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# A BASIC ANALOGUE OF THE GENERALISED $H$-FUNCTION

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In the present paper the authors define a new basic hypergeometric function, which is an extension of the basic $H$-function defined earlier by Saxena et al [11]. As $q \to 1$, it reduces to the $I$-function defined by Saxena [12]. Three basic integral representations for this function are investigated, which provide elegant generalizations of the results given earlier by Saxena et al [11].

## 1. A basic analogue of the generalized $H$-function.

In a recent paper Saxena [12] has introduced the following generalization of the well-known Fox's $H$-function in terms of the $I$-function by means of Mellin-Barnes type integral in the form:

$$I^{m,n}_{A_i,B_i}\left[z \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,n},\ (a_{ji},\alpha_{ji})_{n+1,A_i}\\ (b_j,\beta_j)_{1,m},\ (b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right] \tag{1.1}$$

$$= \frac{1}{2\pi\omega}\int_C \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)z^s}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{B_i}\Gamma(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{A_i}\Gamma(a_{ji}-\alpha_{ji}s)\right\}}\,ds$$

where $0 \le m \le B_i$; $0 \le n \le A_i$; $i = 1, 2, \dots, r$, $r$ is finite and $\omega = \sqrt{-1}$.

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We define a basic analogue of this $I$-function in terms of the Mellin-Barnes type basic contour integral as

$$(1.2)\qquad I^{m,n}_{A_i,B_i}\left[z;q\,\middle|\,\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right]$$

$$=\frac{1}{2\pi\omega}\int_C\frac{\prod_{j=1}^{m}G(q^{b_j-\beta_j s})\prod_{j=1}^{n}G(q^{1-a_j+\alpha_j s})\pi z^s}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{B_i}G(q^{1-b_{ji}+\beta_{ji}s})\prod_{j=n+1}^{A_i}G(q^{a_{ji}-\alpha_{ji}s})\right\}G(q^{1-s})\sin\pi s}\,ds$$

where $0\le m\le B_i$; $0\le n\le A_i$; $i=1,2,\dots,r$; $r$ is finite; $\omega=\sqrt{-1}$; and

$$(1.3)\qquad G(q^\alpha)=\prod_{n=0}^{\infty}\left\{(1-q^{\alpha+n})\right\}^{-1}.$$

Also, $a_j,\beta_j,\alpha_{ji},\beta_{ji}$ are real and positive and $a_j,b_j,a_{ji},b_{ji}$ are complex numbers.

The contour of integration $C$ runs from $-i\infty$ to $+i\infty$ in such a manner that all the poles of $G(q^{b_j-\beta_j s})$; $1\le j\le m$, are to its right, and those of $G(q^{1-a_j+\alpha_j s})$, $1\le j\le n$, are to its left and are at least some $\varepsilon>0$ distance away from the contour $C$. The integral converges if $\mathrm{Re}[s\log(z)-\log\sin\pi s]<0$, for large values of $|s|$ on the contour, that is if $|\arg z|<\pi$. It may be observed that the contour of integration $C$ can be replaced by other suitably indented contours parallel to the imaginary axis.

It is interesting to observe that for $r=1$, $A_1=A$; $B_1=B$; (1.2) yields the $q$-analogue of the $H$-function due to Saxena et al [11], namely,

$$(1.4)\qquad H^{m,n}_{A,B}\left[z;q\,\middle|\,\begin{matrix}(a_1,\alpha_1),\dots,(a_A,\alpha_A)\\(b_1,\beta_1),\dots,(b_B,\beta_B)\end{matrix}\right]$$

$$=\frac{1}{2\pi\omega}\int_C\frac{\prod_{j=1}^{m}G(q^{b_j-\beta_j s})\prod_{j=1}^{n}G(q^{1-a_j+\alpha_j s})\pi z^s}{\prod_{j=m+1}^{B}G(q^{1-b_j+\beta_j s})\prod_{j=n+1}^{A}G(q^{a_j-\alpha_j s})G(q^{1-s})\sin\pi s}\,ds$$

where $0\le m\le B$; $0\le n\le A$; and $\omega=\sqrt{-1}$.

Further, for $r = 1$, $A_1 = A$, $B_1 = B$; $\alpha_j = \beta_i = 1$, $j = 1, \dots, A$; $i = 1, \dots, B$, (1.2) reduces to the basic analogue of Meijer's $G$-function given by Saxena et al [11], p. 139, namely,

$$(1.5) \qquad H^{m,n}_{A,B}\left[z;q \,\middle|\, \begin{matrix}(a_1,1),\dots,(a_A,1)\\(b_1,1),\dots,(b_B,1)\end{matrix}\right] = G^{m,n}_{A,B}\left[z;q \,\middle|\, \begin{matrix}a_1,\dots,a_A\\b_1,\dots,b_B\end{matrix}\right]$$

$$= \frac{1}{2\pi\omega}\int_C \frac{\prod_{j=1}^{m} G(q^{b_j-s}) \prod_{j=1}^{n} G(q^{1-a_j+\alpha_j s})\pi z^s}{\prod_{j=m+1}^{B} G(q^{1-b_j+s}) \prod_{j=n+1}^{A} G(q^{a_j-s})G(q^{1-s})\sin\pi s}\, ds$$

where $0 \le m \le B$; $0 \le n \le A$, and $\omega = \sqrt{-1}$.

A detailed account of the basic hypergeometric functions is available from the monographs of Bailey [3], Slater [13], Agarwal [1], Exton [4] and Gasper and Rahman [5]. In brief, the function $I^{m,n}_{A_i,B_i}(\cdot)$ will be called the $I_q$-function.

## 2. A limit formula.

In this section we will establish the following limit formula for the $I_q$-function as $q \to 1-$,

$$(2.1) \qquad \lim_{q\to 1-}\left[(1-q)^{\sum_{t=1}^{n} a_t - \sum_{t=1}^{m} b_t + m + n - 1 + \sum_{i=1}^{r}\left(\sum_{t=m+1}^{A_i} a_{ti} - \sum_{t=m+1}^{B_i} b_{ti} - A_i\right)}\right.$$

$$\cdot \{G(q)\}^{\sum_{i=1}^{r}(A_i+B_i)-2(m+n-1)} \; I^{m,n}_{A_i,B_i+1}\left[z(1-q)^{\sum_{t=1}^{m}\beta_t - \sum_{t=1}^{n}\alpha_t + \sum_{i=1}^{r}\left(\sum_{t=m+1}^{B_i}\beta_{ti} - \sum_{t=n+1}^{A_i}\alpha_{ti}\right)};q \,\middle|\right.$$

$$\left.\left.\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m}(b_{ji},\beta_{ji})_{m+1,B_i},(1,1)\end{matrix}\right]\right]$$

$$= I^{m,n}_{A_i,B_i}\left[z \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right]$$

where $i = 1, \dots, r$.

To prove (2.1), we consider the expression

(2.2)
$$I^{m,n}_{A_i,B_i+1}\left[z(1-q)^{\sum_{t=1}^{m}\beta_t-\sum_{t=1}^{n}\alpha_t+\sum_{i=1}^{r}\left(\sum_{t=m+1}^{B_i}\beta_{ti}-\sum_{t=n+1}^{A_i}\alpha_{ti}\right)};q\,\middle|\,\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m}(b_{ji},\beta_{ji})_{m+1,B_i},(1,1)\end{matrix}\right]$$

$$=\frac{1}{2\pi\omega}\int_C\frac{\prod_{j=1}^{m}G(q^{b_j-\beta_j s})\prod_{j=1}^{n}G(q^{1-a_j+\alpha_j s})\pi z^s}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{B_i}G(q^{1-b_{ji}+\beta_{ji}s})\prod_{j=n+1}^{A_i}G(q^{a_{ji}-\alpha_{ji}s})\right\}}\cdot$$

$$\cdot\frac{(1-q)^{s\left\{\sum_{t=1}^{m}\beta_t-\sum_{t=1}^{n}\alpha_t+\sum_{i=1}^{r}\left(\sum_{t=m+1}^{B_i}\beta_{ti}-\sum_{t=n+1}^{A_i}\alpha_{ti}\right)\right\}}}{G(q^s)G(q^{1-s})\sin\pi s}\,ds$$

on multiplying both sides of (2.2) by

$$\left[(1-q)^{\sum_{t=1}^{n}a_t-\sum_{t=1}^{m}b_t+m+n-1+\sum_{i=1}^{r}\left(\sum_{t=n+1}^{A_i}a_{ti}-\sum_{t=m+1}^{B_i}b_{ti}-A_i\right)}\{G(q)\}^{\sum_{i=1}^{r}(A_i+B_i)-2(m+n-1)}\right]$$

and then making use of the identity (cf. Askey [2])

(2.3)
$$\Gamma_q(x)=\frac{G(q^x)}{(1-q)^{x-1}G(q)};\quad |q|<1,$$

we find that the R.H.S. of (2.2) reduces to

$$\frac{1}{2\pi\omega}\int_C\frac{\prod_{j=1}^{m}\Gamma_q(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma_q(1-a_j+\alpha_j s)\pi z^s}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{B_i}\Gamma_q(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{A_i}\Gamma_q(a_{ji}-\alpha_{ji}s)\right\}}\cdot$$

$$\cdot\frac{1}{\Gamma_q(s)\Gamma_q(1-s)\sin\pi s}\,ds\,.$$

Now, taking the limit of both sides, as $q \to 1-$, and making use of the result (cf. Askey [2])

$$\lim_{q\to 1-} \left\{\Gamma_q(x)\right\} = \Gamma(x), \tag{2.4}$$

(2.1) follows immediately.

It may be remarked here that for $r = 1$, $A_1 = A$, $B_1 = B$, (2.1) reduces to the known limit formula for basic analogue of Fox's $H$-function due to the authors [9], namely

$$\lim_{q\to 1-} \left[ (1-q)^{\sum_{t=1}^{A} a_t - \sum_{t=1}^{B} b_t - A - 1 + m + n} \{G(q)\}^{A+B-2(m+n-1)} \cdot H_{A,B+1}^{m,n}\left[ z(1-q)^{\sum_{t=1}^{B}\beta_t - \sum_{t=1}^{A}\alpha_t}; q \,\middle|\, \begin{matrix} (a_1,\alpha_1),\dots,(a_A,\alpha_A) \\ (b_1,\beta_1),\dots,(b_B,\beta_B),(1,1) \end{matrix} \right]\right]$$

$$= H_{A,B}^{m,n}\left[ z \,\middle|\, \begin{matrix} (a_1,\alpha_1),\dots,(a_A,\alpha_A) \\ (b_1,\beta_1),\dots,(b_B,\beta_B) \end{matrix} \right]$$

where $0 \le m \le B; 0 \le n \le A$.

A detailed account of the ordinary $G$ and ordinary $H$-functions is available from the monographs due to Mathai and Saxena [7], [8].

## 3. Integral representation for the $I_q$-functions.

Three basic integral representations of the $I_q$-function are to be established here.

$$\frac{G(q)}{1-q} \int_0^1 x^{\sigma-1} E_q(qx) I_{A_i,B_i}^{m,n}\left[ zx^{-\delta}; q \,\middle|\, \begin{matrix} (a_j,\alpha_j)_{1,n}, (a_{ji},\alpha_{ji})_{n+1,A_i} \\ (b_j,\beta_j)_{1,m}, (b_{ji},\beta_{ji})_{m+1,B_i} \end{matrix} \right] d(q;x) = I_{A_i,B_i+1}^{m+1,n}\left[ z; q \,\middle|\, \begin{matrix} (a_j,\alpha_j)_{1,m}, (a_{ji},\alpha_{ji})_{n+1,A_i} \\ (\sigma,\delta), (b_j,\beta_j)_{1,m}, (b_{ji},\beta_{ji})_{m+1,B_i} \end{matrix} \right] \tag{3.1}$$

where $\operatorname{Re}\sigma > 0$, $\operatorname{Re}\delta > 0$, $|\arg z| < \pi$.

$$\frac{G(q)}{1-q} \int_0^1 x^{\sigma-1} (1-qx)_{\delta-\sigma-1} I_{A_i,B_i}^{m,n}\left[ zx^{-\gamma}; q \,\middle|\, \begin{matrix} (a_j,\alpha_j)_{1,n}, \\ (b_j,\beta_j)_{1,m}, \end{matrix} \right. \tag{3.2}$$

$$\left.\begin{matrix}(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right] d(q;x)$$

$$= G(\delta-\sigma) I^{m+1,n}_{A_i+1,B_i+1}\left[z;q \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i},(\delta,\gamma)\\(\sigma,\gamma),(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right]$$

where $\operatorname{Re}\sigma > 0$, $\operatorname{Re}\delta > 0$, $\operatorname{Re}\gamma > 0$, $\operatorname{Re}(\delta-\sigma) > 0$ and $|\arg z| < \pi$.

(3.3) $$\frac{1}{2\pi\omega}\int_C e_q(x)x^{-\sigma} I^{m,n}_{A_i,B_i}\left[zx^{\delta};q \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right]$$

$$= G(q) I^{m,n}_{A_i+1,B_i}\left[z;q \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i},(\sigma,\delta)\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,B_i}\end{matrix}\right]$$

where $\operatorname{Re}\sigma > 0$ and $|\arg z| < \pi$.

*Proof.* In view of (1.2) and on interchanging the order of integration, which is justified in view of the aforesaid conditions, (3.1) transforms into the integral

$$\frac{1}{2\pi\omega}\int_C \frac{\prod_{j=1}^{m} G(q^{b_j-\beta_j s})\prod_{j=1}^{n} G(q^{1-a_j+\alpha_j s})\pi z^s}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{B_i} G(q^{1-b_{ji}+\beta_{ji}s})\prod_{j=n+1}^{A_i} G(q^{a_{ji}-\alpha_{ji}s})\right\} G(q^{1-s})\sin\pi s}\,.$$

$$\cdot\left\{\frac{G(q)}{1-q}\int_0^1 x^{\sigma-\delta s-1}E_q(qx)d(q;x)\right\} ds\,.$$

The result follows immediately if we now use the integral due to Hahn [6], p. 372

(3.4) $$\frac{G(q)}{1-q}\int_0^1 x^{\beta-s-1}E_q(qx)d(q;x) = G(q^{\beta-s})\,.$$

The remaining result (3.2) and (3.3) can respectively be proved by making the use of following integrals due to Hahn [6], namely,

(3.5) $$\frac{1}{1-q}\int_0^1 x^{\alpha-1}(1-qx)_{\beta-1}\, d(q;x) = \prod_{n=0}^{\infty}\frac{(1-q^{\alpha+\beta+n})(1-q^{1+n})}{(1-q^{\alpha+n})(1-q^{\beta+n})}\,,$$

for $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$,
and

(3.6) $$\frac{1}{2\pi\omega}\int_C e_q(sx)(sx)^{-r-1}\, ds = \frac{x^r}{(1-q)_r}$$

where

$$(x+y)_\alpha = x^\alpha \prod_{n=0}^{\infty} \frac{(1+yq^n/x)}{(1+yq^{\alpha+n}/x)}. \tag{3.7}$$

**4. Special cases.**

(i) For $r = 1$, $A_1 = A$; $B_1 = B$; (3.1) through (3.3) respectively yield the known results due to Saxena et al [11], p. 140, eqns. (3.1) - (3.3).

(ii) On the other hand if we set $r = 1$, $A_1 = A$; $B_1 = B$; in (3.1) - (3.3) and then replace $x^{-\delta}$ by $x^{\delta}$ in (3.1), $x^{-\gamma}$ by $x^{\gamma}$ in (3.2) and $x^{\delta}$ by $x^{-\delta}$ in (3.3) respectively then we arrive at the results due to Saxena et al [11], p. 140, eqns. (3.4) - (3.6).

(iii) On taking $r = 1$, $B_1 = m = B$; $n = 0$; $A_1 = A$; $\alpha_j = \beta_i = 1$, $j = 1, \ldots, A$, $i = 1, \ldots, B$; $\delta = 1$ and replacing $z$ by $\dfrac{-1}{t(1-q)^{1+A-B}}$ in (3.1) and using the identity (cf. Saxena and Kumar [10]), namely

$$G_{B,A}^{A,0}\left[\frac{-1}{y}; q \,\middle|\, \begin{matrix} a_1 \ldots, a_A \\ b_1, \ldots, b_B \end{matrix}\right] = \prod \left[\begin{matrix} 1, q^{a_1}, \ldots, q^{a_A}; \\ q^{b_1}, \ldots, q^{b_B}; \end{matrix} \; q\right] \cdot \tag{4.1}$$

$$\cdot\, {}_B\Phi_A\left[\begin{matrix} b_1, \ldots, b_B; \\ a_1, \ldots, a_A; \end{matrix} \; q; y\right]$$

where $y = t(1-q)^{1+A-B}$. The following integral representations for the generalized basic hypergeometric series are obtained.

$$\frac{G(q)}{G(\sigma)(1-q)} \int_0^1 x^{\sigma-1} E_q(qx)\, {}_B\Phi_A\left[\begin{matrix} b_1, \ldots, b_B; \\ a_1, \ldots, a_A; \end{matrix} \; q; y/x\right] d(q;x) \tag{4.2}$$

$$= {}_{B+1}\Phi_A\left[\begin{matrix} \sigma, b_1, \ldots, b_B; \\ a_1, \ldots, a_A; \end{matrix} \; q; y\right]$$

where $y = t(1-q)^{1+A-B}$.

(iv) Next, if we set $r = 1, m = B_1 = B; n = 0, A_1 = A; \gamma = 1; \alpha_j = \beta_i = 1$, $j = 1, \ldots, A; i = 1, \ldots, B$, and replace $z$ by $\dfrac{-1}{t(1-q)^{1+A-B}}$ in (3.2) and make use of (4.1), it yields

$$\frac{G(q)}{1-q} \int_0^1 x^{\sigma-1}(1-qx)_{\delta-\sigma-1}\, {}_B\Phi_A\left[\begin{matrix} b_1, \ldots, b_b; \\ a_1, \ldots, a_A; \end{matrix} \; q; y/x\right] d(q;x) \tag{4.3}$$

$$= \frac{G(\delta-\sigma)G(\sigma)}{G(\sigma)}\ {}_{B+1}\Phi_{A+1}\left[\begin{matrix}\sigma, b_1, \ldots, b_B;\\ \delta, a_1, \ldots, a_A;\end{matrix} q; y\right].$$

(v) Further, for $r = 1$, $m = B_1 = B$; $n = 0$, $A_1 = A$; $\delta = 1$; $\alpha_j = \beta_i = 1$, $j = 1, \ldots, A$, $i = 1, \ldots, B$, $z = \dfrac{-1}{t(1-q)^{1+A-B}}$ and making use of (4.1), (3.3) reduces to

$$\frac{1}{2\pi\omega}\int_C e_q(x)x^{-\sigma}\ {}_B\Phi_A\left[\begin{matrix}b_1, \ldots, b_B;\\ a_1, \ldots, a_A;\end{matrix} q; xy\right]dx \tag{4.4}$$

$$= \frac{G(q)}{G(\sigma)}\ {}_B\Phi_{A+1}\left[\begin{matrix}b_1, \ldots, b_b;\\ \sigma, a_1, \ldots, a_A;\end{matrix} q; y\right]$$

where $y = t(1-q)^{1+A-B}$.

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