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HÖLDER REGULARITY FOR NON LINEAR NON HOMOGENEOUS ELLIPTIC SYSTEMS

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In this paper we study the $L^{2,\mu}$-regularity for the gradient of the solution $u \in H^1(\Omega, \mathbb{R}^N)$ to the following non linear elliptic system:

$$\begin{cases} u - g \in H_0^1(\Omega, \mathbb{R}^N) \\ \sum_{i=1}^{n} D_i\, a^i(x, Du) = \sum_{i=1}^{n} D_i\; f_i(x, u) - f(x, u, Du), \end{cases}$$

where $f(x, u, p)$ and $f_i(x, u)$ have linear growthes and $g \in H^{1,(\mu)}(\Omega, \mathbb{R}^N)$. In particular for $2 \leq n \leq 4$ we obtain the Hölder-regularity for u, extending a result of S. Campanato (see [1]) to the case of systems with non zero free terms.

1. Introduction.

Let Ω be a bounded open set of R^n, $n \geq 2$, for instance of class C^2, with points $x = (x_1, x_2, \dots, x_n)$. N is an integer > 1, $(\cdot \mid \cdot)_k$ and $\|\cdot\|_k$ are the scalar product and the norm in R^k.
We will drop the subscript k when there is no fear of confusion.
If $u : \Omega \to \mathbb{R}^N$, we set $Du = (D_1 u, \dots, D_n u)$, where, as usual, $D_i = \partial/\partial x_i$.

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Clearly, $Du \in \mathbb{R}^{nN}$ and we denote by $p = (p^1, \dots, p^n)$, $p^j \in \mathbb{R}^N$, a typical vector of $\mathbb{R}^{nN}$.
$H^1 = H^{1,2}$ and $H_0^1 = H_0^{1,2}$ are the usual Sobolev spaces. Let $a^i(x, p)$, $i = 1, 2, \dots, n$, be vectors of $\mathbb{R}^N$, defined in $\Omega \times \mathbb{R}^{nN}$, continuous in x and of class C^1 in p, which satisfy the following conditions:

$$\left\{ \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} \left| \frac{\partial a_h^i(x, p)}{\partial p_k^j} \right|^2 \right\}^{\frac{1}{2}} \leq M, \quad \forall (x, p) \in \Lambda = \Omega \times \mathbb{R}^{nN}, \tag{1.1}$$

$$\sum_{i,j=1}^{n} \sum_{h,k=1}^{N} \frac{\partial a_h^i(x, p)}{\partial p_k^j} \xi_h^i \xi_k^j \geq \nu \left\| \xi \right\|^2, \tag{1.2}$$

$$\forall (x, p) \in \Lambda \quad \text{and} \quad \forall \xi \in \mathbb{R}^{nN}$$

where M and ν are suitable positive constants with $\nu < M$, and, moreover,

$$a^i(x, 0) = 0, \quad \forall x \in \Omega. \tag{1.3}$$

From (1.1), (1.3) it follows that

$$\|a^i(x, p)\| \leq M \, \|p\|, \; \forall (x, p) \in \Lambda.$$

Let, moreover, $f(x, u, p)$ and $f_i(x, u)$, $i = 1, \dots, n$, be vectors of $\mathbb{R}^N$, defined, respectively, in $\Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$ and in $\Omega \times \mathbb{R}^N$, measurable in x, continuous in u and p, such that

$$\|f(x, u, p)\| \leq f^0(x) + c(x)\|u\| + b(x)\|p\| \quad \text{a.e. in} \quad \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}, \tag{1.4}$$

$$\|f_i(x, u)\| \leq f^i(x) + \alpha(x)\|u\| \quad \text{a.e. in } \Omega \times \mathbb{R}^N, i = 1, \dots, n, \tag{1.5}$$

with $\alpha(x)$, $c(x)$, $b(x) \in L^\infty(\Omega)$ and

$$f^0(x) \in L^{2r, \mu r}(\Omega), \tag{1.6}$$

where, for $n > 2$, $r = \frac{n}{n+2}$ and for $n = 2$, $2r$ is a number $\in (1, 2)$,

$$f^i(x) \in L^{2,\mu}(\Omega), i = 1, \dots, n, \text{ with } 0 < \mu < \lambda, \tag{1.7}$$

where $\lambda = \min\{2 + \varepsilon, n\}$ is, here and in that follows, the exponent of the fundamental estimates for non linear base-operators (2.12) (cfr.[1] p.292).
Moreover, for every $x, y \in \Omega$ and $p \in \mathbb{R}^{nN}$, we suppose that

$$\left\{ \sum_{i=1}^{n} \left\| a^i(x, p) - a^i(y, p) \right\|^2 \right\}^{\frac{1}{2}} \leq \omega(\|x - y\|) \cdot \|p\| \tag{1.8}$$

where $\omega(t)$, with $t > 0$, is a bounded, non-decreasing function which converges to zero when $t \to 0$.
Under the hypotheses (1.1) - (1.8) we will prove the following:

Theorem 1.1. *Let $u \in H^1(\Omega, \mathbb{R}^N)$ be a solution of the Dirichlet problem*

$$\text{(1.9)} \qquad \begin{cases} u - g \in H_0^1(\Omega, \mathbb{R}^N) \\ \sum_{i=1}^{n} D_i\, a^i(x, Du) = \sum_{i=1}^{n} D_i f_i(x, u) - f(x, u, Du) \end{cases}$$

where $g \in H^{1,(\mu)}(\Omega, \mathbb{R}^N)$, with $0 < \mu < \lambda$ (1). Then we have that $Du \in L^{2,\mu}(\Omega, \mathbb{R}^{nN})$ and

$$\text{(1.10)} \qquad \|Du\|_{L^{2,\mu}(\Omega,\mathbb{R}^{nN})} \le c\Big\{\|g\|_{H^{1,(\mu)}(\Omega,\mathbb{R}^N)} + \|f^0\|_{L^{2r,\mu r}(\Omega)} + + \|u\|_{H^1(\Omega,\mathbb{R}^N)} + \sum_{i=1}^{n} \|f^i\|_{L^{2,\mu}(\Omega)}\Big\}.$$

Moreover $u \in \mathcal{L}^{2,2+\mu}(\Omega, \mathbb{R}^N)$ and if $2 \le n \le 4$ and $n - 2 < \mu < \lambda$, we get

$$u \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N) \text{ with } \alpha = 1 - \frac{n-\mu}{2}.$$

Note that, assuming $w = u - g$, problem (1.9) can be written in the equivalent form

$$\text{(1.11)} \qquad \begin{cases} w \in H_0^1(\Omega, \mathbb{R}^N) \\ \sum_{i=1}^{n} D_i\, a^i(x, Dw + Dg) = \sum_{i=1}^{n} D_i f_i(x, w + g) - \\ \qquad - f(x, w + g, Dw + Dg). \end{cases}$$

We will divide the present paper in the following way: notations and some preliminary results are contained in Section 2; in Section 3 we will prove an interior regularity result, in Section 4 a regularity result near the boundary, and, finally, in Section 5 the global regularity.
The technique we use to reach these regularity results is due to S. Campanato, who in the work [1] was concerned with the same problem in the homogeneous case $f = 0$ $f_i = 0$, $i = 1, \dots, n$.

(1) For the notations see Section 2.

2. Preliminaries and notations.

We define

$$B(x^0,\sigma) = \{x : \|x - x^0\| < \sigma\}; \tag{2.1}$$

moreover, if $x_n^0 = 0$,

$$B^+(x^0,\sigma) = \{x \in B(x^0,\sigma) : x_n > 0\}, \tag{2.2}$$

$$\Gamma(x^0,\sigma) = \{x \in B(x^0,\sigma) : x_n = 0\}. \tag{2.3}$$

We will simply write $B^+(\sigma), \Gamma(\sigma)$ and Γ instead of $B^+(0,\sigma), \Gamma(0,\sigma)$ and $\Gamma(0,1)$, respectively.
Throughout the present paper, Ω will denote a bounded open set of $\mathbb{R}^n$ with diameter d_Ω.
If $u \in L^1(\mathscr{B}, \mathbb{R}^N)$, $\mathscr{B}$ is an open non-empty set of Ω, then

$$u_{\mathscr{B}} = \fint_{\mathscr{B}} u(x)\,dx = \frac{1}{\text{meas}\,\mathscr{B}} \int_{\mathscr{B}} u(x)\,dx. \tag{2.4}$$

If $u \in L^\infty(\Omega, \mathbb{R}^N)$, we define

$$\|u\|_{\infty,\Omega} = \text{ess}\ \sup_{\Omega} \|u(x)\|. \tag{2.5}$$

If $u \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$, $0 < \alpha \leq 1$, we set

$$[u]_{\alpha,\overline{\Omega}} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|u(x) - u(y)\|}{\|x - y\|^\alpha} \tag{2.6}$$

and we will say that $u \in C^{0,\alpha}(\Omega, \mathbb{R}^N)$ if $u \in C^{0,\alpha}(K, \mathbb{R}^N)$ for every compact subset $K \subset \Omega$.
If $u \in L^{p,\mu}(\Omega, \mathbb{R}^N)$ we define, as usual,

$$\|u\|_{L^{p,\mu}(\Omega,\mathbb{R}^N)} = \sup_{\substack{x^0\in\Omega \\ o<\sigma\leq d_\Omega}} \left[\sigma^{-\mu} \int_{\Omega(x^0,\sigma)} \|u(x)\|^p dx\right]^{\frac{1}{p}} \text{ with } \mu \in [0, n]; \tag{2.7}$$

in particular, if $u \in L^{2,\lambda}(\Omega, \mathbb{R}^N)$, $0 \leq \lambda \leq n$, or $u \in \mathscr{L}^{2,\lambda}(\Omega, \mathbb{R}^N)$ with $0 \leq \lambda \leq n + 2$, we define

$$\|u\|^2_{L^{2,\lambda}(\Omega,\mathbb{R}^N)} = \sup_{\substack{x^0\in\Omega \\ o<\sigma\leq d_\Omega}} \sigma^{-\lambda} \int_{\Omega(x^0,\sigma)} \|u(x)\|^2 dx \tag{2.8}$$

(2.9) $$[u]^2_{\mathscr{L}^{2,\lambda}(\Omega,\mathbb{R}^N)} = \sup_{\substack{x^0\in\Omega \\ 0<\sigma\leq d_\Omega}} \sigma^{-\lambda} \int_{\Omega(x^0,\sigma)} \|u(x) - u_{\Omega(x^0,\sigma)}\|^2 dx$$

where $\Omega(x^0, \sigma) = \Omega \cap B(x^0, \sigma)$.
We say that $u \in H^{1,(\lambda)}(\Omega, \mathbb{R}^N)$, $0 \leq \lambda \leq n$, if

$$u \in H^1(\Omega, \mathbb{R}^N) \text{ and } Du \in L^{2,\lambda}(\Omega, \mathbb{R}^{nN}).$$

When $u \in H^1(\Omega, \mathbb{R}^N)$, we define

$$|u|^2_{0,\Omega} = \int_\Omega \|u\|^2 dx$$

(2.10) $$|u|_{1,\Omega} = |Du|_{0,\Omega}.$$

We examine now the non-linear base system

(2.11) $$\sum_{i=1}^n D_i\, a^i(Du) = 0,$$

where the vectors $a^i(p)$ satisfy the conditions (1.1) - (1.3). Let $u \in H^1(\Omega, \mathbb{R}^N)$ be a solution of the system (2.11), in the sense that

(2.12) $$\int_\Omega \sum_{i=1}^n \left(a^i(Du) \mid D_i\varphi\right) dx = 0 \quad \forall \varphi \in H^1_0(\Omega, \mathbb{R}^N).$$

In [1] the following fundamental estimates have been proved:

Theorem 2.1. *If $u \in H^1(\Omega, \mathbb{R}^N)$ is a solution of system* (2.11), *for every ball $B(\sigma) = B(x^0, \sigma) \subset \Omega$ and $\forall t \in (0, 1)$, we have* (see [1] Theorem 3.1)*:*

(2.13) $$|Du|^2_{0,B(t\sigma)} \leq c\, t^\lambda |Du|^2_{0,B(\sigma)}$$

where $\lambda = \min(2 + \varepsilon, n)$, $\varepsilon = \varepsilon(\nu, M, n)$ is a suitable number $0 < \varepsilon < 1$ (which existence is assured by [4], Theorem 8.I, p.90) *and the constant c does not depend on t, σ, x^0.*
Moreover if $2 \leq n \leq 4$ we have (see [1] Theorem 3.2)

(2.14) $$|u|^2_{0,B(t\sigma)} \leq c\, t^n |u|^2_{0,B(\sigma)}.$$

Theorem 2.2. *Let $u \in H^1(B^+(1), \mathbb{R}^N)$ be a solution of the problem*

$$u = 0 \quad on \quad \Gamma$$

$$\sum_{i=1}^{n} D_i\, a^i(Du) = 0 \quad in \quad B^+(1).$$

Then, for every $\sigma \leq 1$ and $\forall t \in (0,1)$

$$|Du|^2_{0,B^+(t\sigma)} \leq c\, t^{\lambda} |Du|^2_{0,B^+(\sigma)} \tag{2.15}$$

where $\lambda = \min(2+\varepsilon, n)$ and the constant c depends neither on t nor on σ.

(See [1], Theorem 5.II).

Afterwards we shall use the following auxiliary result (see [1], Lemma 2.VII):

Theorem 2.3. *Let $a^i(x,p)$, $i = 1, 2, \dots, n$, be vectors of $\mathbb{R}^N$, defined in Λ, measurable in x, continuous in p and such that $\forall x \in \Omega$, $\forall p, \bar{p} \in \mathbb{R}^{nN}$ it results*

$$\begin{cases} a^i(x,0) = 0 \\ \left\{ \sum\limits_{i=1}^{n} \|a^i(x,p) - a^i(x,\bar{p})\|^2 \right\}^{\frac{1}{2}} \leq M\, \|p - \bar{p}\| \\ \sum\limits_{i=1}^{n} \Big(a^i(x,p) - a^i(x,\bar{p}) \,\Big|\, (p^i - \bar{p}^i) \Big) \geq \nu \|p - \bar{p}\|^2 \end{cases} \tag{2.16}$$ (2)

with M and ν suitable positive constants with $\nu < M$.
Let $f^0(x)$, $f^i(x)$, $i = 1, 2, \dots, n$, and $g(x)$ be vectors of $\mathbb{R}^N$, such that

$$f^0(x) \in L^{2r}(\Omega, \mathbb{R}^N)$$

with $1 < 2r < 2$ if $n = 2$ and $r = \frac{n}{n+2}$ if $n > 2$,

$$f^i(x) \in L^2(\Omega, \mathbb{R}^N), \quad i = 1, 2, \dots, n,$$

$$g(x) \in H^1(\Omega, \mathbb{R}^N).$$

Then there exists a unique vector u such that

$$\begin{cases} u \in H^1_0(\Omega, \mathbb{R}^N) \\ \sum\limits_{i=1}^{n} D_i\, a^i(x, Du + Dg) = \sum\limits_{i=1}^{n} D_i f^i(x) - f^0(x) \end{cases} \tag{2.17}$$

(2) The hypotheses (2.16) are more general than (1.1) - (1.3).

and it results

$$(2.18)\quad |Du|^2_{0,\Omega} \le c\ (\nu, M, \bar{c}) \left\{ \sum_{i=1}^{n} \left| f^i - a^i(x, Dg) \right|^2_{0,\Omega} + \|f^0\|^2_{L^{2r}(\Omega,\mathbb{R}^N)} \right\}$$

where $\bar{c}$ is the Sobolev constant (3).

Proof. To proof this theorem is equivalent to show that there exists an unique vector $u \in H^1_0(\Omega, \mathbb{R}^N)$ such that

$$(2.19)\quad \sum_{i=1}^{n} D_i D_i\ u = \sum_{i=1}^{n} D_i D_i\ u - \\ - \frac{\nu}{M^2}\left[\sum_{i=1}^{n} D_i \left(a^i(x, Du + Dg) - f^i(x)\right) + f^0(x) \right]$$

i.e. that the transformation $\tau : H^1_0(\Omega, \mathbb{R}^N) \to H^1_0(\Omega, \mathbb{R}^N)$, defined setting for $\forall u \in H^1_0(\Omega, \mathbb{R}^N)$

$$(2.20)\quad \tau(u) = U,$$

with U solution of the problem

$$(2.21)\quad \begin{cases} U \in H^1_0(\Omega, \mathbb{R}^N) \\ \sum_{i=1}^{n} D_i D_i\ U = \sum_{i=1}^{n} D_i D_i\ u - \\ \qquad - \frac{\nu}{M^2}\left[\sum_{i=1}^{n} D_i \left(a^i(x, Du + Dg) - f^i(x)\right) + f^0(x) \right] \end{cases}$$

admits a fixed point.
(We proof the theorem for $n > 2$, a few changes are necessary for $n = 2$).
For any $u \in H^1_0(\Omega, \mathbb{R}^N)$ the condition

$$D_i\ u - \frac{\nu}{M^2}\left[a^i(x, Du + Dg) - f^i(x)\right] \in L^2(\Omega, \mathbb{R}^N)$$

holds, and then there is a unique solution $U = \tau(u) \in H^1_0(\Omega, \mathbb{R}^N)$ of Dirichlet problem (2.21) and also the inequality

$$(2.22)\quad \|U\|^2_{H^1_0(\Omega,\mathbb{R}^N)} \le \int_\Omega \sum_{i=1}^{n} \left\| D_i\ u - \frac{\nu}{M^2}\left[a^i(x, Du + Dg) - f^i(x)\right] \right\|^2 dx +$$

(3) The Sobolev constant $\bar{c}$ does not depend on Ω.

$$+\frac{2\nu}{M^2}\int_\Omega (f^0 \mid U)\,dx$$

holds.
Now we observe that, setting

$$U=\tau(u) \quad \text{and} \quad V=\tau(v)$$

$U-V$ is the solution of the Dirichlet problem

$$\begin{cases} U-V\in H_0^1(\Omega,\mathbb{R}^N) \\ \sum_{i=1}^n D_iD_i(U-V)=\sum_{i=1}^n D_i\Big\{D_i(u-v)- \\ \qquad\qquad -\frac{\nu}{M^2}\Big[a^i(x,Du+Dg)-a^i(x,Dv+Dg)\Big]\Big\} \end{cases}$$

and we have

$$\|\tau(u)-\tau(v)\|^2_{H_0^1(\Omega,\mathbb{R}^N)}=\int_\Omega\sum_{i=1}^n \|D_i(U-V)\|^2dx\le$$

$$\le\sum_{i=1}^n\int_\Omega \|D_i(u-v)-\frac{\nu}{M^2}\Big[a^i(x,Du+Dg)-a^i(x,Dv+Dg)\Big]\|^2dx=$$

$$=\sum_{i=1}^n\int_\Omega\Big\{\|D_i(u-v)\|^2+\frac{\nu^2}{M^4}\|a^i(x,Du+Dg)-a^i(x,Dv+Dg)\|^2-$$

$$-2\frac{\nu}{M^2}\Big(a^i(x,Du+Dg)-a^i(x,Dv+Dg)\Big|D_i(u-v)\Big)\Big\}dx\le$$

$$\le\|D(u-v)\|^2_{L^2(\Omega,\mathbb{R}^{nN})}+\frac{\nu^2}{M^2}\|D(u-v)\|^2_{L^2(\Omega,\mathbb{R}^{nN})}-2\frac{\nu^2}{M^2}\|D(u-v)\|^2_{L^2(\Omega,\mathbb{R}^{nN})}=$$

$$=\left(1-\frac{\nu^2}{M^2}\right)\|D(u-v)\|^2_{L^2(\Omega,\mathbb{R}^{nN})}=\left(1-\frac{\nu^2}{M^2}\right)\|u-v\|^2_{H_0^1(\Omega,\mathbb{R}^n)},$$

from which it follows that the transformation $\tau: H_0^1(\Omega,\mathbb{R}^N)\to H_0^1(\Omega,\mathbb{R}^N)$ being a contraction, has a unique fixed point u which is the solution of problem (2.19). To achieve the estimate (2.18), taking in account (2.22), we observe that

$$\|u\|^2_{H_0^1(\Omega,\mathbb{R}^N)}\le\int_\Omega\sum_{i=1}^n\left\|D_i\,u-\frac{\nu}{M^2}\Big[a^i(x,Du+Dg)-f^i(x)\Big]\right\|^2dx+$$

$$+\frac{2\nu}{M^2}\int_\Omega (f^0|u)\,dx = \int_\Omega \sum_{i=1}^n \Big\| D_i\, u - \frac{\nu}{M^2}\Big[a^i(x, Du+Dg) - a^i(x, Dg)\Big] -$$

$$- \frac{\nu}{M^2}\Big[a^i(x, Dg) - f^i(x)\Big]\Big\|^2 dx + \frac{2\nu}{M^2}\int_\Omega (f^0\,|\,u)\,dx,$$

from which, using the calculations made before we reach

$$\|u\|_{H_0^1(\Omega,\mathbb{R}^N)} \le \left\{\sum_{i=1}^n \int_\Omega \Big\| D_i\, u - \frac{\nu}{M^2}\Big[a^i(x, Du+Dg) - a^i(x, Dg)\Big]\Big\|^2 dx\right\}^{\frac{1}{2}} +$$

$$+\left\{\frac{\nu}{M^2}\sum_{i=1}^n \int_\Omega \|a^i(x, Dg) - f^i(x)\|^2 dx\right\}^{\frac{1}{2}} + \Big[\frac{2\nu}{M^2}\int_\Omega (f^0\,|\,u)\,dx\Big]^{\frac{1}{2}} \le$$

$$\le \Big(1-\frac{\nu^2}{M^2}\Big)^{\frac{1}{2}}\|u\|_{H_0^1(\Omega,\mathbb{R}^N)} + \Big(\frac{\nu}{M^2}\sum_{i=1}^n \int_\Omega \|a^i(x, Dg) - f^i(x)\|^2 dx\Big)^{\frac{1}{2}} +$$

$$+\Big[\frac{2\nu}{M^2}\int_\Omega (f^0\,|\,u)\,dx\Big]^{\frac{1}{2}} \le$$

$$\le \Big(1-\frac{\nu^2}{M^2}\Big)^{\frac{1}{2}}\|u\|_{H_0^1(\Omega,\mathbb{R}^N)} + \Big(\frac{\nu}{M^2}\sum_{i+1}^N \int_\Omega \|a^i(x, Dg) - f^i(x)\|^2 dx\Big)^{\frac{1}{2}} +$$

$$+\left\{\frac{2\nu}{M^2}\Big[\Big(\int_\Omega \|f^0\|^{\frac{2n}{n+2}}\,dx\Big)^{\frac{n+2}{2n}}\cdot\Big(\int_\Omega \|u\|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{2n}}\Big]\right\}^{\frac{1}{2}} \le$$

$$\le \Big(1-\frac{\nu^2}{M^2}\Big)^{\frac{1}{2}}\|u\|_{H_0^1(\Omega,\mathbb{R}^N)} + \Big(\frac{\nu}{M^2}\sum_{i=1}^n \int_\Omega \|a^i(x, Dg) - f^i(x)\|^2 dx\Big)^{\frac{1}{2}} +$$

$$+\left\{\frac{2\nu}{M^2}\Big[\bar{c}\|f^0\|_{L^{\frac{2n}{n+2}}(\Omega,\mathbb{R}^N)}\|u\|_{H_0^1(\Omega,\mathbb{R}^N)}\Big]\right\}^{\frac{1}{2}} \text{(4)}$$

and, for every $\varepsilon > 0$,

$$\|u\|_{H_0^1(\Omega,\mathbb{R}^N)} \le \Big(1-\frac{\nu^2}{M^2}\Big)^{\frac{1}{2}}\|u\|_{H_0^1(\Omega,\mathbb{R}^N)} +$$

$$+\Big(\frac{\nu}{M^2}\sum_{i=1}^n \int_\Omega \|a^i(x, Dg) - f^i(x)\|^2 dx\Big)^{\frac{1}{2}} +$$

(4) $\bar{c}$ denotes the Sobolev constant.

$$+\frac{\nu\bar{c}}{M^2\sqrt{\varepsilon}}\|f^0\|_{L^{\frac{2n}{n+2}}(\Omega,\mathbb{R}^N)}+\sqrt{\varepsilon}\|u\|_{H_0^1(\Omega,\mathbb{R}^N)}.$$

From here, the estimate follows

$$\|u\|_{H_0^1(\Omega,\mathbb{R}^N)}\leq \tag{2.23}$$

$$\leq c(\bar{c},\nu,M)\Big\{\Big[\sum_{i=1}^{n}\int_\Omega\|a^i(x,Dg)-f^i(x)\|^2dx\Big]^{\frac{1}{2}}+\|f^0\|_{L^{\frac{2n}{n+2}}(\Omega,\mathbb{R}^N)}\Big\}$$

and, consequently, (2.18).

3. An interior regularity result.

Now, we can prove the following interior regularity theorem:

Theorem 3.1. *Let $u\in H^1(\Omega,\mathbb{R}^N)$ be a solution of system*

$$\sum_{i=1}^{n}D_i\,a^i(x,Du+Dg)=\sum_{i=1}^{n}D_if_i(x,u+g)-f(x,u+g,Du+Dg) \tag{3.1}$$

where we suppose that $g\in H^{1,(\mu)}(\Omega,\mathbb{R}^N)$, $0<\mu<\lambda$ (5). Then, under the assumptions (1.1) - (1.8), for every open set $\Omega^*\subset\subset\Omega$, we have that $Du\in L^{2,\mu}(\Omega^*,\mathbb{R}^{nN})$ and the inequality

$$\|Du\|_{L^{2,\mu}(\Omega^*,\mathbb{R}^{nN})}\leq c\mathscr{M} \tag{3.2}$$

holds, where, if $\mu\leq 2$, the constant c depends also on $d=dist(\overline{\Omega}^*,\partial\Omega)$ and

$$\mathscr{M}^2=\|g\|^2_{H^{1,(\mu)}(\Omega,\mathbb{R}^N)}+\|u\|^2_{L^{2,2}(\Omega,\mathbb{R}^N)}+\|f^0\|^2_{L^{2r,\mu r}(\Omega)}+\sum_{i=1}^{n}\|f^i\|^2_{L^{2,\mu}(\Omega)}$$

whereas, if $\mu>2$, denoted by Ω^{**} an open set such that $\Omega^*\subset\subset\Omega^{**}\subset\subset\Omega$ and set $d^*=dist(\overline{\Omega}^*,\partial\Omega^{**})$, the constant c depends on d^* and

$$\mathscr{M}^2=\|g\|^2_{H^{1,(\mu)}(\Omega,\mathbb{R}^N)}+\|u\|^2_{L^{2,4}(\Omega^{**},\mathbb{R}^N)}+\|f^0\|^2_{L^{2r,\mu r}(\Omega)}+\sum_{i=1}^{n}\|f^i\|^2_{L^{2,\mu}(\Omega)}.$$

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(5) λ is defined in (1.7).

Proof. Fix $B(\sigma) = B(x^0, \sigma)$ with $x^0 \in \Omega^*$ and $\sigma \le d$. In $B(\sigma)$ we decompose u :

$$u = v - w$$

where w is the solution of the Dirichlet problem (recall Theorem 2.3)

$$\begin{cases} w \in H_0^1(B(\sigma), \mathbb{R}^N) \\ \sum_{i=1}^n D_i\, a^i(x^0, Dw + Du + Dg) = \sum_{i=1}^n D_i \Big[a^i(x, Du + Dg) - f_i(x, u + g) \Big] + \\ \qquad\qquad + f(x, u + g, Du + Dg) \end{cases}$$

while $v \in H^1(B(\sigma), \mathbb{R}^N)$ is a solution of system

$$\sum_{i=1}^n D_i a^i(x^0, Dv + Dg) = 0.$$

We prove this theorem in the case $n > 2$; the proof needs only little changes in the case $n = 2$ (6).

From Theorem 2.3 with $a^i(x, p)$ constant in x, we get:

$$(3.3) \qquad |Dw|^2_{0,B(\sigma)} \le c(\nu, M, \bar{c}) \Big(\sum_{i=1}^n \big| a^i(x, Du + Dg) -$$

$$- a^i(x^0, Du + Dg) \big|^2_{0,B(\sigma)} + \sum_{i=1}^n |f_i(x, u + g)|^2_{0,B(\sigma)} +$$

$$+ \|f(x, u + g, Du + Dg)\|^2_{L^{\frac{2n}{n+2}}(B(\sigma), \mathbb{R}^n)} \Big) \le$$

$$\le c(\nu, M, \bar{c}) \Big\{ \omega^2(\sigma) \Big(|Du|^2_{0,B(\sigma)} + |Dg|^2_{0,B(\sigma)} \Big) + \|f^0(x)\|^2_{L^{\frac{2n}{n+2}}(B(\sigma))} +$$

$$+ \|u\|^2_{L^{\frac{2n}{n+2}}(B(\sigma), \mathbb{R}^N)} + \|Du\|^2_{L^{\frac{2n}{n+2}}(B(\sigma), \mathbb{R}^{nN})} + \|g\|^2_{L^{\frac{2n}{n+2}}(B(\sigma), \mathbb{R}^N)} +$$

(6) Note that $L^2(B(\sigma), \mathbb{R}^k) \subset L^{\frac{2n}{n+2}}(B(\sigma), \mathbb{R}^k)$ and, for every $\varphi \in L^2(B(\sigma), \mathbb{R}^k)$ we have

$$\|\varphi\|^2_{L^{\frac{2n}{n+2}}(B(\sigma), \mathbb{R}^k)} \le c\, \sigma^2 |\varphi|^2_{0,B(\sigma)}.$$

Likewise, if $n > 2$ and if $u \in H^1(\Omega, \mathbb{R}^N)$, then $u \in L^{\frac{2n}{n-2}}(\Omega, \mathbb{R}^N) \subset L^{2,2}(\Omega, \mathbb{R}^N)$.

$$+\|Dg\|^2_{L^{\frac{2n}{n+2}}(B(\sigma),\mathbb{R}^{nN})}+\sum_{i=1}^{n}|f^i(x)|^2_{0,B(\sigma)}+|u|^2_{0,B(\sigma),\mathbb{R}^N}+|g|^2_{0,B(\sigma)}\Big\}.$$

On the other hand, $v+g$ satisfies the hypotheses of Theorem 2.1, then, $\forall t\in(0,1)$, we have:

$$|Dv+Dg|^2_{0,B(t\sigma)}\le c\,t^\lambda|Dv+Dg|^2_{0,B(\sigma)}$$

and so, $\forall t\in(0,1)$,

$$|Dv|^2_{0,B(t\sigma)}\le c\,t^\lambda|Dv|^2_{0,B(\sigma)}+c|Dg|^2_{0,B(\sigma)}. \tag{3.4}$$

From (3.3) and (3.4), $\forall t\in(0,1)$ it easily follows that

$$\begin{aligned}
&|Du|^2_{0,B(t\sigma)}\le 2|Dv|^2_{0,B(t\sigma)}+2|Dw|^2_{0,B(t\sigma)}\le\\
&\le c\,t^\lambda|Dv|^2_{0,B(\sigma)}+c|Dg|^2_{0,B(\sigma)}+2|Dw|^2_{0,B(t\sigma)}\le\\
&\le c\,t^\lambda|Du|^2_{0,B(\sigma)}+c\,t^\lambda|Dw|^2_{0,B(\sigma)}+c|Dg|^2_{0,B(\sigma)}+2|Dw|^2_{0,B(t\sigma)}\le\\
&\le c\,t^\lambda|Du|^2_{0,B(\sigma)}+c|Dg|^2_{0,B(\sigma)}+c|Dw|^2_{0,B(\sigma)}\le\\
&\le c\Big\{t^\lambda|Du|^2_{0,B(\sigma)}+|Dg|^2_{0,B(\sigma)}+\omega^2(\sigma)\Big(|Du|^2_{0,B(\sigma)}+|Dg|^2_{0,B(\sigma)}\Big)+\\
&+\|f^0\|^2_{L^{\frac{2n}{n+2}}(B(\sigma))}+\|u\|^2_{L^{\frac{2n}{n+2}}(B(\sigma),\mathbb{R}^N)}+\|Du\|^2_{L^{\frac{2n}{n+2}}(B(\sigma),\mathbb{R}^{nN})}+\\
&+\|g\|^2_{L^{\frac{2n}{n+2}}(B(\sigma),\mathbb{R}^N)}+\|Dg\|^2_{L^{\frac{2n}{n+2}}(B(\sigma),\mathbb{R}^{nN})}+\sum_{i=1}^{n}|f^i(x)|^2_{0,B(\sigma)}+\\
&+|u|^2_{0,B(\sigma)}+|g|^2_{0,B(\sigma)}\Big\}\le c\Big\{(t^\lambda+\omega^2(\sigma)+\sigma^2)|Du|^2_{0,B(\sigma)}+\\
&+(1+\omega^2(\sigma)+\sigma^2)|Dg|^2_{0,B(\sigma)}+(1+\sigma^2)|u|^2_{0,B(\sigma)}+\\
&+(1+\sigma^2)|g|^2_{0,B(\sigma)}+\sum_{i=1}^{n}|f^i|^2_{0,B(\sigma)}+\|f^0\|^2_{L^{\frac{2n}{n+2}}(B(\sigma))}\Big\}\le\\
&\le c\Big\{(t^\lambda+\omega^2(\sigma)+\sigma^2)|Du|^2_{0,B(\sigma)}+|Dg|^2_{0,B(\sigma)}+|u|^2_{0,B(\sigma)}+\\
&+|g|^2_{0,B(\sigma)}+\sum_{i=1}^{n}|f^i|^2_{0,B(\sigma)}+\|f^0\|^2_{L^{\frac{2n}{n+2}}(B(\sigma))}\Big\}\le\\
&\le c(t^\lambda+\omega^2(\sigma)+\sigma^2)Du|^2_{0,B(\sigma)}+c\sigma^\mu\Big\{\|g\|^2_{H^{1,(\mu)}(\Omega,\mathbb{R}^N)}+\\
&+\sum_{i=1}^{n}\|f^i\|^2_{L^{2,\mu}(\Omega)}+\|f^0\|^2_{L^{\frac{2n}{n+2},\mu\frac{n}{n+2}}(\Omega)}\Big\}+|u|^2_{0,B(\sigma)};
\end{aligned}$$

and, from $u\in L^{2,2}(\Omega,\mathbb{R}^N)$ and if $\mu\le 2$, it follows that

$$|Du|^2_{0,B(t\sigma)}\le c\left(t^\lambda+\omega^2(\sigma)+\sigma^2\right)|Du|^2_{0,B(\sigma)}+ \tag{3.5}$$

$$+ c\sigma^{\mu}\{\|g\|^2_{H^{1,(\mu)}(\Omega,\mathbb{R}^N)} + \sum_{i=1}^{n}\|f^i\|^2_{L^{2,\mu}(\Omega)} + \|f^0\|^2_{L^{\frac{2n}{n+2},\mu\frac{n}{n+2}}(\Omega)}\} +$$

$$+ c\sigma^2\|u\|^2_{L^{2,2}(\Omega,\mathbb{R}^N)} \leq c(t^{\lambda} + \omega^2(\sigma) + \sigma^2)|Du|^2_{0,B(\sigma)} + c\sigma^{\mu\wedge 2}\mathscr{M}^2.$$

Hence, by Lemma 2.VII of [2], it follows that $\forall\tau \in (0, \lambda - (\mu \wedge 2))$ there exists a positive $\sigma_\tau \leq d$ such that, if $0 < \sigma \leq \sigma_\tau$ and $t \in (0, 1)$

$$|Du|^2_{0,B(t\sigma)} \leq (1 + c)t^{\lambda-\tau}|Du|^2_{0,B(\sigma)} + K(c, \tau, \lambda, \mu)(t\sigma)^{\mu\wedge 2}\mathscr{M}^2.$$

This implies that, for every $\sigma \leq \sigma_\tau$,

$$|Du|^2_{0,B(\sigma)\cap\Omega^*} \leq c\sigma^{\mu\wedge 2}\left\{\sigma_\tau^{-(\mu\wedge 2)}|Du|^2_{0,\Omega} + \mathscr{M}^2\right\} \tag{3.6}$$

and then $Du \in L^{2,\mu\wedge 2}(\Omega^*, \mathbb{R}^{nN})$.
Now if $\mu \leq 2$ the theorem is proved.

On the contrary, if $\mu > 2$, from the property

$$Du \in L^{2,\mu\wedge 2}(\Omega^*, \mathbb{R}^{nN}) = L^{2,2}(\Omega^*, \mathbb{R}^{nN})$$

it follows

$$u \in L^{2,4}(\Omega^*, \mathbb{R}^N).$$

Then, using the result proved above, it follows (7) because $\mu < 4$

$$Du \in L^{2,\mu}(\Omega^*, \mathbb{R}^{nN}).$$

Indeed, if $2 < \mu (< 4)$, we fix $\Omega^* \subset\subset \Omega^{**} \subset\subset \Omega$ and we reason in the same previous way: $\forall t \in (0, 1)$ and $\forall\sigma \in (0, d^*)$, we have that:

$$|Du|^2_{0,B(t\sigma)} \leq c(t^{\lambda} + \omega^2(\sigma) + \sigma^2)|Du|^2_{0,B(\sigma)} + c\sigma^{\mu}\{\|g\|^2_{H^{1,(\mu)}(\Omega;\mathbb{R}^N)} + \sum_{i=1}^{n}\|f^i\|^2_{L^{2,\mu}(\Omega)} + \|f_0\|^2_{L^{\frac{2n}{n+2},\mu\frac{n}{n+2}}(\Omega)}\} + |u|^2_{0,B(\sigma)}. \tag{3.7}$$

On the other hand, we have $Du \in L^{2,2}(\Omega^{**}, \mathbb{R}^{nN})$ and then $u \in L^{2,4}(\Omega^{**}, \mathbb{R}^N)$ and

$$|u|^2_{0,B(\sigma)} \leq \sigma^4|u|^2_{L^{2,4}(\Omega^{**},\mathbb{R}^N)}. \tag{3.8}$$

(7) We remark that the proof on the estimate (3.6) employs only local properties.

From (3.7) and (3.8) follows

$$|Du|^2_{0,B(t\sigma)} \leq c(t^\lambda + \omega^2(\sigma) + \sigma^2)|Du|^2_{0,B(\sigma)} + c\sigma^\mu \mathscr{M}^2 \tag{3.9}$$

and then, reasoning in the same previous way, we conclude that

$$Du \in L^{2,\mu}(\Omega^*; \mathbb{R}^{nN})$$

and we have the inequality (3.2) with

$$\mathscr{M}^2 = \|g\|^2_{H^{1,(\mu)}(\Omega,\mathbb{R}^N)} + \sum_{i=1}^n \|f^i\|^2_{L^{2,\mu}(\Omega)} + \|f^0\|^2_{L^{\frac{2n}{n+2},\mu\frac{n}{n+2}}(\Omega)} + \|u\|^2_{L^{2,4}(\Omega^{**},\mathbb{R}^N)}.$$

Therefore Theorem 3.1 is proved.

4. Regularity near the boundary.

Let us consider the operator

$$\sum_{i=1}^n D_i\, a^i(x, Du),$$

where $a^i(x, p)$ are vectors of $\mathbb{R}^N$, defined in $\Lambda^+ = B^+(1) \times \mathbb{R}^{nN}$, continuous in x, of class C^1 in p, which satisfy conditions (1.1) - (1.3), (1.8), for all $(x, p) \in \Lambda^+$ and for all $\xi \in \mathbb{R}^{nN}$.

Let $f(x, u, p)$ and $f_i(x, u)$, $i = 1, 2, \ldots, n$, be vectors of $\mathbb{R}^N$ defined in $B^+(1)\times\mathbb{R}^N\times\mathbb{R}^{nN}$ and in $B^+(1)\times\mathbb{R}^N$, respectively, measurable in x, continuous in u and p, which satisfy conditions (1.4) - (1.7), where Ω is replaced by $B^+(1)$.

Then, we want to prove the following

Theorem 4.1. *Let $u \in H^1(B^+(1), \mathbb{R}^n)$ be a solution of the problem*

$$\begin{cases} u = 0 \quad on \quad \Gamma \\ \sum_{i=1}^n D_i\, a^i(x, Du + Dg) = \sum_{i=1}^n D_i f_i(x, u + g) - \\ \qquad\qquad - f(x, u + g, Du + Dg) \ in\ B^+(1). \end{cases} \tag{4.1}$$

Let us suppose that $g \in H^{1,(\mu)}(B^+(1), \mathbb{R}^N)$ with $0 < \mu < \lambda$. Then, for every $R < 1$, $Du \in L^{2,\mu}(B^+(R), \mathbb{R}^{nN})$ and the inequality

$$\|Du\|_{L^{2,\mu}(B^+(R),\mathbb{R}^{nN})} \leq c\, \mathscr{M}' \tag{4.2}$$

holds, where

$$\mathscr{M}'^2 = \|g\|^2_{H^{1,(\mu)}(B^+(1),\mathbb{R}^N)} + \|u\|^2_{L^{2,2}(B^+(1),\mathbb{R}^N)} + \|f^0\|^2_{L^{2r,\mu r}(B^+(1))} +$$

$$+\sum_{i=1}^{n} \|f^i\|^2_{L^{2,\mu}(B^+(1))} \quad \text{if } \mu \leq 2$$

whereas, denoted by R^* *a number such that* $0 < R < R^* < 1$,

$$\mathscr{M}'^2 = \|g\|^2_{H^{1,(\mu)}(B^+(1),\mathbb{R}^N)} + \|u\|^2_{L^{2,4}(B^+(R^*),\mathbb{R}^N)} + \|f^0\|^2_{L^{2r,\mu r}(B^+(1))} +$$

$$+\sum_{i=1}^{n} \|f^i\|^2_{L^{2,\mu}(B^+(1))} \quad \text{is } \mu > 2,$$

and where the constant c *depends on* R *if* $\mu \leq 2$ *or on* R^* *if* $\mu > 2$.

Proof. Fix R, $0 < R < 1$. In any hemisphere $B^+(x^0, \sigma)$, with $\sigma < 1 - R$ and centred in $x^0 \in \Gamma(R)$ we write $u = V - W$, where $V = v + g \in H^1(B^+(x^0, \sigma), \mathbb{R}^N)$ is a solution of the problem

$$\text{(4.3)} \qquad \begin{cases} V = v + g = 0 & \text{on } \quad \Gamma(x^0, \sigma) \\ \sum_{i=1}^{n} D_i\, a^i(x^0, DV) = 0 & \text{in } B^+(x^0, \sigma) \end{cases}$$

whereas $W = w + g$ is the solution of the Dirichlet problem

$$\text{(4.4)} \qquad \begin{cases} W = w + g \in H_0^1(B^+(x^0, \sigma), \mathbb{R}^N) \\ \sum_{i=1}^{n} D_i\, a^i(x^0, DW + Du) = \sum_{i=1}^{n} D_i[a^i(x, Du + Dg) - f_i(x, u + g)] \\ \qquad\qquad + f(x, u + g, Du + Dg) \end{cases}$$

Taking into account (2.18) where u is replaced by W, g is replaced by u, $f^i(x)$ is replaced by $a^i(x, Du + Dg) - f_i(x, u + g)$, $a^i(x, Dg)$ is replaced by $a^i(x^0, Du)$, $f^0(x)$ is replaced by $f(x, u + g, Du + Dg)$, Ω is replaced by

$B^+(x^0, \sigma)$, in virtue of the assumption (1.8), we have

$$|D(w+g)|^2_{0,B^+(x^0,\sigma)} \le c(\nu, M, \bar{c})\left\{ \sum_{i=1}^{n} \left| a^i(x, Du+Dg) - a^i(x^0, Du)\right|^2 + \right.$$

$$\left. + \sum_{i=1}^{n} \left| f_i(x, u+g)\right|^2_{0,B^+(x^0,\sigma)} + \|f(x, u+g, Du+Dg)\|^2_{L^{\frac{2n}{n+2}}(B^+(x^0,\sigma),\mathbb{R}^N)} \right\} \le$$

$$\le c(\nu, M, \bar{c})\left\{ \sum_{i=1}^{n} \left| a^i(x, Du+Dg) - a^i(x^0, Du+Dg) + \right.\right.$$

$$+ a^i(x^0, Du+Dg) - a^i(x^0, Du)\Big|^2_{0,B^+(x^0,\sigma)} + \sum_{i=1}^{n} \left| f_i(x, u+g)\right|^2_{0,B^+(x^0,\sigma)} +$$

$$\left. + \|f(x, u+g, Du+Dg)\|^2_{L^{\frac{2n}{n+2}}(B^+(x^0,\sigma),\mathbb{R}^N)} \right\} \le$$

$$\le c(\nu, M, \bar{c})\left\{ \omega^2(\sigma)\left(|Du|^2_{0,B^+(x^0,\sigma)} + |Dg|^2_{0,B^+(x^0,\sigma)}\right) + \right.$$

$$+ M^2|Dg|^2_{0,B^+(x^0,\sigma)} +$$

$$\left. + \sum_{i=1}^{n} |f_i(x, u+g)|^2_{0,B^+(x^0,\sigma)} + \|f(x, u+g, Du+Dg)\|^2_{L^{\frac{2n}{n+2}}(B^+(x^0,\sigma),\mathbb{R}^N)} \right\}$$

and then

$$(4.5) \qquad |Dw|^2_{0,B^+(x^0,\sigma)} \le 2|D(w+g)|^2_{0,B^+(x^0,\sigma)} + 2|Dg|^2_{0,B^+(x^0,\sigma)} \le$$

$$\le c(\nu, M, \bar{c})\left\{ \omega^2(\sigma)|Du|^2_{0,B^+(x^0,\sigma)} + |Dg|^2_{0,B^+(x^0\sigma)} + \right.$$

$$\left. + \sum_{i=1}^{n} |f_i(x, u+g)|^2_{0,B^+(x^0,\sigma)} + \|f(x, u+g, Du+Dg\|^2_{L^{\frac{2n}{n+2}}(B^+(x^0,\sigma),\mathbb{R}^N)} \right\}.$$

On the other hand, the fundamental estimate (2.15) imply that, $\forall t \in (0, 1)$,

$$|Dv|^2_{0,B^+(x^0,t\sigma)} \le 2|D(v+g)|^2_{0,B^+(x^0,t\sigma)} + 2|Dg|^2_{0,B^+(x^0,\sigma)} \le$$
$$\le c\, t^\lambda |D(v+g)|^2_{0,B^+(x^0,\sigma)} + 2|Dg|^2_{0,B^+(x^0,\sigma)} \le c\, t^\lambda |Dv|^2 + c|Dg|^2$$

with $\lambda = \min\{2+\varepsilon, n\}$
Now, taking into account that $u = V - W$, we go on as in Theorem 3.1 to obtain (3.5) and (3.9) and we have, $\forall t \in (0,1)$:

$$(4.6)\quad |Du|^2_{0,B^+(x^0,t\sigma)} \le c\left(t^\lambda + \omega^2(\sigma) + \sigma^2\right)|Du|^2_{0,B^+(x^0,\sigma)} + c\sigma^{\mu\wedge 2}\mathcal{M}'^2 \ (^8).$$

Hence by lemma 2.VII of [2], it follows that $\forall \tau \in (0, \lambda - (\mu \wedge 2))$ there exists a positive $\sigma_\tau \le 1 - R$ such that, if $\sigma \le \sigma_\tau$, $\forall t \in (0,1)$:

$$|Du|^2_{0,B^+(x^0,t\sigma)} \le (1+c)t^{\lambda-\tau}|Du|^2_{0,B^+(x^0,\sigma)} + K(t\sigma)^{\mu\wedge 2}\mathcal{M}'^2$$

and so, $\forall \sigma \le \sigma_\tau$

$$(4.7)\qquad |Du|^2_{0,B^+(x^0,\sigma)} \le c\,\sigma^{\mu\wedge 2}\left\{\sigma_\tau^{-(\mu\wedge 2)}|Du|^2_{0,B^+(1)} + \mathcal{M}'^2\right\}.$$

After this inequality, we go on as in Theorem 6.I of [1].

5. A global regularity result.

Let $u \in H^1(\Omega, \mathbb{R}^N)$ be the solution of the Dirichlet problem

$$(5.1)\qquad \begin{cases} u - g \in H^1_0(\Omega, \mathbb{R}^N) \\ \sum\limits_{i=1}^n D_i\, a^i(x, Du) = \sum\limits_{i=1}^n D_i f_i(x,u) + f(x,u,Du) \text{ in } \Omega \end{cases}$$

where $g \in H^{1,(\mu)}(\Omega, \mathbb{R}^n)$, with $0 < \mu < \lambda$, the open set Ω is of class C^2 and the vector mappings $a^i(x,p)$, f_i, $i = 1, \dots, n$, f satisfy assumptions (1.1)-(1.8).

Note that, assuming $w = u - g$, Problem (5.1) can be written in the equivalent form

$$(5.2)\qquad \begin{cases} w \in H^1_0(\Omega, \mathbb{R}^N) \\ \sum\limits_{i=1}^n D_i\, a^i(x, Dw + Dg) = \sum\limits_{i=1}^n D_i f_i(x, w+g) + \\ \qquad\qquad + f(x, w+g, Dw+Dg) \text{ in } \Omega \end{cases}$$

Let us premise some notations and remarks. As Ω is of class C^2, if $x^0 \in \partial\Omega$, there exists an open neighbourhood Ω_r of x^0, such that $\overline{\Omega}_r$ is mapped, by a mapping

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(8) We denote with c constants with different values and omit to indicate the arguments on wich it depend.

$y = \mathscr{F}_r(x)$ of class C^2 together with its inverse, onto the ball $\overline{B(0,1)}$ and, in particular, $\Omega \cap \Omega_r$ is sent in $B^+(1)$ and $\partial\Omega \cap \Omega_r$ in Γ (cfr [1] e [5] for a similar reasoning).
A solution w to problem (5.2) satisfies, in particular, the problem

$$(5.3) \qquad -\sum_{i=1}^{n} \int_{\Omega\cap\Omega_r} (a^i(x, Dw + Dg) \mid D_i\varphi)dx =$$

$$-\sum_{i=1}^{n} \int_{\Omega\cap\Omega_r} (f_i(x, w + g) \mid D_i\varphi)dx +$$

$$+ \int_{\Omega\cap\Omega_r} (f(x, w + g, Dw + Dg) \mid \varphi)dx, \ \forall \varphi \in H_0^1(\Omega \cap \Omega_r, \mathbb{R}^N).$$

Let us set

$$\frac{\partial \mathscr{F}_r(x)}{\partial x} = \left\{\frac{\partial \mathscr{F}_{r,i}(x)}{\partial x_j}\right\}, \quad J(x) = \left|\det \frac{\partial \mathscr{F}_r(x)}{\partial x}\right|$$

and, for all $y \in B(0,1)$, $u \in \mathbb{R}^N$ and $p \in \mathbb{R}^{nN}$, let us define

$$\alpha_{ij}(y) = \left(\frac{\partial \mathscr{F}_{r,i}}{\partial x_j}\right)\left(\mathscr{F}_r^{-1}(y)\right)$$

$$\beta_{hi}(y) = \left(\frac{\partial \mathscr{F}_{r,h}}{\partial x_i}\frac{1}{J}\right)\left(\mathscr{F}_r^{-1}(y)\right)$$

$$q^j(y, p) = \sum_{i=1}^{n} \alpha_{ij}(y)p^i$$

$$q = (q^1, \dots, q^n)$$

$$A^h(y, p) = \sum_{i=1}^{n} a^i\left(\mathscr{F}_r^{-1}(y), q(y, p)\right)\beta_{hi}(y)$$

$$F_h(y, u) = \sum_{i=1}^{n} f_i\left(\mathscr{F}_r^{-1}(y), u\right)\beta_{hi}(y)$$

$$F(y, u, p) = f\left(\mathscr{F}_r^{-1}(y), u, q(y, p)\right)\frac{1}{J\left(\mathscr{F}_r^{-1}(y)\right)}.$$

Clearly, q^i and A^h are vectors of $\mathbb{R}^N$ defined in $B(0,1) \times \mathbb{R}^{nN}$; F_h are vectors of $\mathbb{R}^N$ defined in $B(0,1) \times \mathbb{R}^N$ and F is a vector of $\mathbb{R}^N$ defined in $B(0,1) \times$

$\mathbb{R}^N \times \mathbb{R}^{nN}$; moreover α_{ij} and β_{hi} are functions of class $C^1\overline{(B(0,1))}$.
Then, by assumptions (1.1) - (1.8), it is not difficult to prove that the vectors $A^h(y,p)$, $F_h(y,u)$, $F(y,u,p)$ verify the same conditions of $a^i(x,p)$, $f_i(x,u)$, $f(x,u,p)$ in which constants and coefficients are multiplied for a suitable positive constant $c(\mathscr{F}_r)$ and f^i and f^0 are replaced by F^i and F^0.
Then setting

$$(5.5)\qquad \begin{cases} \tilde{w}(y) = w\left(\mathscr{F}_r^{-1}(y)\right) & \text{and so} \quad w(x) = \tilde{w}\left(\mathscr{F}_r(x)\right) \\ \tilde{g}(y) = g\left(\mathscr{F}_r^{-1}(y)\right) & \text{and so} \quad g(x) = \tilde{g}\left(\mathscr{F}_r(x)\right) \\ \tilde{\varphi}(y) = \varphi\left(\mathscr{F}_r^{-1}(y)\right) & \text{and so} \quad \varphi(x) = \tilde{\varphi}\left(\mathscr{F}_r(x)\right), \end{cases}$$

being

$$(5.6)\qquad \begin{cases} D_i w(x) = \sum\limits_{h=1}^{n} D_h\tilde{w}\left(\mathscr{F}_r(x)\right) \cdot D_i\mathscr{F}_{r,h}(x) \\ D_i\varphi(x) = \sum\limits_{h=1}^{n} D_h\tilde{\varphi}\left(\mathscr{F}_r(x)\right) \cdot D_i\mathscr{F}_{rh}(x), \end{cases}$$

from (5.3) and taking into account (5.5) and (5.6), we get, by means of the variables change $x = \mathscr{F}_r^{-1}(y)$:

$$(5.7)\qquad -\sum_{i=1}^{n}\int_{B^+(1)}\Bigg(a^i\Big(\mathscr{F}_r^{-1}(y), \sum_{j=1}^{n}\alpha_{j1}(y)\Big(D_j\tilde{w}(y)+D_j\tilde{g}(y)\Big),\ldots,$$

$$\sum_{j=1}^{n}\alpha_{jn}(y)\Big(D_j\tilde{w}(y)+D_j\tilde{g}(y)\Big) \,\Big|\, \sum_{h=1}^{n}\beta_{hi}(y)D_h\tilde{\varphi}(y)\Bigg)dy =$$

$$= -\sum_{i=1}^{n}\int_{B^+(1)}\Bigg(\Big(f_i\Big(\mathscr{F}_r^{-1}(y), \tilde{w}(y)+\tilde{g}(y)\Big) \,\Big|\, \sum_{h=1}^{n}\beta_{hi}(y)D_h\tilde{\varphi}(y)\Bigg)dy +$$

$$+\int_{B^+(1)}\Bigg(f\Bigg(\mathscr{F}_r^{-1}(y), \tilde{w}(y)+\tilde{g}(y), \sum_{j=1}^{n}\alpha_{j1}\Big(D_j\tilde{w}(y)+D_j\tilde{g}(y),\ldots,$$

$$\sum_{j=1}^{n}\alpha_{jn}(y)\Big(D_j\tilde{w}(y)+D_j\tilde{g}(y)\Big) \,\Big|\, \tilde{\varphi}(y)\Bigg)\frac{1}{J(\mathscr{F}_r^{-1}(y))}dy.$$

Hence $\tilde{w}$ is a solution of the problem:

$$-\int_{B^+(1)}\sum_{h=1}^{n}\Big(A^h(y, D\tilde{w}+D\tilde{g}) \,\big|\, D_h\tilde{\varphi}\Big)dy =$$

$$= - \int_{B^+(1)} \sum_{h=1}^{n} \Big(F_h(y, \tilde{w} + \tilde{g}) \mid D_h \tilde{\varphi} \Big) dy +$$

$$+ \int_{B^+(1)} \Big(F(y, \tilde{w} + \tilde{g}, D\tilde{w} + D\tilde{g}) \mid \tilde{\varphi} \Big) dy \quad \forall \tilde{\varphi} \in H_0^1(B^+(1), \mathbb{R}^N),$$

that is

$$(5.8) \qquad \begin{cases} \tilde{w} \in H^1(B^+(1), \mathbb{R}^N) \\ \tilde{w} = 0 \quad \text{on} \quad \Gamma \\ \sum\limits_{h=1}^{n} D_h A^h(y, D\tilde{w} + D\tilde{g}) = \sum\limits_{h=1}^{n} D_h F_h(y, \tilde{w} + \tilde{g}) + \\ \qquad\qquad + F(y, \tilde{w} + \tilde{g}, D\tilde{w} + D\tilde{g}) \\ \text{in} \quad B^+(1). \end{cases}$$

Since $\mathscr{F}_r$ is of class C^2 and $g \in H^{1,(\mu)}(\Omega_r \cap \Omega, \mathbb{R}^N)$, $u \in H^1(\Omega_r \cap \Omega, \mathbb{R}^N)$, $f^0(x) \in L^{2r,\mu r}(\Omega_r \cap \Omega)$, $f^i \in L^{2,\mu}(\Omega_r \cap \Omega)$, $i = 1, \dots, n$, then if we put

$$\tilde{u}(y) = u\left(\mathscr{F}_r^{-1}(y)\right)$$

also $\tilde{g} \in H^{1,(\mu)}(B^+(1), \mathbb{R}^N)$, $\tilde{u} \in H^1(B^+(1), \mathbb{R}^N)$, $F^0 \in L^{2r,\mu r}(B^+(1))$, $F^i \in L^{2,\mu}(B^+(1))$, $i = 1 \dots, n$, and we get

$$(5.9) \qquad \begin{aligned} \|\tilde{g}\|^2_{H^{1,(\mu)}(B^+(1),\mathbb{R}^N)} &\le c\,(\mathscr{F}_r)\, \|g\|^2_{H^{1,(\mu)}(\Omega_r \cap \Omega, \mathbb{R}^N)} \\ \|\tilde{u}\|^2_{H^1(B^+(1),\mathbb{R}^N)} &\le c(\mathscr{F}_r) \|u\|^2_{H^1(\Omega_r \cap \Omega, \mathbb{R}^N)} \\ \|F^0\|_{L^{2r,\mu r}(B^+(1)} &\le c\,(\mathscr{F}_r)\, \|f^0\|_{L^{2r,\mu r}(\Omega_r \cap \Omega)} \\ \|F^i\|_{L^{2,\mu}(B^+(1))} &\le c\,(\mathscr{F}_r)\, \|f^i\|_{L^{2,\mu}(\Omega_r \cap \Omega)}, \quad i = 1, \dots, n. \end{aligned}$$

Then, from Theorem 4.1, $\forall R \in (0, 1)$ we obtain $D\tilde{w} \in L^{2,\mu}(B^+(R), \mathbb{R}^{nN})$ and

$$(5.10) \qquad \|D\tilde{w}\|_{L^{2,\mu}(B^+(R),\mathbb{R}^{nN}))} \le c \Big\{ \|\tilde{g}\|_{H^{1,(\mu)}(B^+(1),\mathbb{R}^n)} + \|\tilde{w}\|_{H^1(B^+(1),\mathbb{R}^N)} +$$

$$+ \|F^0\|_{L^{2r,\mu r}(B^+(1))} + \sum_{i=1}^{n} \|F^i\|_{L^{2,\mu}(B^+(1))} \Big\}.$$

Therefore, being $\tilde{w} = \tilde{u} - \tilde{g}$, we have:

$$(5.11) \qquad \|D\tilde{u}\|_{L^{2,\mu}(B^+(R),\mathbb{R}^{nN})} \le c \Big\{ \|\tilde{g}\|_{H^{1,(\mu)}(B^+(1),\mathbb{R}^N)} +$$

$$+ \|\tilde{u}\|_{H^1(B^+(1),\mathbb{R}^N)} + \|F^0\|_{L^{2r,\mu r}(B^+(1))} + \sum_{i=1}^{n} \|F^i\|_{L^{2,\mu}(B^+(1))} \Big\}.$$

Denoting by $\mathscr{B}(R)$ the inverse image of $B(0, R)$ and taking into account that the mapping $\mathscr{F}_r$ of class C^2 preserves the desidered $\mathscr{L}^{2,\lambda}$-properties ([3], Theorem V, pag. 375), from (5.11), we derive

$$(5.12) \qquad [u]_{\mathscr{L}^{2,\mu+2}(\Omega\cap\mathscr{B}(R),\mathbb{R}^N)} + \|Du\|_{L^{2,\mu}(\Omega\cap\mathscr{B}(R),\mathbb{R}^{nN})} \leq$$

$$\leq c\Big\{\|g\|_{H^{1,(\mu)}(\Omega,\mathbb{R}^N)} + \|u\|_{H^1(\Omega,\mathbb{R}^N)} + \|f^0\|_{L^{2r,\mu r}(\Omega)} + \sum_{i=1}^{n} \|f^i\|_{L^{2,\mu}(\Omega)} \Big\}.$$

Using this local regularity result near the boundary together with Theorem 3.1, we can prove, by an usual covering argument, the global regularity result which follows:

Theorem 5.1. *Let $u \in H^1(\Omega, \mathbb{R}^N)$ be the solution of the Dirichlet problem* (5.1) *and suppose that*

$$(5.13) \qquad \Omega \quad \textit{is of class } C^2$$

$$(5.14) \qquad g \in H^{1,(\mu)}(\Omega, \mathbb{R}^N) \quad \textit{with} \quad 0 < \mu < \lambda,$$

a^i, f, f_i, $i = 1, 2, \dots, n$, *satisfy conditions* (1.1) - (1.8)*; then it results*

$$(5.15) \qquad u \in H^{1,(\mu)}(\Omega, \mathbb{R}^N) \cap \mathscr{L}^{2,\mu+2}(\Omega, \mathbb{R}^N)$$

and we have

$$(5.16) \qquad [u]_{\mathscr{L}^{2,\mu+2}(\Omega,\mathbb{R}^N)} + \|Du\|_{L^{2,\mu}(\Omega,\mathbb{R}^{nN})} \leq$$

$$\leq c\Big\{\|g\|_{H^{1,(\mu)}(\Omega,\mathbb{R}^N)} + \|u\|_{H^1(\Omega,\mathbb{R}^N)} + \|f^0\|_{L^{2r,\mu r}(\Omega)} + \sum_{i=1}^{n} \|f^i\|_{L^{2,\mu}(\Omega)} \Big\}.$$

In particular, if

$2 \leq n \leq 4$ *and* $n-2 < \mu < \lambda$, *then* $u \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$, *with* $\alpha = 1 - \dfrac{n-\mu}{2}$

and the inequality

$$(5.17) \qquad [u]_{\alpha,\overline{\Omega}} \leq c\Big\{\|g\|_{H^{1,(\mu)}(\Omega,\mathbb{R}^N)} + \|u\|_{H^1(\Omega,\mathbb{R}^N)} +$$

$$+ \|f^0\|_{L^{2r,\mu r}(\Omega)} + \sum_{i=1}^{n} \|f^i\|_{L^{2,\mu}(\Omega)} \Big\}$$

holds.

This theorem can be showed with easy variations of calculations and using the same technique used in the proof of Theorem 7.1 of [1].

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