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MEASURES WITH RELATIVELY NORM COMPACT RANGE AND ∞- NUCLEAR OPERATORS

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We prove that measures with relatively norm compact range corresponds to the ∞-nuclear operators and that to a p-Bochner integrable function (resp. to a p-Pettis integrable function) correspond a p-nuclear operator (resp. a q-nuclear operator).

Let S be a set, Σ be a σ-field of subsets of S, X be a Banach space $G : \Sigma \longrightarrow X$ a countably additive vector measure, $\|G\|(S)$ the semivariation of G, $B(\Sigma)$ the Banach space of all scalarly totally measurable functions under the supnorm. Recall also, see [2], that given $1 \leq r \leq \infty$ with r^* the conjugate of r, i.e. $\frac{1}{r} + \frac{1}{r^*} = 1$, an operator $U \in L(X, Y)$ is called r-nuclear operator if U has a representation of the form: $U = \sum_{n=1}^{\infty} \sigma_n x_n^* \otimes y_n$ with $(\sigma_n) \in l_r$, (resp. $(\sigma_n) \in c_0$ for $r = \infty$), $(x_n^*) \in w_\infty(X^*)$, $(y_n) \in w_{r^*}(Y)$ and the r-nuclear norm of U is : $N_r(U) = \inf l_r(\sigma_n) w_\infty(x_n^*) w_{r^*}(y_n)$ where the infimum is taken over all possible representation of U as above. As is well known the class of all r-nuclear operators is a normed ideal of operators with respect to the r-nuclear norm denoted by (Nuc_r, N_r) (see also [2]). For all notations and notions used and not defined we refer the reader to [1], [2].

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Theorem 1. *Let $G : \Sigma \longrightarrow X$ be a countably additive vector measure and $U : B(\Sigma) \longrightarrow X$ be the operator $U(f) = \int_S f dG$. Then G has relatively norm compact range if and only if U is a ∞-nuclear operator. In this case:* $N_\infty(U) = \|G\|(S)$. *In particular, G has a representation of the form:* $G(E) = \sum_{n=1}^{\infty} \sigma_n \lambda_n(E) x_n$, $E \in \Sigma$, *where* $(\sigma_n) \in c_0$, $(\lambda_n) \in w_\infty\left(B(\Sigma)^*\right)$, $(x_n) \in w_1(X)$.

Proof. $\Longrightarrow$ There exists a probability measure $\mu : \Sigma \longrightarrow [0,1]$ such that $G << \mu$ ([1], p. 263). Let $f = \sum_{j=1}^{k} y_j \chi_{E_j}$ be a simple function, $F : \Sigma \longrightarrow X$, $F(E) = \int_E f d\mu$ and $V : B(\Sigma) \longrightarrow X$, $V(g) = \int_S g f d\mu$. Then $V = \sum_{j=1}^{k} \lambda_j \otimes y_j$, where $\lambda_j(g) = \int_{E_j} g d\mu$, $g \in B(\Sigma)$. We have evidently $\|\lambda_j\| = \mu(E_j)$, hence $V \in Nuc_\infty(B(\Sigma), X)$ and $N_\infty(V) \leq w_1\left(y_j \mu\left(E_j\right)\right) = \|f\|_{Pettis} = \|F\|(S)$. Let now $V = \sum_{n=1}^{\infty} \sigma_n \nu_n \otimes x_n$ be a ∞-nuclear representation of V. Then $y_j \mu(E_j) = V(\chi_{E_j}) = \sum_{n=1}^{\infty} \sigma_n \nu_n(E_j) x_n$ and thus for each $x^* \in X^*$ we have $\sum_{j=1}^{k} |x^*(y_j)| \mu(E_j) \leq \sum_{n=1}^{\infty} |\sigma_n| \sum_{j=1}^{k} |\nu_n(E_j)|\, |x^*(x_n)| \leq \sum_{n=1}^{\infty} |\sigma_n| \|\nu_n\| |x^*(x_n)|$. This implies that $\|f\|_{Pettis} \leq N_\infty(V)$. Hence $N_\infty(V) = \|F\|(S) = \|f\|_{Pettis}$, for f a simple function. If G has relatively norm compact range then: $\|G_\pi - G\|(S) \longrightarrow 0$ where $G_\pi(E) = \int_E g_\pi d\mu$ and $g_\pi = \sum_{E \in \pi} \frac{G(E)}{\mu(E)} \chi_E$ (see [1]). Let U_π be the operator associated to g_π as above. Then U_π is a ∞-nuclear operator and $N_\infty(U_\pi) = \|g_\pi\|_{Pettis} = \|G_\pi\|(S)$. Since $U_{\pi_1} - U_{\pi_2}$ is the operator associated to $g_{\pi_1} - g_{\pi_2}$ we have again $N_\infty(U_{\pi_1} - U_{\pi_2}) = \|g_{\pi_1} - g_{\pi_2}\|_{Pettis} = \|G_{\pi_1} - G_{\pi_2}\|(S) \longrightarrow 0$, i.e. $(U_\pi) \subset Nuc_\infty(B(\Sigma), X)$ is a Cauchy net for the ∞-nuclear norm. Since (Nuc_∞, N_∞) is a normed ideal of operators then, there exists $T \in Nuc_\infty(B(\Sigma), X)$ such that $N_\infty(U_\pi - T) \longrightarrow 0$. As evidently $U_\pi \longrightarrow U$ with respect to the operatorial norm we must have $U = T$, i.e. U will be a nuclear operator. Since $N_\infty(U_\pi) \longrightarrow N_\infty(U)$ and $N_\infty(U_\pi) = \|G_\pi\|(S) \longrightarrow \|G\|(S)$ we obtain also $N_\infty(U) = \|G\|(S)$. Conversely if U is a ∞-nuclear operator then U will be a compact operator hence, as $G(E) = U(\chi_E)$, the range of G will be relatively norm compact. The last part of the theorem is clear.

Corollary 2. *Let $\mu : \Sigma \longrightarrow R_+$ be a countably additive measure. Then the injective tensor product $L_1(\mu) \otimes_\varepsilon X$ coincide with $Nuc_\infty(L_\infty(\mu), X)$.*

Proof. It is well known that $L_1(\mu) \otimes_\varepsilon X = K(\mu, X) = \Big\{G : \Sigma \longrightarrow X \mid G$ countably additive with the relatively norm compact range and $G << \mu\Big\}$ (see [1]). By Theorem 1, $K(\mu, X) = Nuc_\infty(L_\infty(\mu), X)$.

In [3] for a unconditionally norm convergent series $\sum_{n=1}^{\infty} x_n$ in X is defined the countably sum of segments:

$$\sum_{n=1}^{\infty}[-x_n, x_n] = \left\{\sum_{n=1}^{\infty} \alpha_n x_n \mid (\alpha_n) \in l_\infty, \|\alpha_n\| \leq 1\right\}.$$

Using Theorem 1 we obtain also the following result which is proved in [3], Prop. 1.4 with a different proof.

Corollary 3. *Let $G : \Sigma \longrightarrow X$ be a countably additive vector measure with relatively norm compact range. Then there exists an unconditionally norm convergent series $\sum_{n=1}^{\infty} y_n$ in X such that $G(\Sigma) \subset \sum_{n=1}^{\infty}[-y_n, y_n]$.*

Proof. By Theorem 1 there exists $(\sigma_n) \in c_0$, $(\lambda_n) \in w_\infty\left(B(\Sigma)^*\right)$, $(x_n) \in w_1(X)$ such that $G(E) = \sum_{n=1}^{\infty} \sigma_n \lambda_n(E) x_n$, $E \in \Sigma$. Then the series $\sum_{n=1}^{\infty} y_n$ where $y_n = \sigma_n \|\lambda_n\| x_n$ is unconditionally norm convergent in X and $G(E) \in \sum_{n=1}^{\infty}[-y_n, y_n]$ for each $E \in \Sigma$.

In the following theorem, which completes the above results, for a (S, Σ, μ) a finite measure space and $1 \leq p < \infty$ we denote by $P_p(\mu, X)$ the space of all functions $f : S \longrightarrow X$ μ-Pettis integrable for which $x^* f \in L_p(\mu)$ for every $x^* \in X^*$ which is a normed space when endowed with the so called p-Pettis norm $\|f\|_{pPe} = \sup_{\|x^*\| \leq 1} \left(\int_S |x^* f|^p\right)^{1/p}$ and by $I_p(\mu, X) \subset P_p(\mu, X)$ the subspace of $P_p(\mu, X)$ formed by those functions $f \in P_p(\mu, X)$ such that there exists a sequence (f_n) of simple functions such that $\|f_n - f\|_{pPe} \longrightarrow 0$. By $L_p(\mu, X)$ we denote the space of all p-Bochner integrable function.

Theorem 4. *Let $1 \leq p < \infty$, $f : S \longrightarrow X$ be a Pettis integrable function and $U_f : L_\infty(\mu) \longrightarrow X$ be the operator $U_f(g) = P - \int_S gf\,d\mu$.*

a) If $f \in L_p(\mu, X)$ then U_f is a p-nuclear operator.

b) If $f \in I_p(\mu, X)$ then U_f is a q-nuclear operator.
(here q is the conjugate of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$).

Proof. a) Let $f = \sum_{i=1}^{n} x_i \chi_{E_i} \in L_p(\mu, X)$ be a simple function. Then we have $U_f = \sum_{i=1}^{n} \lambda_i \otimes x_i$ where $\lambda_i(g) = \int_{E_i} g d\mu$, $g \in L_\infty(\mu)$ and hence $U_f = \sum_{i=1}^{n} \sigma_i \alpha_i \otimes y_i$ where $\sigma_i = \|x_i\|(\mu(E_i))^{\frac{1}{p}}$, $\alpha_i = \frac{\lambda_i}{\mu(E_i)}$, $y_i = \frac{x_i}{\|x_i\|}(\mu(E_i))^{\frac{1}{q}}$; thus U_f is p-nuclear and $N_p(U_f) \leq l_p(\sigma_i) w_\infty(\alpha_i) w_q(y_i) = \|f\|_p (\mu(S))^{\frac{1}{q}}$. Let $f \in L_p(\mu, X)$ and (f_n) be a sequence of simple functions such that $\|f_n - f\|_p \longrightarrow 0$; hence $\|f_n - f_m\|_p \longrightarrow 0$. We have $N_p(U_{f_n} - U_{f_m}) = N_p(U_{f_n - f_m}) \leq \|f_n - f_m\|_p (\mu(S))^{\frac{1}{q}}$ and hence (U_{f_n}) is a Cauchy sequence with respect to the p-nuclear norm. Since $U_{f_n} \longrightarrow U_f$ in operatorial norm and (Nuc_p, N_p) is a normed ideal of operators we obtain that U_f is a p-nuclear operator.

b) If $f = \sum_{i=1}^{n} x_i \chi_{E_i} \in I_p(\mu, X)$ is a simple function then $U_f = \sum_{i=1}^{n} \lambda_i \otimes x_i$. As $l_q((\mu(E_i))^{\frac{1}{q}}) = (\mu(S))^{\frac{1}{q}}$, $w_\infty\left(\frac{\lambda_i}{\mu(E_i)}\right) = 1$, $w_p(x_i(\mu(E_i))^{\frac{1}{p}}) = \|f\|_{pPe}$ we obtain that U_f is q-nuclear and $N_q(U_f) \leq \|f\|_{pPe}(\mu(S))^{\frac{1}{q}}$. From this point the proof of b) is similar with that of a) so we omit it.

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