}$ with all their descriptions, we will show how $\mathfrak{B}$ can be described as a hierarchy of hierarchies of descriptions of sets.

The partitioning of descriptions in classes outlined above is exploited in the definition of the hierarchy: we construct a hierarchy $\mathfrak{B}_0$ for the descriptions of class 0, a hierarchy $\mathfrak{B}_1$ for the descriptions of class 1, and so on. The final

[20]We require that, if $p > 0$, the constant objects substituted in φ and ψ either are of class less than p, or are not the only representative of class p in χ, φ and ψ. This also must hold for the construction of the intermediate hierarchies $\mathfrak{B}_p$, as outlined below.

hierarchy $\mathfrak{B}$ is obtained by taking the union of all levels of all hierarchies of finite class.

Let us designate by $L^0_{\mathfrak{B}_i}, L^1_{\mathfrak{B}_i}, \ldots$ the levels of the intermediate hierarchies $\mathfrak{B}_i$. The overall construction is illustrated in the following.

- The hierarchy $\mathfrak{B}_0$ has the following construction:

$$
\begin{aligned}
L^0_{\mathfrak{B}_0} &= \{\emptyset, Z_0, A, a_1, \bar{a}_1, a_2, \bar{a}_2, \ldots\},\\
L^1_{\mathfrak{B}_0} &= L^0_{\mathfrak{B}_0} \cup \big\{\{\varphi,\psi\} : \varphi,\psi \in L^0_{\mathfrak{B}_0}\big\} \cup \big\{\mathcal{P}(\varphi) : \varphi \in L^0_{\mathfrak{B}_0}\big\} \cup\\
&\quad \cup \big\{\textstyle\bigcup \varphi : \varphi \in L^0_{\mathfrak{B}_0}\big\},\\
L^2_{\mathfrak{B}_0} &= L^1_{\mathfrak{B}_0} \cup \big\{\{\varphi,\psi\} : \varphi,\psi \in L^1_{\mathfrak{B}_0}\big\} \cup \big\{\mathcal{P}(\varphi) : \varphi \in L^1_{\mathfrak{B}_0}\big\} \cup\\
&\quad \cup \big\{\textstyle\bigcup \varphi : \varphi \in L^1_{\mathfrak{B}_0}\big\},\\
&\ \vdots
\end{aligned}
$$

- For $p \geqslant 0$, $\mathfrak{B}_{p+1}$ is obtained from $\mathfrak{B}_p$ as follows.

  For every $i = 0, 1, \ldots$, let $H^i_{\mathfrak{B}_{p+1}}$ be the collection of all sets $\{x \in \chi \mid \varphi(x) \in \psi(x)\}$, $\{x \in \chi \mid \varphi(x) \notin \psi(x)\}$ such that χ is in $L^i_{\mathfrak{B}_p}$ and $\varphi(x)$, $\psi(x)$ are obtained from some descriptions φ, ψ in $L^i_{\mathfrak{B}_p}$, by substituting x for a primitive object (if $p = 0$) or for a constant object (in case $p > 0$). Then the levels of $\mathfrak{B}_{p+1}$ are

$$
\begin{aligned}
L^0_{\mathfrak{B}_{p+1}} &= H^0_{\mathfrak{B}_{p+1}} \cup L^0_{\mathfrak{B}_p},\\
L^1_{\mathfrak{B}_{p+1}} &= H^1_{\mathfrak{B}_{p+1}} \cup L^1_{\mathfrak{B}_p} \cup L^0_{\mathfrak{B}_{p+1}} \cup \big\{\{\varphi,\psi\} : \varphi,\psi \in L^0_{\mathfrak{B}_{p+1}}\big\} \cup\\
&\quad \cup \big\{\mathcal{P}(\varphi) : \varphi \in L^0_{\mathfrak{B}_{p+1}}\big\} \cup \big\{\textstyle\bigcup \varphi : \varphi \in L^0_{\mathfrak{B}_{p+1}}\big\},\\
L^2_{\mathfrak{B}_{p+1}} &= H^2_{\mathfrak{B}_{p+1}} \cup L^2_{\mathfrak{B}_p} \cup L^1_{\mathfrak{B}_{p+1}} \cup \big\{\{\varphi,\psi\} : \varphi,\psi \in L^1_{\mathfrak{B}_{p+1}}\big\} \cup\\
&\quad \cup \big\{\mathcal{P}(\varphi) : \varphi \in L^1_{\mathfrak{B}_{p+1}}\big\} \cup \big\{\textstyle\bigcup \varphi : \varphi \in L^1_{\mathfrak{B}_{p+1}}\big\},\\
&\ \vdots
\end{aligned}
$$

Finally, $\mathfrak{B}$ is obtained from $\mathfrak{B}_0, \mathfrak{B}_1, \ldots$ by putting for $i = 0, 1, \ldots$,

$$L^i_{\mathfrak{B}} = \bigcup_{p=0}^{\infty} L^i_{\mathfrak{B}_p}.$$

With each description φ of the hierarchy $\mathfrak{B}$, it is possible to associate a pair of numbers $(\mathrm{cl}(\varphi), \mathrm{lv}(\varphi))$ indicating the minimal intermediate hierarchy and the minimal level of the hierarchy containing φ. Then $\mathrm{cl}(\varphi)$ and $\mathrm{lv}(\varphi)$ can be recursively computed over the structure of φ as follows:

1. $\mathrm{cl}(\varphi) = 0$ and $\mathrm{lv}(\varphi) = 0$, for every primitive object φ.
2. If $\varphi = \{\psi, \chi\}$, then

   (2a) $\mathrm{cl}(\varphi) = \max\{\mathrm{cl}(\psi), \mathrm{cl}(\chi)\}$, and
   (2b) $\mathrm{lv}(\varphi) = \max\big\{\mathrm{lv}(x) : x \in \{\psi, \chi\} \mid \mathrm{cl}(x) = \mathrm{cl}(\varphi)\big\} + 1$.
3. If $\varphi = \bigotimes \psi$, where $\bigotimes \in \big\{\mathcal{P}, \bigcup\big\}$, then

   (3a) $\mathrm{cl}(\varphi) = \mathrm{cl}(\psi)$, and
   (3b) $\mathrm{lv}(\varphi) = \mathrm{lv}(\psi) + 1$.
4. If either $\varphi = \{x \in \psi \mid \chi(x) \in \xi(x)\}$ or $\varphi = \{x \in \psi \mid \chi(x) \notin \xi(x)\}$, then

   (4a) $\mathrm{cl}(\varphi) = \max\{\mathrm{cl}(\psi), \mathrm{cl}(\chi), \mathrm{cl}(\xi)\} + 1$, and
   (4b) $\mathrm{lv}(\varphi) = \max\big\{\mathrm{lv}(x) : x \in \{\psi, \chi, \xi\} \mid \mathrm{cl}(x) = \mathrm{cl}(\varphi)\big\}$.

As a simple example, let us consider $\varphi = \{x \in Z_0 \mid \emptyset \in x\}$ (respectively, $\varphi' = \{x \in Z_0 \mid \emptyset \notin x\}$) which describes the set $\{\{\emptyset\}\}$ (resp. $Z_0 \setminus \{\{\emptyset\}\}$). It is easy to check that $\mathrm{cl}(\varphi) = 1$ and $\mathrm{lv}(\varphi) = 0$ (resp., $\mathrm{cl}(\varphi') = 1$ and $\mathrm{lv}(\varphi') = 0$).

One may wonder whether iterated applications of the axiom of union, having destructive effects (for instance, for $i \geqslant 2$, $\bigcup^{(i)} A$ is a description of the null set), leads to a "stabilization" level in the hierarchy, in which no new sets are generated anymore. In fact, it can easily be verified that there is no such stabilization level: for example, in the construction of the level $k+1$ from the previous level k, $\mathcal{P}^{(k+1)}(A)$ and $\mathcal{P}^{(k+1)}(Z_0)$ are always produced. These describe sets that cannot be obtained from descriptions generated in previous levels of the hierarchy.

Analogously, there is no stabilization class. In fact, each class contains descriptions of sets that cannot be obtained from descriptions produced in previous classes. For instance, let us consider, for $i \geqslant 1$, the description $\mathcal{P}^{(i)}(\bigcup A)$ of class 0 and level $i+1$. From $\mathcal{P}^{(i)}(\bigcup A)$, by separation, it is possible to construct the set $X_1 = \{x \in \mathcal{P}^{(i)}(\bigcup A) \mid \{a_1\}^{(i-1)} \notin x\}$ of class 1 and level $i+1$, and, more generally, for $j \geqslant 2$, the set $X_j = \{x \in X_{j-1} \mid \{a_j\}^{(i-1)} \notin x\}$ of class j and level $i+1$. Each X_j describes a set that cannot be produced with less than j applications of Axiom III.[21]

[21] In order to obtain the set described by X_j with a single application of the separation axiom we would need a more liberal set former schema, allowing us to produce the description $\{x \in \mathcal{P}^{(i)}(\bigcup A) \mid \{a_1\}^{(i-1)} \notin x \vee \ldots \vee \{a_j\}^{(i-1)} \notin x\}$.

We close the section by mentioning the following interesting problem, that we are currently investigating: is there any algorithm to test whether two descriptions in the hierarchy reported above denote the same set?

16. Referee scenarios on cumulative hierarchies

In the centennial anniversary of completion of Peano's *Formulario* [42] and of publication of Zermelo's epochal paper [47] on the axiomatization of set theory, we like to report on a proof verifier under development, ÆtnaNova, aka Referee ('Ref' for brevity), see [9, 11, 12, 35] and the URLs

http://www.settheory.com/Setl2/Ref_user_manual.html .
http://setl.dyndns.org/EtnaNova/login/Ref_user_manual.html .

Ref is based on a version of the Zermelo-Fraenkel set theory; it receives as input script files, called *scenarios*, consisting of successive definitions (even inductive, [43]) and theorems (cf., e.g., Figure 2), which it either certifies as constituting a valid sequence or rejects as defective.[22] Thanks to the power of its set-theoretic basis, Ref is not oriented to a specific field of mathematics; on the contrary, it can cover a very broad spectrum of applications.

The study outlined in these pages served, among others, to prepare a series of definitions and theorems, which were gradually incorporated in our main scenario and then used to test the Ref system. Most of the laws on rank reported at the end of Sec. 6, of the closure properties enjoyed by superstructures as discussed in Sec. 4, etc., have been proved with the help of Ref—cf. Figures 2, 3, 4, and 5. For instance, the theorem proved in Sec. 11 appears (after having reproduced in Ref its development in a satisfactory way) on the interface of the THEORY shown in Figure 4 (see [38] for a discussion of the construct THEORY).

The interface of another useful THEORY that exploits the notion of rank is presented in Figure 5. This theory, signedSymbols, is a very useful preliminary to theoretical investigations (for example on questions regarding the completeness or incompleteness of a formal deductive system) that require the arithmetization of a syntax, where, in our specialized context, "arithmetization" is intended as the representation by means of operations over *sets* of the constructs of a symbolic language.

Example. The language of a mono-modal propositional logic whose set of literals has cardinality of atms, can be modelled with the sets of the superstructure

[22] In the case of rejection, the verifier attempts to pinpoint the troublesome locations within a scenario, so that errors can be located and repaired. Step timings are produced for all correct proofs, to help the user in spotting places where appropriate modifications could speed up proof processing.

$$\mathcal{V}_X =_{\mathrm{Def}} \bigcup\{\mathcal{P}(\mathcal{V}_y) : y \in \mathsf{X}\}$$
$$\mathsf{HF} =_{\mathrm{Def}} \mathcal{V}_{\mathbb{N}}$$

$$\emptyset \in \mathsf{HF} \;\&\; \mathbf{arb}(\emptyset) = \emptyset$$
$$\mathsf{S} \in \mathsf{HF} \;\&\; \mathsf{X} \in \mathsf{S} \rightarrow \mathsf{X} \in \mathsf{HF} \setminus \mathbf{arb}(\mathsf{S}) \;\&\; \mathbf{arb}(\mathsf{S}) \in \mathsf{HF} \cap \mathsf{S}$$
$$\mathsf{S} \in \mathsf{HF} \rightarrow \mathcal{P}(\mathsf{S}), \{\mathsf{S}\}, \bigcup \mathsf{S} \in \mathsf{HF}$$
$$\mathsf{S}, \mathsf{T} \in \mathsf{HF} \rightarrow \mathsf{S} \cup \mathsf{T}, \mathsf{S} \setminus T, \{\mathsf{S}, \mathsf{T}\} \in \mathsf{HF}$$
$$\mathsf{S} \in \mathsf{HF} \;\&\; \mathsf{T} \subseteq \mathsf{S} \rightarrow \mathsf{T} \in \mathsf{HF}$$
$$\mathsf{S} \in \mathsf{HF} \leftrightarrow \mathsf{Finite}(\mathsf{S}) \;\&\; \{\mathsf{u} \in \mathsf{S} \mid \mathsf{u} \notin \mathsf{HF}\} = \emptyset$$

Figure 2: Definition of the set HF of hereditarily finite sets, along with theorems

$\mathcal{V}_\omega^A$, where A is the collection $\{\mathsf{aff}_\Theta(\mathsf{x}) : \mathsf{x} \in \mathsf{atms}\}$ produced by signedSymbols applied to atms. (It follows from our considerations in Sec. 10 that the elements of A may act as ur-elemets, though being sets).

Indeed, each element e of $\mathcal{V}_\omega^A \setminus A$ may be decomposed in an "operand" $e \setminus \mathcal{V}_\omega$ and an "*opcode*" $e \cap \mathcal{V}_\omega$, by regarding the first one as a disjunction and the second one as one of the three constructs of affirmation, negation, and necessitation. In agreement with such conventions, we can therefore specify, by means of the following recursive *global* formula (that is, defined over the whole $\mathcal{V}$), how sets in a structure *à la* Kripke, constituted of "sets of worlds" W, relation R of "accessibility between worlds" and interpretation M for the propositional letters (so that $R \subseteq W \times W$ and $A \xmapsto{M} \mathcal{P}(W)$), are evaluated:

$$\begin{array}{rcl} m\_vl(Op,U,W,R) & =_{\mathrm{Def}} & \mathsf{if}\ Op = 0\ \mathsf{then}\ U\ \mathsf{elseif}\ Op = 1\ \mathsf{then}\ W \setminus U \\ & & \mathsf{else}\ \big\{x \in W \mid U \supseteq R \upharpoonright \{x\}\big\}\ \mathsf{fi}; \\ m\_eval(E,W,R,M) & =_{\mathrm{Def}} & \mathsf{if}\ E \in \mathsf{domain}(M)\ \mathsf{then}\ M \upharpoonright E\ \mathsf{else} \\ \multicolumn{3}{r}{m\_vl\big(\mathsf{arb}(E \cap \mathcal{V}_\omega), \bigcup\{m\_eval(y,W,R,M) : y \in E \setminus \mathcal{V}_\omega\}, W, R, M\big)} \\ & & \mathsf{fi}\,. \end{array}$$

□

17. Construction of the Ackermann hierarchy

The following SETL program [45] carries out, up to a given level, the construction of the hierarchy of pure hereditarily finite sets. By regarding such domain as ordered *à la* Ackermann (see Sec. 7), we can represent each set by means of a position, to which it is advantageous (for instance for *pretty-printing* purposes) to associate a pair of natural numbers: the position of the maximum of such set and the position of the set formed by its remaining elements.

$$
\begin{array}{l}
\textsf{THEORY herfin\_induction}(s_0, P(x)) \\
\quad s_0 \in \mathsf{HF} \\
\quad P(s_0) \\
\Longrightarrow (\mathsf{hf0}_\Theta, \mathsf{hf1}_\Theta) \\
\quad \langle \forall k \subseteq \mathsf{hf0}_\Theta \mid k \neq \mathsf{hf0}_\Theta \rightarrow \neg P(k) \rangle \;\&\; \mathsf{hf0}_\Theta \subseteq s_0 \;\&\; P(\mathsf{hf0}_\Theta) \\
\quad \langle \forall k \mid k \in \mathsf{hf1}_\Theta \vee (k \subseteq \mathsf{hf1}_\Theta \;\&\; k \neq \mathsf{hf1}_\Theta) \rightarrow \neg P(k) \rangle \;\&\; \mathsf{hf1}_\Theta \in \mathsf{HF} \;\&\; P(\mathsf{hf1}_\Theta) \\
\textsf{END herfin\_induction}
\end{array}
$$

Figure 3: Induction principles concerning the hereditarily finite sets

$$
\begin{array}{l}
\textsf{THEORY globalizeTog}\ (T) \\
\quad \mathsf{Svm}(T) \;\&\; T^{\leftarrow} = T \;\&\; \{p \in T \mid p^{[1]} = p^{[2]}\} = \emptyset \\
\Longrightarrow (\mathsf{tog}_\Theta) \\
\quad \langle \forall x \in \mathsf{domain}(T) \mid T \restriction x \neq x \;\&\; T \restriction (T \restriction x) = x \rangle \\
\quad \langle \forall x \in \mathsf{domain}(T) \mid \mathsf{tog}_\Theta(x) = T \restriction x \rangle \\
\quad \langle \forall x \mid \mathsf{tog}_\Theta(x) \neq x \;\&\; \mathsf{tog}_\Theta(\mathsf{tog}_\Theta(x)) = x \rangle \\
\textsf{END globalizeTog}
\end{array}
$$

Figure 4: A tool to make a toggle, T, global

```
program lexOrd;

   const M := 4;        -- Maximal rank that will be considered by this program.

   var SY,              -- (Scalar) symbolic representation of particular transitive sets.
       HF;              -- List of the hereditarily finite sets, up to a certain maximal rank.

-- This SETL program generates all the sets of the cumulative hierarchy of
-- von Neumann having rank not exceeding M, ordering them
-- in a lexicographic way à la Ackermann to form a list HF. Inside HF,
-- each nonempty set x is represented as a pair [i, j] where
-- j is the position of the maximal element of x and i is the position
-- of the set x − {max(x)} resulting from the removal of such
-- maximum from x. Positions are counted, inside HF, starting from 1;
-- the first of them is occupied by 1 = 0. By convention we indicate
-- with 0 the position of the empty set (coinciding with the natural
-- number 0); in position 1 thus we will find,
-- the pair [0,0], which represents the singleton 0.

-- The first layer embraces the interval of positions included between 1
-- and 1. Once completed the generation of a layer, whose elements have
```

-- This theory receives a set atms of which it will univocally convert every element x into a positive literal by the affirmation operation $\mathsf{aff}_\Theta(\mathsf{x})$ and into a negative literal by the negation operation $\mathsf{neg}_\Theta\big(\mathsf{aff}_\Theta(\mathsf{x})\big)$, where neg_Θ is a Galois correspondence. For a finite set atms, positive integer encodings and their negatives would suffice, but a more general approach will be taken here, and $\mathsf{aff}_\Theta, \mathsf{neg}_\Theta$ will be defined globally. Along with the functions $\mathsf{aff}_\Theta, \mathsf{neg}_\Theta$, this theory returns a collection lits_Θ of positive and negative literals including all the affirmed and negated images of the "symbols" in atms. Moreover, it returns a designation false_Θ for falsehood such that the pair $\mathsf{false}_\Theta, \mathsf{neg}_\Theta(\mathsf{false}_\Theta)$ of complementary truth values does not intersect lits_Θ. All literals, as well as the logical constants, will share the same rank exceeding the ordinal $\mathbb{N}$.

THEORY $\mathsf{signedSymbols}(\mathsf{atms})$

$\Longrightarrow (\mathsf{aff}_\Theta, \mathsf{neg}_\Theta, \mathsf{lits}_\Theta, \mathsf{false}_\Theta)$

$\langle \forall \mathsf{x}, \mathsf{y} \mid \mathsf{x} \neq \mathsf{y} \rightarrow \mathsf{aff}_\Theta(\mathsf{x}) \neq \mathsf{aff}_\Theta(\mathsf{y}) \rangle$

$\langle \forall \mathsf{x} \mid \mathsf{neg}_\Theta\big(\mathsf{neg}_\Theta(\mathsf{x})\big) = \mathsf{x} \,\&\, \mathsf{neg}_\Theta(\mathsf{x}) \neq \mathsf{x} \rangle$

$\langle \forall \mathsf{x}, \mathsf{y} \mid \mathsf{aff}_\Theta(\mathsf{x}) \neq \mathsf{neg}_\Theta\big(\mathsf{aff}_\Theta(\mathsf{y})\big) \rangle$

$\{\mathsf{aff}_\Theta(\mathsf{x}) : \mathsf{x} \in \mathsf{atms}\} \subseteq \mathsf{lits}_\Theta$

$\{\mathsf{neg}_\Theta(\mathsf{x}) : \mathsf{x} \in \mathsf{lits}_\Theta\} = \mathsf{lits}_\Theta$

$\mathsf{false}_\Theta \notin \mathsf{lits}_\Theta$

$\{\mathsf{rk}(\mathsf{x}) : \mathsf{x} \in \mathsf{lits}_\Theta\} \subseteq \{\mathsf{rk}(\mathsf{false}_\Theta)\}$

$\langle \forall \mathsf{n} \mid \mathsf{n} \in \mathsf{next}(\mathbb{N}) \rightarrow \mathsf{n} \in \mathsf{rk}(\mathsf{false}_\Theta) \rangle$

$\langle \forall \mathsf{x} \mid \mathsf{rk}(\mathsf{x}) \notin \mathsf{rk}(\mathsf{false}_\Theta) \rightarrow \mathsf{rk}\big(\mathsf{neg}_\Theta(\mathsf{x})\big) = \mathsf{rk}(\mathsf{x}) \rangle$

END $\mathsf{signedSymbols}$

Figure 5: Theory to generate a basis of literals of assigned cardinality

```
-- positions included between i and j (extremes included), to construct the
-- subsequent layer we will proceed in this way: for every value h of the interval
-- [i...j], we will compose a "foil" formed by all the elements of the new
-- layer having the h-th element as their maximum.

   HF := [[0,0]];        -- number 1 intended à la von Neumann, having rank 1
   SY := { };            -- table of the positions which deserve a particular symbol
   i := 1;               -- lower edge of the last generated layer

   for r in [2...M] loop    -- cycle generating all the sets of rank r
      j := # HF;            -- upper edge of the last layer generated
      for h in [i...j] loop -- generation of the foil formed by those
                            -- sets having as their maximum
                            -- the h-th set of the hierarchy
         k := # HF;         -- lower edge of the foil in construction
         HF +:= [[p,h]: p in [0..k]];
      end loop;
      SY(j) := 1 - r;       -- position of the set that represents
                            -- the r − 1-th level of the hierarchy
      i := j + 1;           -- shift of the lower edge of the most recent layer
   end loop;

   NM := 0;                 -- position (nominal, given that the 0 is not enrolled in HF)
                            -- of a natural number inserted in the hierarchy
   SY(0) := 0;              -- position 0 is the one that is due to the number 0
   r := 1;                  -- rank of the next natural number that will be introduced in SY
   for h in [1...#HF] loop -- pinpointing of the numerals, ...
      if HF(h)(1) = NM & HF(h)(2) = NM then
         SY(h) := r; r +:= 1; NM := h;       -- ... with annotation in SY
      end if;
   end loop;
   SY(#HF) ?:= -M;          -- coding of a further level of the hierarchy
   oufile := open("lexOrd.txt","TEXT-OUT");
   for s = SY(p) loop
      printa(oufile, s, " = ", pos2set(p,"") + "}");
   end loop;
   printa(oufile, -(M + 1), " = {");
   for h in [0..#HF-1] loop
      printa(oufile,pos2set(h,"}") + ",");
   end loop;
   printa(oufile,pos2set(#HF,"}") + "}");
   close(oufile);

   procedure pos2set(p,b);

   -- This procedure prepares the printing of a set representation
```

```
-- as a string, where some transitive sets are expressed
-- by means of numerical constants (these are non-negative
-- in the case of natural numbers, whereas they are negative in the case of levels).

    return if b /= "," & b /= "" & SY(p) ≠ Ω then str(SY(p))
        elseif p = 0 then "{"
        elseif b /= "," & HF(p)(1) /= 0 &
                HF(HF(p)(1))(1) = 0 & HF(HF(p)(2))(1) = HF(p)(1) then
            "(" + str(pos2set(HF(HF(p)(2))(2),"}")) + "," +
            str(pos2set(HF(HF(p)(1))(2),"}")) + ")"
        elseif b /= "," & SY(p) = Ω & HF(p)(1) = HF(p)(2) then
            str(pos2set(HF(p)(1),"}")) + "+"
        else pos2set(HF(p)(1), ",") +
            pos2set(HF(p)(2), "}") +
            b
        end if;

end pos2set;

end lexOrd;
```

18. Handling the Ackermann hierarchy in Maple

We report here a Maple [23] implementation of the basic operations over pure hereditarily finite sets, in which sets are represented by their positions in the Ackermann lexicographic ordering. The rough lines of the specification below have been already traced in Sections 8 and 9. In some cases we could have provided quicker translations than the ones given: but the specifications proposed here tend, rather than to a maximal descriptive parsimony, to ensure some saving in computation times.

Even the *complements* in $\mathscr{V}_\omega$ of the hereditarily finite sets (see the example of Sec. 13) are correctly manipulated by this Maple library, though not all operations can be performed on such infinite sets (to mention one, in the case of such sets it does not always makes sense to select a minimal element with respect to inclusion; additionally, the possibility of forming subsets by separation and replacement will be guaranteed only in particularly restrictive circumstances). As numerical representation of the complement of $\widehat{p}$ we have made the obvious choice $-(p+1)$ that, in binary codification in two's complement, corresponds to the membership function of the complement. In infinite binary strings that represent these new aggregates, there is a finite number of zeros.[23]

[23] In our implementation of ordered pairs, each component can be finite or cofinite. To achieve this, we have adopted a new pair representation, which is finite or cofinite according as whether

Our implementation could serve as a platform to which one can reduce, either manually or by automatic synthesis (and, if not always, at least in most cases), the algorithmic manipulations relative to computable universes of sets (see Sec. 13). Among these are, for example, the universes of hypersets (see Sec. 14).

Once enriched with associative maps constructs (that, in any case, are sets, though oriented to more specific applications), the library proposed here would almost coincide with the core of a programming language based on sets, as there are many of them. Our language, however, would be characterized by a greater versatility, since its “sets” and “maps” do not have to be intended in a predetermined sense, but as changeable instances of two very particular abstract data types, which are as widespread in the practice of programming as they are relevant in the investigations on the foundations of exact reasoning.

Ackermann hierarchy: library of functions for finite and cofinite sets

Utilities

Arithmetic operations appear only in this section

An arbitrary set

```
any0:=proc() 0 end:
```

Successor of a finite set in the hierarchy

```
succ:=proc(p::nonnegint) p+1 end:
```

Predecessor of a finite set in the hierarchy

```
pred:=proc(p::posint) p-1 end:
```

Complement

```
cmp0:=proc(p::integer) -(p+1) end:
```

Head and tail of a finite set

---

the same holds for the first pair component; moreover, the empty set belongs to our representation if and only if the second component if cofinite. When the components are finite: if they are equal, our pair construction yields the same result as that of Kuratowski; otherwise, it yields the same result as the construction proposed in [19]—cf.Sec. 4.

```
hd:=proc(p::posint,t::evaln) local h,x;
   # The head is returned as the function value,
   # the tail as the ‘‘t’’ parameter.
   # The ‘‘evaln’’ type requires that the parameter be a name,
   # which can be assigned a value.
  x:=p;
  for h from 0 do x:=iquo(x,2); if x=0 then break fi od;
   # Logarithm in base 2 is computed by iterating division
   # for precision
  t:=p-2^h; return h;
end:

tl:=proc(p::posint,h::evaln) local t; h:=hd(p,t); return t end:
```

Minimum element of a set or class

```
mn:=proc(p::integer) local f,m,r,x;
  if   p=0 then error ‘‘the argument p is null’’
  elif p>0 then x:=p; f:=1
  else          x:=cmp0(p); f:=0
  fi;
  for m from 0 do x:=iquo(x,2,’r’); if r=f then break fi od;
  return m
end:
```

Add element “q” to “p”, when “q” not in “p”

```
wth0:=proc(p::integer,q::integer)
  if q<0 then p else p+2^q fi  # If ‘‘q’’ is a class, do nothing
end:
```

Union of disjoint sets or classes

```
un0:=proc(p::integer,q::integer)
  if p<0 and q<0
    then error ‘‘both args. are classes, which are never disjoint’’ fi;
  p+q
end:
```

Difference p\q when "q" included in "p"

```
df0:=proc(p::integer,q::integer)
  if q>=0 then p-q fi;   # If ‘‘q’’ is a class, do nothing
end:
```

Symmetric difference and intersection

```
synt0:=proc(pp::nonnegint,qq::nonnegint,nt::evaln)
   # Basic function for finite sets.
   # The difference is returned as the function value,
   # the intersection as the ‘‘nt’’ parameter
  local p,q,hp,hq,tp,tq,sy,n;
  p:=pp; q:=qq;
  if   p=0 then nt:=0; sy:=q
  elif q=0 then nt:=0; sy:=p
  else sy:=0; n:=0;
       hp:=hd(p,tp);  hq:=hd(q,tq);
       do if hp<hq  then sy:=wth0(sy,hq); q:=tq;
            if q<>0 then hq:=hd(q,tq) else break fi
          elif hq<hp then sy:=wth0(sy,hp); p:=tp;
            if p<>0 then hp:=hd(p,tp) else break fi
          else n:=wth0(n,hq); p:=tp; q:=tq;
            if tp<>0 then hp:=hd(tp,tp) else break fi;
            if tq<>0 then hq:=hd(tq,tq) else break fi
          fi
       od;
       nt:=n;
       if p>q then sy:=un0(sy,p) else sy:=un0(sy,q) fi
  fi
end:

sy0:=proc(p::integer,q::integer,nt::evaln) local n;
   # Symmetric difference for both finite sets and classes.
   # The intersection is returned as the ‘‘nt’’ parameter.
   # ‘‘%’’ indicates the last element in the Maple stack,
   # ‘‘%%’’ the penultimate
  if p>=0 and q>=0 then synt0(p,q,nt)
  elif p>=0 and q<0 then
    synt0(p,cmp0(q),n); nt:=df0(p,n); un0(n,cmp0(un0(%%,n)))
  elif p<0 and q>=0 then
    synt0(q,cmp0(p),n); nt:=df0(q,n); un0(n,cmp0(un0(%%,n)))
  else synt0(cmp0(p),cmp0(q),n); nt:=cmp0(un0(%,n)); return %%
  fi
end:
```

```
nt0:=proc(p::integer,q::integer,sy::evaln) local n;
   # Intersection for both finite sets and classes.
   # The symmetric difference is returned as the ‘‘sy’’ parameter.
  sy:=sy0(p,q,n); return n
end:
```

Predicates

Is “p” null?

```
Is_null:=proc(p::integer) p=null() end:
```

Is “p” a class?

```
Is_class:=proc(p::integer) mlt(p)=null() end:
 # A class cannot be the member of a set
```

Is “p” a number?

```
Is_num:=proc(p::integer) p=rk(p) end:
```

Is “q” in “p”?

```
In:=proc(q::integer,p::integer) local s;
  if Is_class(q) then false
  else mlt(q); nt0(%,p,s)=%
  fi
end:
```

Is “p” a pair?

```
Is_pair:=proc(p::integer) p=opr(sn(p),dx(p)) end:
```

Is “q” a subset of “p”?

```
Is_sub:=proc(q::integer,p::integer) local s; nt0(p,q,s)=q end:
```

Is “p” transitive?

```
Is_trans:=proc(p::integer) Is_sub(U(p),p) end:
```

Set constructors: basic functions

Empty set

```
null:=proc() df0(any0(),any0()) end:
```

Class of all sets

```
all:=proc() cmp0(null()) end:
```

Add element "q" to "p"

```
wth:=proc(p::integer,q::integer)
  if In(q,p) then p else wth0(p,q) fi;
   # If ‘‘q’’ is a class, then ‘‘p’’ is returned
end:
```

Multleton: build the set made of the elements given in the arguments set

```
mlt:=proc() local p,q,qset;
   # Classes, if any, are disregarded
  qset:={args};  # Retrieve the arguments set
  p:=null(); for q in qset do p:=wth0(p,q) od;
end:
```

Take element "q" out of "p"

```
less:=proc(p::integer,q::integer)
  if In(q,p) then df0(p,mlt(q)) else p fi
end:
```

Special transitive sets

Next of set "p", i.e.: p union {p}

```
nx:=proc(p::nonnegint)  wth0(p,p)  end:
```

Natural number
Builds the von Neumann number corresponding to the Maple number in the argument

```
num:=proc(N::nonnegint)
  if N=0 then null() else nx(num(N-1)) fi
end:
```

Cumulative hierarchy "Vn"

```
cum:=proc(N::nonnegint) local sng0;
  sng0:=proc(N::nonnegint)
     # Singleton {...{0}...} with ‘‘N’’ levels of parenthesis
     # nesting
    if N=0 then 0 else mlt(sng0(N-1)) fi
  end;
  pred(sng0(N+1))
end:
```

Boolean set operations

Symmetric difference

```
sy:=proc(p::integer,q::integer) local n; sy0(p,q,n) end:
```

Intersection

```
nt:=proc(p::integer,q::integer) local s; nt0(p,q,s) end:
```

Complement

```
cmp:=proc(p::integer) local n; sy0(p,all(),n) end:
```

Union

```
un:=proc(p::integer,q::integer) local n; un0(sy0(p,q,n),n) end:
```

Difference

```
df:=proc(p::integer,q::integer) local s; df0(p,nt0(p,q,s)) end:
```

Aggregators

Union of all elements of a set or class

```
U:=proc(p::integer) local t;
  if Is_null(p) then null()
  elif Is_class(p) then all()
  else un(hd(p,t),U(t)) fi
end:
```

Power set

```
P:=proc(p::nonnegint) local h,pt,S;
  S:=proc(r::nonnegint,q::nonnegint) local h;
     # Add element ‘‘q’’ to all elements of set ‘‘r’’.
     # ‘‘q’’ should not be in the elems. of ‘‘r’’.
    if Is_null(r) then pt
       else wth0(S(tl(r,h),q), wth0(h,q)) fi
  end;
  if Is_null(p) then num(1)
  else pt:=P(tl(p,h)); S(pt,h)
  fi
end:
```

Pairing and its conjugated projections

Ordered pair

```
opr:=proc(p::integer,q::integer) local t;
  if Is_class(q) then mlt(null(),mlt(cmp(q)))
                 else mlt(mlt(q)) fi;
  if Is_class(p) then wth(%,wth(hd(%,t),cmp(p))); cmp(%)
                 else wth(%,wth(hd(%,t),p))
  fi;
end:
```

Left projection of a pair (for any set or class)

```
sn0:=proc(pq::nonnegint) local t,h,r,x;
   # Left projection of a set
  if Is_sub(rk(pq),num(1)) then null()
  else h:=hd(wth(pq,null()),t); r:=hd(t,x);
       if Is_null(r) then hd(h,x)
       else less(hd(pq,x),hd(r,x)); hd(%,x)
       fi
  fi
end:

sn:=proc(pq::integer)
   # Left projection of a set or class
  if Is_class(pq) then cmp(sn0(cmp(pq))) else sn0(pq) fi
end:
```

Right projection of a pair (for any set or class)

```
dx0:=proc(pq::nonnegint) local p,r,t;
   # Right projection of a set
  if Is_null(pq) then null()
  else p:=sn0(pq);  r:=less(hd(pq,t),p);
       if Is_null(r) then p else hd(r,t) fi
  fi
end:

dx:=proc(pq::integer) local r;
   # Right projection of a set or class
  if Is_class(pq) then r:=cmp(pq) else r:=pq fi;
  if In(null(),r) then cmp(dx0(r)) else dx0(r) fi
end:
```

Dimensions of sets: depth, width, length

Rank

```
rk:=proc(p::integer) local n,q;
   # Returns a von Neumann number (set)
  if Is_class(p) then omega
  else q:=p; n:=null();
    while q<>null() do tl(q,q); n:=nx(n) od;
     # iterated head extraction
    return n
  fi
end:

Rk:=proc(p::integer) local J,q;
   # Returns a Maple number (integer)
  if Is_class(p) then infinity
  else q:=p;
    for J from 0 while q<>null() do tl(q,q) od;
    return J
  fi
end:
```

Cardinality as Maple number (integer)

```
Card:=proc(p::integer) local J,q;
  if Is_class(p) then infinity
  else q:=p;
    for J from 0 while q<>null() do hd(q,q) od;
     # iterated tail extraction
    return J;
  fi
end:
```

Right length as Maple number (integer)

```
Rlen:=proc(p::integer) local J,q;
  if Is_class(p) then infinity
  else q:=p;
    for J from 0 while q<>null() do q:=dx(q) od;
    return J
  fi
end:
```

Left length as Maple number (integer)

```
Llen:=proc(p::integer) local J,q;
  if Is_class(p) then infinity
  else q:=p;
    for J from 0 while q<>null() do q:=sn(q) od;
    return J
  fi
end:
```

Selectors

Selection of an arbitrary element of a set/class not intersecting the set/class

```
arb:=proc(p::integer) if Is_null(p) then null() else mn(p) fi end:
```

Choice of an element from each set in a set

```
ch:=proc(p::nonnegint) local h,t;
   # The highest element in each block is selected
  if Is_null(p) or p=num(1) then null()
  else wth0(ch(tl(p,h)),hd(h,t))
  fi
end:

eta:=proc(p::nonnegint,q::nonnegint)
  if p=q then arb(p)
    else df(p,q); if Is_null(%) then arb(df(q,p)) else arb(%) fi
  fi
end:
```

Intensionally characterized subsets (only for sets)

Negative separation à la Fraenkel

```
spf:=proc(p::nonnegint,f::procedure,g::procedure) local h;
  if Is_null(p) then p
  else spf(tl(p,h),f,g);
       if not In(f(h),g(h)) then wth0(%,h) else % fi
  fi
end:
```

Positive separation

```
sppf:=proc(p::nonnegint,f::procedure,g::procedure) local q;
  spf(p, q->q, q->spf(p,q->f(q),q->g(q)))
end:
```

Negative replacement à la Fraenkel

```
rpf:=proc(p::nonnegint,f::procedure,g::procedure,h::procedure)
     local t;
  if Is_null(p) then p
  else hd(p,t);
    if In(f(%),g(%)) or In(%,rpf(t,f,g,h))
    then rpf(t,f,g,h)
    else wth0(rpf(t,f,g,h),%)
    fi
  fi
end:
```

Positive replacement

```
rppf:=proc(p::nonnegint,f::procedure,g::procedure,h::procedure)
  rpf(p, q->q, q->rpf(p,q->f(q),q->g(q),q->q), q->h(q));
end:
```

Replacement à la Skolem

```
rpl:=proc(p::nonnegint,f::procedure,h::procedure) local t;
    if Is_null(p) then p
    else hd(p,t);
      if f(%) then wth0(rpl(t,f,h),%) else rpl(t,f,h) fi
    fi
end:
```

Displaying functions

Table of special sets [position, symbol] up to rank “R”

```
symb:=proc(R::nonnegint)  local i;
  [seq([num(i),i],i=0..R), seq([cum(i),V||i],i=3..R)];
end:
```

Standard set representation by a character string

```
pos2string:=proc(p::integer,ty::string) local q,h,sel,str;
             # p:      position of the set
             # ty=‘‘,’’: the set is a tail
             # ty=‘‘’’ : the set is a head
             # ty=‘‘}’’: expand, used to display symbols
  if Is_class(p) then q:=cmp(p) else q:=p fi;
  sel:=select(x->op(1,x)=q, symb(Rk(q)));
    # extract symbol from table, if present
  if ty=‘‘’’ and nops(sel)>0 then cat(‘‘’’,op(sel)[2])
```

```
  elif Is_null(q) and ty=``,'' then ``''
  elif Is_null(q) and ty=``}'' then ``{}''
  else if ty=``,'' then str:=``'' else str:=``{'' fi;
       str:= cat(str, pos2string(tl(q,h),``,''), pos2string(h,``''));
       if ty=``'' then cat(str,``}'') else cat(str,ty) fi;
  fi;
  if Is_class(p) then cat(``-'',%) else % fi;
end:
```

Standard set representation (Maple "set" type)

```
p2s:=proc(p::nonnegint) local h;
   # Basic funtion for recursive set construction
  if Is_null(p) then {}
  else p2s(tl(p,h)) union {p2s(h)}
  fi
end:

pos2set:=proc(p::integer) local q,R,subslist;
   # Representation with special sets
  if Is_class(p) then q:=cmp(p) else q:=p fi;
  R:=Rk(q);
  subslist:=seq(p2s(cum(j))=V||j, j=3..R-1);              # levels
  subslist:=subslist, seq({seq(j,j=0..i-2)}=i-1, i=1..R); # ordinals
  subs(subslist,p2s(q));
  if Is_class(p) then -% else % fi;
end:
```

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