THE VORONOVS'KA THEOREM FOR SOME OPERATORS OF THE SZASZ-MIRAKJAN TYPE

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We give the Voronovskaya theorem for some operators of the Szasz-Mirakjan type defined in the space of functions continuous on \([0, +\infty)\) and having the polynomial growth at infinity.

Some approximation properties of these operators are given in [2], [4].

1. Notations.

1.1. Let \(\mathbb{R}_0 := [0, +\infty), \mathbb{N} := \{1, 2, \ldots\}, \mathbb{N}_0 := \mathbb{N} \cup \{0\}\) and let \(w_q(\cdot), q \in \mathbb{N}_0\), be the weight function defined on \(\mathbb{R}_0\) by the formula

\[
(1) \quad w_0(x) := 1, \quad w_q(x) := (1 + x^q)^{-1} \quad \text{if} \quad q \geq 1.
\]

Analogously as in [1], we denote by \(C_q, q \in \mathbb{N}_0\), the space of real-valued functions \(f\) defined on \(\mathbb{R}_0\) and such that \(w_q f\) is uniformly continuous and bounded function on \(\mathbb{R}_0\). The norm in \(C_q\) is defined by

\[
\|f\|_q := \sup_{x \in \mathbb{R}_0} w_q(x) |f(x)|.
\]

Entrato in Redazione il 29 agosto 1995.

AMS Subject classification: 41A36

Key words: Linear positive operator, Voronovskaya theorem.
For a fixed \( q \in \mathbb{N}_0 \) let

\[
C_q^2 := \{ f \in C_q : f', f'' \in C_q \}.
\]

1.2. In [2] were introduced the operators \( L_n \) and \( U_n \) of the Szasz-Mirakjan type for functions \( f \in C_q, q \in \mathbb{N}_0 \),

(2) \[
L_n (f; x) := \sum_{k=0}^{\infty} p_{n,k}(x) f \left( \frac{2k}{n} \right),
\]

(3) \[
U_n (f; x) := \sum_{k=0}^{\infty} p_{n,k}(x) \frac{n}{2} \int_{I_{n,k}} f (t) \, dt,
\]

\( n \in \mathbb{N}, x \in \mathbb{R}_0 \), where

(4) \[
p_{n,k}(x) := \frac{1}{\cosh nx} \frac{(nx)^{2k}}{(2k)!}, \quad k \in \mathbb{N}_0,
\]

\( \sinh x, \cosh x, \tanh x \) are the elementary hyperbolic functions and

\[
I_{n,k} := \left[ \frac{2k}{n}, \frac{2k+2}{n} \right], \quad k \in \mathbb{N}_0.
\]

In [4] we have introduced the operators \( A_n \) an \( B_n \) in the space \( C_q \):

(5) \[
A_n (f; x) := \frac{f (0)}{1 + \sinh nx} + \sum_{k=0}^{\infty} q_{n,k}(x) f \left( \frac{2k+1}{n} \right),
\]

(6) \[
B_n (f; x) := \frac{f (0)}{1 + \sinh nx} + \sum_{k=0}^{\infty} q_{n,k}(x) \frac{n}{2} \int_{I_{n,k}^*} f (t) \, dt,
\]

\( n \in \mathbb{N}, x \in \mathbb{R}_0 \), where

(7) \[
q_{n,k}(x) := \frac{1}{1 + \sinh nx} \frac{(nx)^{2k+1}}{(2k+1)!},
\]

and

\[
I_{n,k}^* := \left[ \frac{2k+1}{n}, \frac{2k+3}{n} \right], \quad k \in \mathbb{N}_0.
\]
It is clear that $L_n, U_n, A_n$ and $B_n, n \in \mathbb{N}$, are a linear positive operators defined on every space $C_q, q \in \mathbb{N}_0$, and

\begin{equation}
L_n (1; x) = U_n (1; x) = A_n (1; x) = B_n (1; x) = 1
\end{equation}

for each $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.

In [2] and [4] was proved that $L_n, U_n, A_n$ and $B_n, n \in \mathbb{N}$, are a operators from $C_q$ into $C_q$. Moreover, in [2] and [4] some approximation properties of these operators were given. In particular in [2] it was proved that if $f \in C_q$, $q \in \mathbb{N}_0$, there exists a positive constant $M(q)$ depending only on $q$ and such that for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$ holds

\begin{equation}
w_q (x) |L_n (f; x) - f (x)| \leq M(q) \omega \left( f; \sqrt[1 + 1]{x} \right),
\end{equation}

where $\omega(f; \cdot)$ is the modulus of continuity of $f$, i.e.

$$\omega(f; t) := \sup_{0 < h \leq t} \| f (\cdot + h) - f (\cdot) \|_q.$$ 

The estimation (9) is true also for the operators $A_n, B_n$ and $U_n$ ([2], [4]).

2. Auxiliary results.

In this section we shall give some properties of the above operators, which we shall apply to the proofs of the main theorems. We set

\begin{equation}
S(nx) := \frac{\sinh nx}{1 + \sinh nx}, \quad T(nx) := \frac{\cosh nx}{1 + \sinh nx},
\end{equation}

$$V(nx) := 1 - \tanh nx,$$

for $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$.

From (2) - (8) and (10), and by the elementary calculations, we obtain
Lemma 1. For each \( x \in \mathbb{R}_0 \) and \( n \in \mathbb{N} \) we have

\[
L_n (t - x; x) = -x V(nx),
\]

\[
L_n \left( (t - x)^2; x \right) = \left( 2x^2 - \frac{x}{n} \right) V(nx) + \frac{x}{n},
\]

\[
L_n \left( (t - x)^4; x \right) = \left( 8x^4 - \frac{12x^3}{n} + \frac{4x^2}{n^2} - \frac{x}{n^3} \right) V(nx) + \frac{3x^2}{n^2} + \frac{x}{n^3},
\]

\[
U_n (t - x; x) = -x V(nx) + \frac{1}{n},
\]

\[
U_n \left( (t - x)^2; x \right) = \left( 2x^2 - \frac{3x}{n} \right) V(nx) + \frac{x}{n} + \frac{4}{3n^2},
\]

\[
U_n \left( (t - x)^4; x \right) = \left( 8x^4 - \frac{28x^3}{n} + \frac{32x^2}{n^2} - \frac{21x}{n^3} \right) V(nx) + \frac{12x}{n^3} + \frac{16}{5n^4},
\]

(11)

\[
A_n (t - x; x) = x \left( T(nx) - 1 \right),
\]

\[
A_n \left( (t - x)^2; x \right) = x^2 \left( S(nx) - 2T(nx) + 1 \right) + \frac{x}{n} T(nx),
\]

\[
A_n \left( (t - x)^4; x \right) = x^4 \left( 7S(nx) - 8T(nx) + 1 \right) + \frac{12x^3}{n} (T(nx) - S(nx)) + \frac{x^2}{n^2} (7S(nx) - 4T(nx)) + \frac{x}{n^3} T(nx),
\]

\[
B_n (t - x; x) = x \left( T(nx) - 1 \right) + \frac{1}{n} S(nx),
\]

\[
B_n \left( (t - x)^2; x \right) = x^2 \left( S(nx) - 2T(nx) + 1 \right) + \frac{2x}{n} (T(nx) - S(nx)) + \frac{4}{3n^2} S(nx),
\]

\[
B_n \left( (t - x)^4; x \right) = x^4 \left( 7S(nx) - 8T(nx) + 1 \right) + \frac{28x^3}{n} (T(nx) - S(nx)) + \frac{x^2}{n^2} (35 S(nx) - 32 T(nx)) + \frac{17x}{n^3} T(nx).
\]

Applying Lemma 1, we shall prove two lemmas.
Lemma 2. For every fixed \( x_0 \in \mathbb{R}_0 \) holds

\[
\begin{align*}
\lim_{n \to \infty} n L_n (t - x_0; x_0) &= 0 = \lim_{n \to \infty} n A_n (t - x_0; x_0), \\
\lim_{n \to \infty} n U_n (t - x_0; x_0) &= 1 = \lim_{n \to \infty} n B_n (t - x_0; x_0),
\end{align*}
\]

(12)

\[
\lim_{n \to \infty} n \begin{cases} 
L_n ((t - x_0)^2; x_0) \\
U_n ((t - x_0)^2; x_0) \\
A_n ((t - x_0)^2; x_0) \\
B_n ((t - x_0)^2; x_0)
\end{cases} = x_0.
\]

(13)

Proof. We shall give the proof only for \( L_n \), because the proofs for \( U_n \), \( A_n \) and \( B_n \) are analogous.

By Lemma 1 we get

\[
nL_n (t - x; x) = \frac{-2nx}{e^{2nx} + 1},
\]

\[
nL_n ((t - x)^2; x) = x \left( \frac{4nx - 2}{e^{2nx} + 1} + 1 \right),
\]

for \( x \in \mathbb{R}_0 \) and \( n \in \mathbb{N} \), which immediately yield (12) and (13) for \( L_n \).

Lemma 3. For every fixed \( x_0 \in \mathbb{R}_0 \) there exists a positive constant \( M_1 (x_0) \), depending only on \( x_0 \), such that for each \( n \in \mathbb{N} \) holds

\[
\begin{cases} 
L_n ((t - x_0)^4; x_0) \\
U_n ((t - x_0)^4; x_0) \\
A_n ((t - x_0)^4; x_0) \\
B_n ((t - x_0)^4; x_0)
\end{cases} \leq M_1 (x_0) n^{-2}.
\]

(14)

Proof. For example we shall give the proof for \( A_n \).

From (10) we get for \( x \geq 0 \) and \( n, r \in \mathbb{N} \)

\[
x^r |S (nx) - T (nx)| = \frac{2x^r e^{-nx}}{2 + e^{nx} - e^{-nx}} \leq \frac{2x^r}{e^{nx} + 1} \leq 2r! n^{-r},
\]

\[
x^r |1 - S (nx)| = \frac{2x^r}{2 + e^{nx} - e^{-nx}} \leq \frac{2x^r}{e^{nx} + 1} \leq 2r! n^{-r},
\]
and similarly

\[ \begin{align*}
    x^r |1 - T(nx)| &\leq 2r! n^{-r}, \\
    0 &\leq S(nx) \leq 1, \\
    0 &< T(nx) \leq 1.
\end{align*} \]

Using these inequalities, we obtain from (11)

\[
A_n \left((t - x_0)^4; x_0\right) \leq x_0^4 \left(7S(nx_0) - T(nx_0)\right) + 12x_0^3 \left|S(nx_0) - T(nx_0)\right| + \frac{x_0^2}{n} \left[7 |S(nx_0) - T(nx_0)| + 3T(nx_0)\right] + \left(\frac{3}{n}\right) T(nx_0) \leq 16 \left(4! n^{-4}\right) + 24 \left(3! n^{-4}\right) + n^{-2} \left(28n^{-2} + 3x_0^2\right) + x_0 n^{-3} \leq (3x_0^2 + x_0 + 556) n^{-2} \equiv M_1(x_0) n^{-2}
\]

for every fixed \(x_0 \geq 0\) and \(n \in \mathbb{N}\). Hence the proof of (14) is completed.

The following lemma is proved in [4].

**Lemma 4.** For every fixed \(p \in \mathbb{N}\) there exist the positive coefficients \(a_{p,k}\) and \(b_{p,k}\), \(1 \leq k \leq \left\lfloor \frac{p+1}{2} \right\rfloor\), depending only on \(p, k\) and \(a_{2m,m} = 1, b_{2m+1,m+1} = 1\) for \(m \in \mathbb{N}\) and such that for all \(n \in \mathbb{N}\) and \(x \in \mathbb{R}_0\) holds

\[
A_n \left(t^p; x\right) = S(nx) \sum_{k=1}^{\left\lfloor \frac{p}{2} \right\rfloor} a_{p,k} x^{2k} + T(nx) \sum_{k=1}^{\left\lfloor \frac{p+1}{2} \right\rfloor} b_{p,k} x^{2k-1},
\]

where \(S(nx), T(nx)\) are defined by (10), \([y]\) denotes the integral part of \(y\).

(As usual we assume that \(\sum_{k=1}^{j} c_k = 0\) if \(i > j\), for each \(c_k \in \mathbb{R}\)).

**Remark.** By the mathematical induction we proved the formulae similar to (15) for \(B_n(t^p; x)\) ([4]) and for \(L_n\left(t^p; x\right)\) and \(U_n\left(t^p; x\right)\) ([2]), \(n, p \in \mathbb{N}, x \in \mathbb{R}_0\).

The full formulae for \(a_{p,k}\) and \(b_{p,k}\) given in (15) are not necessary for the next properties of considered operators.

By Lemma 4 the following inequalities were derived in the papers [2] (for \(L_n\) and \(U_n\)) and [4] (for \(A_n\) and \(B_n\)).

**Lemma 5.** For every fixed \(q \in \mathbb{N}_0\) there exists a positive constant \(M_2(q)\) depending only on \(q\) such that for all \(x \in \mathbb{R}_0\) and \(n \in \mathbb{N}\) holds

\[
w_q(x) L_n \left(\frac{(t-x)^2}{w_q(t)}; x\right) \leq M_2(q) \frac{x + 1}{n},
\]
$w_q(x)U_n \left( \frac{(t-x)^2}{w_q(t)} ; x \right) \leq M_2(q) \frac{x+1}{n}.$

Now, using Lemma 5, we can prove.

**Lemma 6.** Let $x_0$ be a fixed point on $\mathbb{R}_0$ and $\varphi (\cdot ; x_0)$ be a given function belonging to the space $C_q$, with some $q \in \mathbb{N}_0$, and such that

\[
\lim_{t \to x_0} \varphi (t; x_0) = 0 \quad \left( \lim_{t \to 0^+} \varphi (t; 0) = 0 \right).
\]

Then

\[
\lim_{n \to \infty} L_n (\varphi (t; x_0) ; x_0) = 0,
\]

(17)

\[
\lim_{n \to \infty} U_n (\varphi (t; x_0) ; x_0) = 0 = \lim_{n \to \infty} A_n (\varphi (t; x_0) ; x_0) = \lim_{n \to \infty} B_n (\varphi (t; x_0) ; x_0).
\]

(18)

**Proof.** We shall prove only (17), because the proof of (18) is analogous. By (2) we have for every fixed $x_0 \geq 0$ and $n \in \mathbb{N}$

\[
w_q (x_0) L_n (\varphi (t; x_0) ; x_0) = w_q (x_0) \sum_{k=0}^{\infty} p_{n,k} (x_0) \varphi \left( \frac{2k}{n} ; x_0 \right).
\]

(19)

Choose $\varepsilon > 0$. By the properties of $\varphi (\cdot ; x_0)$ there exists a positive constant $\delta = \delta (\varepsilon)$ such that $|\varphi (t; x_0)| < \frac{\varepsilon}{2}$ if $|t - x_0| < \delta$ and $t \geq 0$. Moreover, there exists a positive constant $M_3 \equiv M_3 (q)$ such that

\[w_q (t) |\varphi (t; x_0)| \leq M_3 \quad \text{for} \quad t \geq 0.\]

Hence we get from (19)

\[w_q (x_0) |L_n (\varphi (t; x_0) ; x_0)| \leq w_q (x_0) \sum_{k \in \mathbb{Q}_{n,1}} p_{n,k} (x_0) \left| \varphi \left( \frac{2k}{n} ; x_0 \right) \right| + \]
\[ + w_q(x_0) \sum_{k \in Q_{n,2}} p_{n,k}(x_0) \varphi \left( \frac{2k}{n}; x_0 \right) := \Sigma_1 + \Sigma_2, \]

where \( Q_{n,1} := \{ k \in \mathbb{N}_0 : \left| \frac{2k}{n} - x_0 \right| < \delta \} \) and \( Q_{n,2} := \{ k \in \mathbb{N}_0 : \left| \frac{2k}{n} - x_0 \right| \geq \delta \}. \)

Moreover, by (1) - (3) and by the above properties of \( \varphi(\cdot; x_0) \), we have

\[ \sigma_1 < \frac{\varepsilon}{2} \sum_{k=0}^{\infty} p_{n,k}(x_0) = \frac{\varepsilon}{2}, \]

\[ \Sigma_2 \leq M_3 w_q(x_0) \sum_{k \in Q_{n,2}} p_{n,k}(x_0) \left( w_q \left( \frac{2k}{n} \right) \right)^{-1}. \]

But if \( \left| \frac{2k}{n} - x_0 \right| \geq \delta \), then \( 1 \leq \delta^{-2} \left( \frac{2k}{n} - x_0 \right)^2 \) and further we get

\[ \Sigma_2 \leq M_3 \delta^{-2} w_q(x_0) \sum_{k \in Q_{n,2}} p_{n,k}(x_0) \left( w_q \left( \frac{2k}{n} \right) \right)^{-1} \left( \frac{2k}{n} - x_0 \right)^2 \]

\[ \leq M_3 \delta^{-2} w_q(x_0) L_n \left( \frac{(t - x_0)^2}{w_q(t)}; x_0 \right), \]

which by (16) yields \( \Sigma_2 \leq M_3 M_2(q) \frac{x_0 + 1}{n \delta^2} \) for all \( n \in \mathbb{N} \).

It is obvious that, for a fixed \( x_0, \varepsilon, \delta, M_2(q), M_1 \), there exists a natural number \( n_0 \equiv n_0(\varepsilon, \delta, x_0, M_2(q), M_3) \) such that for all \( n_0 < n \in \mathbb{N} \) holds

\[ M_3 M_2(q) \frac{x_0 + 1}{n \delta^2} < \frac{\varepsilon}{2}. \]

Hence,

\[ \Sigma_2 < \frac{\varepsilon}{2} \quad \text{for all} \quad n > n_0. \]

Summing up, we obtain

\[ w_q(x_0) |L_n(\varphi(t; x_0); x_0)| < \varepsilon \quad \text{for} \quad n > n_0, \]

i.e.

\[ \lim_{n \to \infty} w_q(x_0) L_n(\varphi(t; x_0); x_0) = 0. \]

From this and by (1) we obtain the desired assertion (17).
3. The Voronovskaya theorems.

The Voronovskaya theorem for the Bernstein operator is given in [3]. In this part we prove the similar theorems for our operators.

**Theorem 1.** If \( f \in C^2_q \) with some \( q \in \mathbb{N}_0 \), then

\[
(20) \quad \lim_{n \to \infty} n \{ L_n (f; x) - f(x) \} = \frac{x}{2} f''(x)
\]

for every fixed \( x \in \mathbb{R}_0 \).

**Proof.** Let \( x_0 \in \mathbb{R}_0 \) be a fixed point. Then, by the Taylor formula, we can write for every \( t \in \mathbb{R}_0 \)

\[
(21) \quad f(t) = f(x_0) + f'(x_0)(t - x_0) + \frac{1}{2} f''(x_0)(t - x_0)^2 + \psi(t; x_0)(t - x_0)^2,
\]

where \( \psi(\cdot; x_0) \) is a function belonging to \( C_q \) and such that

\[
\lim_{t \to x_0} \psi(t; x_0) = 0.
\]

From (21) and by (2) and (8) we get

\[
(22) \quad L_n (f(t); x_0) - f(x_0) = f'(x_0) L_n (t - x_0; x_0) + \frac{1}{2} f''(x_0) L_n ((t - x)^2; x_0) + L_n (\psi(t; x_0)(t - x_0)^2; x_0)
\]

for every \( n \in \mathbb{N} \). By Lemma 2 follows

\[
(23) \quad \lim_{n \to \infty} n L_n (t - x_0; x_0) = 0,
\]

\[
(24) \quad \lim_{n \to \infty} n L_n ((t - x_0)^2; x_0) = x_0.
\]

By (2) and the Hölder inequality we get for every \( n \in \mathbb{N} \)

\[
(25) \quad \left| L_n (\psi(t; x_0)(t - x_0)^2; x_0) \right| \leq \left\{ L_n (\psi^2(t; x_0); x_0) \right\}^{\frac{1}{2}} \left\{ L_n ((t - x_0)^4; x_0) \right\}^{\frac{1}{2}}.
\]
Moreover, the function \( \varphi (t; x_0) := \psi^2(t; x_0) \), \( t \geq 0 \), satisfies the following conditions: \( \varphi (\cdot; x_0) \in C_{2q} \) and

\[
\lim_{t \to x_0} \varphi (t; x_0) = 0.
\]

From this and in view of Lemma 6 we get

\[
\lim_{n \to \infty} L_n (\psi^2(t; x_0); x_0) \equiv \lim_{n \to \infty} L_n (\varphi(t; x_0); x_0) = 0.
\]

Using (26) and (14) we obtain from (25)

\[
\lim_{n \to \infty} nL_n (\psi(t; x_0)(t-x_0)^2; x_0) = 0.
\]

Now, applying (23), (24) and (27), we immediately derive from (22)

\[
\lim_{n \to \infty} n \left\{ L_n (f(t); x_0) - f(x_0) \right\} = \frac{x_0}{2} f''(x_0).
\]

Thus the proof of (20) is finished.

Analogously, using Lemmas 2, 3 and 6 and by (21), we can prove the following Voronovskaya theorem for the operators \( U_n, A_n \) and \( B_n \).

**Theorem 2.** Let \( f \in C_q^2 \) with some \( q \in \mathbb{N}_0 \). Then, for every fixed \( x \geq 0 \), we have

\[
\lim_{n \to \infty} n \left\{ U_n (f(t); x) - f(x) \right\} = f'(x) + \frac{x}{2} f''(x),
\]

\[
\lim_{n \to \infty} n \left\{ A_n (f(t); x) - f(x) \right\} = \frac{x}{2} f''(x),
\]

\[
\lim_{n \to \infty} n \left\{ B_n (f(t); x) - f(x) \right\} = f'(x) + \frac{x}{2} f''(x).
\]
REFERENCES


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