LE MATEMATICHE
Vol. L (1995) – Fasc. II, pp. 251-261

THE VORONOVSKAYA THEOREM FOR SOME OPERATORS OF THE SZASZ-MIRAKJAN TYPE

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We give the Voronovskaya theorem for some operators of the Szasz-Mirakjan type defined in the space of functions continuous on $[0, +\infty)$ and having the polynomial growth at infinity.

Some approximation properties of these operators are given in [2], [4].

1. Notations.

1.1. Let $\mathbb{R}_0 := [0, +\infty)$, $\mathbb{N} := \{1, 2, ...\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and let $w_q(\cdot)$, $q \in \mathbb{N}_0$, be the weight function defined on $\mathbb{R}_0$ by the formula

$$(1) \qquad w_0(x) := 1, \qquad w_q(x) := \left(1 + x^q\right)^{-1} \quad \text{if} \quad q \geq 1.$$

Analogously as in [1], we denote by C_q, $q \in \mathbb{N}_0$, the space of real-valued functions f defined on $\mathbb{R}_0$ and such that $w_q f$ is uniformly continuous and bounded function on $\mathbb{R}_0$. The norm in C_q is defined by

$$\|f\|_q := \sup_{x \in \mathbb{R}_0} w_q(x) \, |f(x)| .$$

Entrato in Redazione il 29 agosto 1995.

AMS Subject classification: 41A36
Key words: Linear positive operator, Voronowskaya theorem.

For a fixed $q \in \mathbb{N}_0$ let

$$C_q^2 := \{f \in C_q : f', f'' \in C_q\} .$$

1.2. In [2] were introduced the operators L_n and U_n of the Szasz-Mirakjan type for functions $f \in C_q$, $q \in \mathbb{N}_0$,

$$L_n(f; x) := \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{2k}{n}\right), \tag{2}$$

$$U_n(f; x) := \sum_{k=0}^{\infty} p_{n,k}(x) \frac{n}{2} \int_{I_{n,k}} f(t)\, dt, \tag{3}$$

$n \in \mathbb{N}$, $x \in \mathbb{R}_0$, where

$$p_{n,k}(x) := \frac{1}{\cosh nx} \frac{(nx)^{2k}}{(2k)!}, \quad k \in \mathbb{N}_0, \tag{4}$$

$\sinh x$, $\cosh x$, $\tanh x$ are the elementary hyperbolic functions and

$$I_{n,k} := \left[\frac{2k}{n}, \frac{2k+2}{n}\right], \quad k \in \mathbb{N}_0 .$$

In [4] we have introduced the operators A_n an B_n in the space C_q:

$$A_n(f; x) := \frac{f(0)}{1+\sinh nx} + \sum_{k=0}^{\infty} q_{n,k}(x) f\left(\frac{2k+1}{n}\right), \tag{5}$$

$$B_n(f; x) := \frac{f(0)}{1+\sinh nx} + \sum_{k=0}^{\infty} q_{n,k}(x) \frac{n}{2} \int_{I_{n,k}^*} f(t)\, dt, \tag{6}$$

$n \in \mathbb{N}$, $x \in \mathbb{R}_0$, where

$$q_{n,k}(x) := \frac{1}{1+\sinh nx} \frac{(nx)^{2k+1}}{(2k+1)!}, \tag{7}$$

and

$$I_{n,k}^* := \left[\frac{2k+1}{n}, \frac{2k+3}{n}\right], \quad k \in \mathbb{N}_0 .$$

It is clear that L_n, U_n, A_n and B_n, $n \in \mathbb{N}$, are a linear positive operators defined on every space C_q, $q \in \mathbb{N}_0$, and

$$L_n(1;x) = U_n(1;x) = A_n(1;x) = B_n(1;x) = 1 \tag{8}$$

for each $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.

In [2] and [4] was proved that L_n, U_n, A_n and B_n, $n \in \mathbb{N}$, are a operators from C_q into C_q. Moreover, in [2] and [4] some approximation properties of these operators were given. In particular in [2] it was proved that if $f \in C_q$, $q \in \mathbb{N}_0$, there exists a positive constant $M(q)$ depending only on q and such that for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$ holds

$$w_q(x)\,|L_n(f;x) - f(x)| \le M(q)\,\omega\left(f;\sqrt{\frac{x+1}{n}}\right), \tag{9}$$

where $\omega(f;\cdot)$ is the modulus of continuity of f, i.e.

$$\omega(f;t) := \sup_{0<h\le t} \|f(\cdot+h) - f(\cdot)\|_q .$$

The estimation (9) is true also for the operators A_n, B_n and U_n ([2], [4]).

2. Auxiliary results.

In this section we shall give some properties of the above operators, which we shall apply to the proofs of the main theorems. We set

$$S(nx) := \frac{\sinh nx}{1+\sinh nx}, \qquad T(nx) := \frac{\cosh nx}{1+\sinh nx}, \tag{10}$$

$$V(nx) := 1 - \tanh nx,$$

for $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$.

From (2) - (8) and (10), and by the elementary calculations, we obtain

Lemma 1. *For each $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$ we have*

$$
\begin{aligned}
& L_n(t-x;x) = -xV(nx)\,,\\
& L_n\left((t-x)^2;x\right) = \left(2x^2 - \frac{x}{n}\right)V(nx) + \frac{x}{n}\,,\\
& L_n\left((t-x)^4;x\right) = \left(8x^4 - \frac{12x^3}{n} + \frac{4x^2}{n^2} - \frac{x}{n^3}\right)V(nx) +\\
& \qquad + \frac{3x^2}{n^2} + \frac{x}{n^3}\,,\\
& U_n(t-x;x) = -x\,V(nx) + \frac{1}{n}\,,\\
& U_n\left((t-x)^2;x\right) = \left(2x^2 - \frac{3x}{n}\right)V(nx) + \frac{x}{n} + \frac{4}{3n^2}\,,\\
& U_n\left((t-x)^4;x\right) = \left(8x^4 - \frac{28x^3}{n} + \frac{32x^2}{n^2} - \frac{21x}{n^3}\right)V(nx) +\\
& \qquad + \frac{12x}{n^3} + \frac{16}{5n^4}\,,\\
& A_n(t-x;x) = x\left(T(nx) - 1\right),\\
& A_n\left((t-x)^2;x\right) = x^2\left(S(nx) - 2T(nx) + 1\right) + \frac{x}{n}T(nx)\,,\\
& A_n\left((t-x)^4;x\right) = x^4\left(7S(nx) - 8T(nx) + 1\right) +\\
& \qquad + \frac{12x^3}{n}\left(T(nx) - S(nx)\right) + \frac{x^2}{n^2}\left(7S(nx) - 4T(nx)\right) + \frac{x}{n^3}T(nx)\,,\\
& B_n(t-x;x) = x\left(T(nx) - 1\right) + \frac{1}{n}S(nx)\,,\\
& B_n\left((t-x)^2;x\right) = x^2\left(S(nx) - 2T(nx) + 1\right) +\\
& \qquad + \frac{2x}{n}\left(T(nx) - S(nx)\right) + \frac{4}{3n^2}S(nx)\,,\\
& B_n\left((t-x)^4;x\right) = x^4\left(7S(nx) - 8T(nx) + 1\right) + \frac{28x^3}{n}\left(T(nx) -\right.\\
& \qquad \left. - S(nx)\right) + \frac{x^2}{n^2}\left(35\,S(nx) - 32\,T(nx)\right) + \frac{17x}{n^3}T(nx)\,.
\end{aligned}
\tag{11}
$$

Applying Lemma 1, we shall prove two lemmas.

Lemma 2. *For every fixed $x_0 \in \mathbb{R}_0$ holds*

$$(12)\qquad \begin{aligned} \lim_{n\to\infty} n\, L_n\,(t-x_0;x_0) = 0 = \lim_{n\to\infty} nA_n\,(t-x_0;x_0)\ ,\\ \lim_{n\to\infty} n\, U_n\,(t-x_0;x_0) = 1 = \lim_{n\to\infty} nB_n\,(t-x_0;x_0)\ , \end{aligned}$$

$$(13)\qquad \lim_{n\to\infty} n \begin{cases} L_n\left((t-x_0)^2;x_0\right)\\ U_n\left((t-x_0)^2;x_0\right)\\ A_n\left((t-x_0)^2;x_0\right)\\ B_n\left((t-x_0)^2;x_0\right) \end{cases} = x_0\,.$$

Proof. We shall give the proof only for L_n, because the proofs for U_n, A_n and B_n are analogous.

By Lemma 1 we get

$$\begin{aligned} nL_n\,(t-x;x) &= \frac{-2nx}{e^{2nx}+1}\,,\\ nL_n\left((t-x)^2;x\right) &= x\left(\frac{4nx-2}{e^{2nx}+1}+1\right)\,, \end{aligned}$$

for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$, which immediately yield (12) and (13) for L_n.

Lemma 3. *For every fixed $x_0 \in \mathbb{R}_0$ there exists a positive constant $M_1\,(x_0)$, depending only on x_0, such that for each $n \in \mathbb{N}$ holds*

$$(14)\qquad \begin{cases} L_n\left((t-x_0)^4;x_0\right)\\ U_n\left((t-x_0)^4;x_0\right)\\ A_n\left((t-x_0)^4;x_0\right)\\ B_n\left((t-x_0)^4;x_0\right) \end{cases} \le M_1\,(x_0)\,n^{-2}\,.$$

Proof. For example we shall give the proof for A_n.

From (10) we get for $x \ge 0$ and $n, r \in \mathbb{N}$

$$\begin{aligned} x^r\,|S\,(nx)-T\,(nx)| &= \frac{2x^r e^{-nx}}{2+e^{nx}-e^{-nx}} \le \frac{2x^r}{e^{nx}+1} \le 2r!\,n^{-r}\,,\\ x^r\,|1-S\,(nx)| &= \frac{2x^r}{2+e^{nx}-e^{-nx}} \le \frac{2x^r}{e^{nx}+1} \le 2r!\,n^{-r}\,, \end{aligned}$$

and similarly

$$x^r\ |1 - T\,(nx)| \le 2r!\,n^{-r}\,,$$

$$0 \le S\,(nx) \le 1, \qquad 0 < T\,(nx) \le 1\,.$$

Using these inequalities, we obtain from (11)

$$A_n\left((t-x_0)^4\,;x_0\right) \le x_0^4\,\{7|S\,(nx_0) - T\,(nx_0)\,| + |1 - T\,(nx_0)\,|\} +$$
$$+\frac{12x_0^3}{n}\,|S(nx_0) - T(nx_0)| + \frac{x_0^2}{n^2}\,\{7\ |S\,(nx_0) - T\,(nx_0)| + 3\,T\,(nx_0)\} +$$
$$+\frac{x_0}{n^3}\,T\,(nx_0) \le 16\left(4!\,n^{-4}\right) + 24\left(3!\,n^{-4}\right) + n^{-2}\left(28n^{-2} + 3x_0^2\right) +$$
$$+x_0 n^{-3} \le \left(3x_0^2 + x_0 + 556\right) n^{-2} \equiv M_1\,(x_0)\,n^{-2}$$

for every fixed $x_0 \ge 0$ and $n \in \mathbb{N}$. Hence the proof of (14) is completed.

The following lemma is proved in [4].

Lemma 4. *For every fixed* $p \in \mathbb{N}$ *there exist the positive coefficients* $a_{p,k}$ *and* $b_{p,k}$, $1 \le k \le \left[\frac{p+1}{2}\right]$, *depending only on* p, k *and* $a_{2m,m} = 1$, $b_{2m+1,m+1} = 1$ *for* $m \in \mathbb{N}$ *and such that for all* $n \in \mathbb{N}$ *and* $x \in \mathbb{R}_0$ *holds*

$$(15) \qquad A_n\left(t^p;x\right) = S\,(nx)\sum_{k=1}^{\left[\frac{p}{2}\right]} a_{p,k}\frac{x^{2k}}{n^{p-2k}} + T\,(nx)\sum_{k=1}^{\left[\frac{p+1}{2}\right]} b_{p,k}\frac{x^{2k-1}}{n^{p-(2k-1)}}\,,$$

where $S(nx)$, $T(nx)$ *are defined by (10),* $[y]$ *denotes the integral part of* y. *(As usual we assume that* $\sum_{k=i}^{j} c_k = 0$ *if* $i > j$, *for each* $c_k \in \mathbb{R}$*).*

Remark. By the mathematical induction were proved the formulae similar to (15) for $B_n(t^p;x)$ ([4]) and for $L_n\,(t^p;x)$ and $U_n\,(t^p;x)$ ([2]), $n,\ p \in \mathbb{N}$, $x \in \mathbb{R}_0$.

The full formulae for $a_{p,k}$ and $b_{p,k}$ given in (15) are not necessary for the next properties of considered operators.

By Lemma 4 the following inequalities were derived in the papers [2] (for L_n and U_n) and [4] (for A_n and B_n).

Lemma 5. *For every fixed* $q \in \mathbb{N}_0$ *there exists a positive constant* $M_2(q)$ *depending only on* q *such that for all* $x \in \mathbb{R}_0$ *and* $n \in \mathbb{N}$ *holds*

$$(16) \qquad w_q(x)\,L_n\left(\frac{(t-x)^2}{w_q(t)};x\right) \le M_2(q)\,\frac{x+1}{n}\,,$$

$$\left.\begin{array}{l} w_q(x)U_n\left(\dfrac{(t-x)^2}{w_q(t)};x\right) \\ \\ w_q(x)A_n\left(\dfrac{(t-x)^2}{w_q(t)};x\right) \\ \\ w_q(x)B_n\left(\dfrac{(t-x)^2}{w_q(t)};x\right) \end{array}\right\} \leq M_2(q)\frac{x+1}{n}.$$

Now, using Lemma 5, we can prove.

Lemma 6. *Let x_0 be a fixed point on $\mathbb{R}_0$ and $\varphi(\cdot;x_0)$ be a given function belonging to the space C_q, with some $q \in \mathbb{N}_0$, and such that*

$$\lim_{t\to x_0} \varphi(t;x_0) = 0 \qquad \left(\lim_{t\to 0+} \varphi(t;0) = 0\right).$$

Then

$$\lim_{n\to\infty} L_n(\varphi(t;x_0);x_0) = 0, \tag{17}$$

$$\lim_{n\to\infty} U_n(\varphi(t;x_0);x_0) = 0 = \lim_{n\to\infty} A_n(\varphi(t;x_0);x_0) = \lim_{n\to\infty} B_n(\varphi(t;x_0);x_0). \tag{18}$$

Proof. We shall prove only (17), because the proof of (18) is analogous.
By (2) we have for every fixed $x_0 \geq 0$ and $n \in \mathbb{N}$

$$w_q(x_0)L_n(\varphi(t;x_0);x_0) = w_q(x_0)\sum_{k=0}^{\infty} p_{n,k}(x_0)\,\varphi\left(\frac{2k}{n};x_0\right). \tag{19}$$

Choose $\varepsilon > 0$. By the properties of $\varphi(\cdot;x_0)$ there exists a positive constant $\delta \equiv \delta(\varepsilon)$ such that $|\varphi(t;x_0)| < \frac{\varepsilon}{2}$ if $|t-x_0| < \delta$ and $t \geq 0$. Moreover, there exists a positive constant $M_3 \equiv M_3(q)$ such that

$$w_q(t)\,|\varphi(t;x_0)| \leq M_3 \qquad \text{for} \quad t \geq 0.$$

Hence we get from (19)

$$w_q(x_0)\,|L_n(\varphi(t;x_0);x_0)| \leq w_q(x_0)\sum_{k\in Q_{n,1}} p_{n,k}(x_0)\left|\varphi\left(\frac{2k}{n};x_0\right)\right| +$$

$$+w_q(x_0)\sum_{k\in Q_{n,2}} p_{n,k}(x_0)\left|\varphi\left(\frac{2k}{n};x_0\right)\right| := \Sigma_1+\Sigma_2,$$

where $Q_{n,1} := \left\{k\in\mathbb{N}_0 : \left|\frac{2k}{n}-x_0\right|<\delta\right\}$ and $Q_{n,2} := \left\{k\in\mathbb{N}_0 : \left|\frac{2k}{n}-x_0\right|\geq\delta\right\}$. Moreover, by (1) - (3) and by the above properties of $\varphi(\cdot;x_0)$, we have

$$\sigma_1<\frac{\varepsilon}{2}\sum_{k=0}^{\infty} p_{n,k}(x_0)=\frac{\varepsilon}{2},$$

$$\Sigma_2\leq M_3 w_q(x_0)\sum_{k\in Q_{n,2}} p_{n,k}(x_0)\left(w_q\left(\frac{2k}{n}\right)\right)^{-1}.$$

But if $\left|\frac{2k}{n}-x_0\right|\geq\delta$, then $1\leq\delta^{-2}\left(\frac{2k}{n}-x_0\right)^2$ and further we get

$$\Sigma_2\leq M_3\,\delta^{-2}w_q(x_0)\sum_{k\in Q_{n,2}} p_{n,k}(x_0)\left(w_q\left(\frac{2k}{n}\right)\right)^{-1}\left(\frac{2k}{n}-x_0\right)^2$$

$$\leq M_3\delta^{-2}w_q(x_0)\,L_n\left(\frac{(t-x_0)^2}{w_q(t)};x_0\right),$$

which by (16) yields $\Sigma_2\leq M_3M_2(q)\frac{x_0+1}{n\delta^2}$ for all $n\in\mathbb{N}$.

It is obvious that, for a fixed x_0, ε, δ, $M_2(q)$, M_1, there exists a natural number $n_0\equiv n_0(\varepsilon,\delta,x_0,M_2(q),M_3)$ such that for all $n_0<n\in\mathbb{N}$ holds

$$M_3\,M_2(q)\,\frac{x_0+1}{n\delta^2}<\frac{\varepsilon}{2}.$$

Hence,

$$\Sigma_2<\frac{\varepsilon}{2}\qquad\text{for all}\qquad n>n_0.$$

Summing up, we obtain

$$w_q(x_0)\left|L_n(\varphi(t;x_0);x_0)\right|<\varepsilon\qquad\text{for}\quad n>n_0,$$

i.e.

$$\lim_{n\to\infty} w_q(x_0)\,L_n(\varphi(t;x_0);x_0)=0.$$

From this and by (1) we obtain the desired assertion (17).

3. The Voronovskaya theorems.

The Voronovskaya theorem for the Bernstein operator is given in [3]. In this part we prove the similar theorems for our operators.

Theorem 1. *If $f \in C_q^2$ with some $q \in \mathbb{N}_0$, then*

$$\lim_{n\to\infty} n\left\{L_n(f;x) - f(x)\right\} = \frac{x}{2} f''(x) \tag{20}$$

for every fixed $x \in \mathbb{R}_0$.

Proof. Let $x_0 \in \mathbb{R}_0$ be a fixed point. Then, by the Taylor formula, we can write for every $t \in \mathbb{R}_0$

$$f(t) = f(x_0) + f'(x_0)(t - x_0) + \frac{1}{2} f''(x_0)(t - x_0)^2 + \tag{21}$$
$$+ \psi(t; x_0)(t - x_0)^2,$$

where $\psi(\cdot; x_0)$ is a function belonging to C_q and such that

$$\lim_{t\to x_0} \psi(t; x_0) = 0.$$

From (21) and by (2) and (8) we get

$$L_n(f(t); x_0) - f(x_0) = f'(x_0)\, L_n(t - x_0; x_0) + \tag{22}$$
$$+ \frac{1}{2} f''(x_0)\, L_n\left((t-x)^2; x_0\right) + L_n\left(\psi(t; x_0)(t - x_0)^2; x_0\right)$$

for every $n \in \mathbb{N}$. By Lemma 2 follows

$$\lim_{n\to\infty} n\, L_n(t - x_0; x_0) = 0, \tag{23}$$

$$\lim_{n\to\infty} n\, L_n\left((t - x_0)^2; x_0\right) = x_0. \tag{24}$$

By (2) and the Hölder inequality we get for every $n \in \mathbb{N}$

$$\left|L_n\left(\psi(t; x_0)(t - x_0)^2; x_0\right)\right| \le \tag{25}$$
$$\le \left\{L_n\left(\psi^2(t; x_0); x_0\right)\right\}^{\frac{1}{2}} \left\{L_n\left((t - x_0)^4; x_0\right)\right\}^{\frac{1}{2}}.$$

Moreover, the function $\varphi(t;x_0) := \psi^2(t;x_0)$, $t \geq 0$, satisfies the following conditions: $\varphi(\cdot\,;x_0) \in C_{2q}$ and

$$\lim_{t\to x_0} \varphi(t;x_0) = 0 .$$

From this and in view of Lemma 6 we get

$$\lim_{n\to\infty} L_n\left(\psi^2(t;x_0);x_0\right) \equiv \lim_{n\to\infty} L_n\left(\varphi(t;x_0);x_0\right) = 0 . \tag{26}$$

Using (26) and (14) we obtain from (25)

$$\lim_{n\to\infty} nL_n\left(\psi(t;x_0)(t-x_0)^2;x_0\right) = 0 . \tag{27}$$

Now, applying (23), (24) and (27), we immediately derive from (22)

$$\lim_{n\to\infty} n\left\{L_n(f(t);x_0) - f(x_0)\right\} = \frac{x_0}{2} f''(x_0) .$$

Thus the proof of (20) is finished.

Analogously, using Lemmas 2, 3 and 6 and by (21), we can prove the following Voronovskaya theorem for the operators U_n, A_n and B_n.

Theorem 2. *Let $f \in C_q^2$ with some $q \in \mathbb{N}_0$. Then, for every fixed $x \geq 0$, we have*

$$\lim_{n\to\infty} n\left\{U_n(f(t);x) - f(x)\right\} = f'(x) + \frac{x}{2} f''(x) ,$$
$$\lim_{n\to\infty} n\left\{A_n(f(t);x) - f(x)\right\} = \frac{x}{2} f''(x) ,$$
$$\lim_{n\to\infty} n\left\{B_n(f(t);x) - f(x)\right\} = f'(x) + \frac{x}{2} f''(x) .$$

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