LE MATEMATICHE
Vol. L (1995) – Fasc. II, pp. 311-321

MONOTONE ITERATIVE METHOD FOR CARATHEODORY SOLUTIONS OF DIFFERENTIAL-FUNCTIONAL EQUATIONS

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The paper deals with initial problems for differential-functional equations. The sufficient conditions for the existence of some monotone function sequences $\{u_n\}$, $\{v_n\}$ are given. The sequences are uniformly convergent to the solution $\bar{y}$ of the given problem. For every n the functions u_n, v_n are the solutions of some linear initial problems. The error estimation of the approximate solutions is given. The Caratheodory solutions of the differential-functional equations are considered. The differential inequalities technique is applied.

1. Introduction.

Let $X \in \mathbb{R}$ be an interval. We will denote by $C(X)$ the set of all functions $\eta : X \to \mathbb{R}$ continuous on X. Let $AC(X) \subset C(X)$ denote the subspace of absolutely continuous functions from X into $\mathbb{R}$. For $X = [a, b]$ we will denote by $L(X)$ the set of all functions $\eta : X \to \mathbb{R}_+$, $\mathbb{R}_+ = [0, +\infty)$, measurable on $[a, b]$ and such that $\int_a^b \eta(t)dt < +\infty$.

Let us adopt the following notation: $I_0 = [-\tau, 0]$, $I = [0, a]$, $I^* = I_0 \cup I$, where $\tau \in \mathbb{R}_+$ and $a > 0$. For a given function $\eta : [-\tau, a] \to \mathbb{R}$ and a given point $x \in [0, a]$, we define the function $\eta_x : I_0 \to \mathbb{R}$ in the following way $\eta_x(t) = \eta(x + t)$, $t \in I_0$. We denote by $\| \cdot \|_0$ the supremum norm in the space

Entrato in Redazione l'8 gennaio 1996.

$C(I_0)$. We will need the definition of the inequality between two real functions $\alpha, \beta : X \to \mathbb{R}$. We say that $\alpha \le \beta$ if and only if $\alpha(x) \le \beta(x)$ for every $x \in X$.

Suppose that $f : I \times \mathbb{R} \times C(I_0) \to \mathbb{R}$, $\omega_0 \in C(I_0)$ are given functions. We consider the following Cauchy problem:

$$y'(x) = f(x, y(x), y_x) \text{ for a.e. } x \in I, \tag{1}$$

$$y(x) = \omega_0(x), \quad x \in I_0. \tag{2}$$

In this paper we consider the Caratheodory solutions of (1), (2). A function $\overline{y} : [-\tau, a_0] \to \mathbb{R}$, $0 < a_0 \le a$, is said to be the Caratheodory solution of problem (1), (2) if

(i) $\overline{y}$ is continuous on $[-\tau, a_0]$ and absolutely continuous on $[0, a_0]$,

(ii) $\overline{y}$ satisfies equation (1) for almost every $x \in [0, a_0]$,

(iii) $\overline{y}$ satisfies initial condition (2).

It is required that f satisfies the following Caratheodory conditions.

Assumption A1.

1^0 *$f(\cdot, p, w)$ is measurable for every $(p, w) \in \mathbb{R} \times C(I_0)$,*

2^0 *$f(x, \cdot, \cdot)$ is continuous for almost every $x \in I$,*

3^0 *for every fixed $(\bar{x}, \bar{p}, \overline{w})$ there exist $r > 0$ and a function $m \in L([\bar{x} - r, \bar{x} + r])$ such that $|f(x, p, w)| \le m(x)$ for $p \in (\bar{p} - r, \bar{p} + r)$, $\|w - \overline{w}\|_0 < r$ and for almost every $x \in (\bar{x} - r, \bar{x} + r)$.*

The following result is a special case of the theorem given in [5].

Proposition. *If Assumption* A1 *is satisfied and $\omega_0 \in C(I_0)$, then there is $0 < a_0 \le a$ such that the solution $\overline{y}$ of problem* (1), (2) *exists on* $[-\tau, a_0]$.

The theorems on the uniqueness and on the continuous dependence can be found in [3].

In this paper we give sufficient conditions for the existence of two function sequences $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $u_n, v_n : [-\tau, a] \to \mathbb{R}$ satisfying the conditions:

(iv) for every $n = 1, 2, \dots$ the functions u_n, v_n are the Caratheodory solutions of some linear differential-functional equations,

(v) the sequences $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ are monotone:

$$u_{n+1} \ge u_n, \quad v_{n+1} \le v_n, \quad \text{for } n = 0, 1, \dots$$

and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = \overline{y}$ uniformly on the interval $[-\tau, a]$, where $\overline{y}$ is the solution of problem (1), (2) on the interval $[-\tau, a]$.

We will apply the theorems on differential and differential-functional inequalities. The classical theory of ordinary differential inequalities is described in the monographes [10], [8], [11]. Classical differential-functional inequalities were considered in [9], [1]. Differential inequalities in the Caratheodory sense were studied in [10], Theorem 16.2. In this paper we will apply the following theorem on differential-functional inequalities in the Caratheodory sense which can be easily derived from Theorem 5 [2].

Comparison Theorem. *Let* $\gamma \in C(I^*)$, $\gamma \in AC(I)$ *and satisfies the conditions*

1^0 $\gamma(x) \leq \omega_0(x)$, $x \in I_0$,

2^0 $\gamma'(x) \leq f(x, \gamma(x), \gamma_x)$ *almost everywhere in* I.

Then we have

$$\gamma(x) \leq \omega(x), \ x \in I^*,$$

where $\omega : I^* \to \mathbb{R}$ *is the maximal solution of the comparison problem*

$$\omega'(x) = f(x, \omega(x), \omega_x) \text{ for a.e. } x \in I,$$

$$\omega(x) = \omega_0(x), \ x \in I_0.$$

The extremal solutions of differential-functional equations in the Caratheodory sense were considered in [4], [2].

The approximation method of solving problem (1), (2) is considered in this paper. The method is called the Chaplygin method and it was initiated in [6]. The monographes [8], [10] contain the results. The essence of the method consists in the consideration together with the differential equation another equation (or equations) which is (are) obtained by the linearization of the first one. The solutions of the linear equations approximate the solutions of the given problem.

The Chaplygin method for the classical solutions of the ordinary differential-functional equations is considered in [7], where only fragmentary linearization of the right side of the equation is performed. The auxiliary equation for the Chaplygin method was obtained by the linearization with respect to the argument representing the value of the function at the point. The linearization with respect to the functional argument was not considered in the paper. It has an effect on the error estimate of the method. The Chaplygin method studied in [7] is of the first order.

In this paper we construct linear differential-functional equations, which solutions satisfy conditions (iv), (v). We give the error estimate of the method. The result of this paper is a generalization of the results in [10] (Theorems 31.1-31.3), [8], [7] on the differential-functional equations and the Caratheodory solutions.

The result of this paper is new even for the classical solutions of the differential-functional equations.

2. Construction of monotone iterative sequences.

The following assumptions will be needed throughout the paper.

Assumption A2. *Suppose that the function f satisfies the conditions*

1^0 *f is nondecreasing with respect to the functional variable,*

2^0 *there exist the derivatives $D_p f(x, p, w)$, $D_w f(x, p, w)$ for $(p, w) \in \mathbb{R} \times C(I_0)$ and for almost every $x \in I$,*

3^0 *the functions $D_p f(x, \cdot, \cdot)$, $D_w f(x, \cdot, \cdot)$ are continuous on $\mathbb{R} \times C(I_0)$ for almost every fixed $x \in I$,*

4^0 *$D_p f(x, p, w) \geq 0$ for $(p, w) \in \mathbb{R} \times C(I_0)$ and for almost every $x \in I$, and the function $D_p f(x, \cdot, \cdot)$ is nondecreasing on $\mathbb{R} \times C(I_0)$ for almost every fixed $x \in I$,*

5^0 *$D_w f(x, p, w)h \geq 0$ for $h \in C(I_0)$, $h \geq 0$, $(p, w) \in \mathbb{R} \times C(I_0)$, almost every fixed $x \in I$, and if additionally $w, \overline{w} \in C(I_0)$, $w \leq \overline{w}$, $p, \bar{p} \in \mathbb{R}$, $p \leq \bar{p}$, $h \in C(I_0)$, $h \geq 0$, then $D_w f(x, p, w)h \leq D_w f(x, \bar{p}, \overline{w})h$ for almost every fixed $x \in I$.*

Assumption A3. *Suppose that*

1^0 *the initial functions $\alpha, \beta \in C(I_0)$ satisfy the inequalities $\alpha \leq \omega_0 \leq \beta$,*

2^0 *the functions $u, v \in C(I^*)$, $u, v \in AC(I)$ satisfy the initial inequalities*

$$u(x) \leq \alpha(x) \leq \omega_0(x) \leq \beta(x) \leq v(x), \quad x \in I_0, \tag{3}$$

and the differential-functional inequalities

$$u'(x) \leq f(x, u(x), u_x) \quad \text{for a.e.} \quad x \in I, \tag{4}$$

$$v'(x) \geq f(x, v(x), v_x) \quad \text{for a.e.} \quad x \in I. \tag{5}$$

For the given functions u, v satisfying Assumption A3 we define the functions $G(\cdot, \cdot, \cdot; u)$, $H(\cdot, \cdot, \cdot; u, v) : I \times \mathbb{R} \times C(I_0) \to \mathbb{R}$ in the following way

$$G(x, p, w; u) = f(x, u(x), u_x) + D_p f(x, u(x), u_x)(p - u(x)) + \\ + D_w f(x, u(x), u_x)(w - u_x)$$

$$H(x, p, w; u, v) = f(x, v(x), v_x) + D_p f(x, u(x), u_x)(p - v(x)) + \\ + D_w f(x, u(x), u_x)(w - v_x)$$

for $x \in I$, $p \in \mathbb{R}$, $w \in C(I_0)$. Moreover we define the operator $\Omega^{\alpha,\beta}$, where $\alpha, \beta \in C(I_0)$ satisfy condition 1^0 of Assumption A3, as follows. If u, v satisfy condition

2^0 of Assumption A3, then $\bar{u}, \bar{v} \in C(I^*), \bar{u}, \bar{v} \in AC(I)$ are the solutions of the problems

(6)
$$y'(x) = G(x, y(x), y_x; u) \text{ for a.e. } x \in I, \quad y(x) = \alpha(x), \ x \in I_0,$$

(7)
$$y'(x) = H(x, y(x), y_x; u, v) \text{ for a.e. } x \in I, \quad y(x) = \beta(x), x \in I_0,$$

respectively and we define $\Omega^{\alpha,\beta}[u, v] = [\bar{u}, \bar{v}]$. If Assumptions A1-A3 are satisfied, then there exist the (unique) solutions on I^* of problems (6), (7).

We can now state the first result.

Theorem 1. *Suppose that* A1-A3 *are satisfied and* $[\bar{u}, \bar{v}] = \Omega^{\alpha,\beta}[u, v]$.

Then the inequalities

(8)
$$u(x) \le \bar{u}(x) \le \overline{y}(x) \le \bar{v}(x) \le v(x), x \in I,$$

(9)
$$\bar{u}'(x) \le f(x, \bar{u}(x), \bar{u}_x) \textit{ for a.e. } x \in I,$$

(10)
$$\bar{v}'(x) \ge f(x, \bar{v}(x), \bar{v}_x) \textit{ for a.e. } x \in I,$$

hold.

Proof. Our proof starts with the observation that

$$u'(x) \le G(x, u(x), u_x; u) \text{ for a.e. } x \in I,$$
$$v'(x) \ge H(x, v(x), v_x; u, v) \text{ for a.e. } x \in I.$$

Since $\bar{u}, \bar{v}$ are the unique solutions of problems (6), (7) respectively, we can apply the Comparison Theorem and get $u \le \bar{u}, \bar{v} \le v$. In the same way, applying (3), (4), (5) we check at once that $u \le \overline{y} \le v$. It remains to prove the inequalities $\bar{u} \le \overline{y}, \overline{y} \le \bar{v}$. First we will prove (9). Applying the Hadamard Mean Value Theorem we have

(11)
$$\begin{aligned} f(x, \bar{u}(x), \bar{u}_x) - \bar{u}'(x) &= f(x, \bar{u}(x), \bar{u}_x) - f(x, u(x), u_x) - \\ & D_p f(x, u(x), u_x)(\bar{u}(x) - u(x)) - D_w f(x, u(x), u_x)(\bar{u}_x - u_x) = \\ & \int_0^1 D_p f(x, u(x) + s(\bar{u}(x) - u(x)), u_x + s(\bar{u}_x - u_x))ds(\bar{u}(x) - u(x)) + \\ & \int_0^1 D_w f(x, u(x) + s(\bar{u}(x) - u(x)), u_x + s(\bar{u}_x - u_x))ds(\bar{u}_x - u_x) - \\ & \int_0^1 D_p f(x, u(x), u_x)ds(\bar{u}(x) - u(x)) - \\ & \int_0^1 D_w f(x, u(x), u_x)ds(\bar{u}_x - u_x) \text{ for a.e. } x \in I. \end{aligned}$$

Since $u \le \bar{u}$ and $s \in [0,1]$, it follows that $s(\bar{u}(x)-u(x)) \ge 0$ and $s(\bar{u}_x-u_x) \ge 0$. From this, (11) and conditions 4^0, 5^0 of Assumption A2 we conclude (9). Now we should prove (10). Likewise,

$$(12) \qquad f(x,\bar{v}(x),\bar{v}_x) - \bar{v}'(x) = f(x,\bar{v}(x),\bar{v}_x) - f(x,v(x),v_x) - D_p f(x,u(x),u_x)(\bar{v}(x)-v(x)) - D_w f(x,u(x),u_x)(\bar{v}_x - v_x) = \int_0^1 D_p f(x, v(x)+s(\bar{v}(x)-v(x)), v_x + s(\bar{v}_x - v_x))ds(\bar{v}(x)-v(x)) + \int_0^1 D_w f(x, v(x)+s(\bar{v}(x)-v(x)), v_x + s(\bar{v}_x - v_x))ds(\bar{v}_x - v_x) - \int_0^1 D_p f(x,u(x),u_x)ds(\bar{v}(x)-v(x)) - \int_0^1 D_w f(x,u(x),u_x)ds(\bar{v}_x - v_x) \text{ for a.e. } x \in I.$$

Since $\overline{v} \le v$ and $s \in [0,1]$, it suffices to show

$$(13) \qquad v + s(\overline{v} - v) \ge u.$$

To prove (13) we will show

$$(14) \qquad \bar{u} \le \overline{v}.$$

Since (7), it is sufficient to prove

$$(15) \qquad \bar{u}'(x) \le H(x,\bar{u}(x),\bar{u}_x; u, v) \text{ for a.e. } x \in I.$$

We have

$$H(x,\bar{u}(x),\bar{u}_x;u,v) - \bar{u}'(x) = f(x,v(x),v_x) + D_p f(x,u(x),u_x)(\bar{u}(x)-v(x)) + D_w f(x,u(x),u_x)(\bar{u}_x - v_x) - f(x,u(x),u_x) - D_p f(x,u(x),u_x)(\bar{u}(x)-u(x)) - D_w f(x,u(x),u_x)(\bar{u}_x - u_x) = \int_0^1 D_p f(x,u(x)+s(v(x)-u(x)), u_x + s(v_x - u_x))ds(v(x)-u(x)) + \int_0^1 D_w f(x,u(x)+s(v(x)-u(x)), u_x + s(v_x - u_x))ds(v_x - u_x) - \int_0^1 D_p f(x,u(x),u_x)ds(v(x)-u(x)) - \int_0^1 D_w f(x,u(x),u_x)ds(v_x - u_x) \quad \text{for a.e.} \quad x \in I.$$

From this and from the inequalities $u \leq \overline{y} \leq v$, $s(v(x)-u(x)) \geq 0$, $s(v_x-u_x) \geq 0$, we get (15) which, after applying the Comparison Theorem, gives (14). From (14) we have (13) which together with (12) gives (10). Repeated application of the Comparison Theorem enables us to deduce the assertion of Theorem 1.

Corollary (Monotone iterative sequence). *Suppose that Assumptions* A1, A2 *are satisfied and*

1^0 *the functions* $u_0, v_0 \in C(I^*)$ *are such that* $u_0, v_0 \in AC(I)$ *and*

$$u_0'(x) \leq f(x, u_0(x), (u_0)_x) \text{ for a.e. } x \in I, \tag{16}$$
$$v_0'(x) \geq f(x, v_0(x), (v_0)_x) \text{ for a.e. } x \in I,$$

2^0 $\{\varphi_n, \psi_n\}_{n=0}^{\infty}$ *is the sequence of the initial functions satisfying* $\varphi_n, \psi_n \in C(I_0)$, $u_0 = \varphi_0$ *on* I_0, $v_0 = \psi_0$ *on* I_0 *and*

$$\varphi_n \leq \varphi_{n+1} \leq \omega_0 \leq \psi_{n+1} \leq \psi_n, \quad n = 0, 1, 2, \ldots \tag{17}$$

3^0 $\{u_n, v_n\}_{n=0}^{\infty}$ *is the monotone iterative sequence defined as follows*

$$[u_{n+1}, v_{n+1}] = \Omega^{\varphi_{n+1}, \psi_{n+1}}[u_n, v_n], \quad n = 0, 1, \ldots \tag{18}$$

Under the above assumptions we get the following inequalities

$$u_n \leq u_{n+1} \leq \overline{y} \leq v_{n+1} \leq v_n, \quad n = 0, 1, 2, \ldots$$

The corollary is an immediate consequence of Theorem 1.

3. Error estimation.

We denote by $\|\cdot\|$ the norm in the space of all linear and continuous operators $\Gamma : C(I_0) \to \mathbb{R}$. Now we state the following

Theorem 2. *Suppose that*

1^0 *Assumptions* A1, A2 *are satisfied,*

2^0 *the sequences* $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{\varphi_n\}_{n=0}^{\infty}$, $\{\psi_n\}_{n=0}^{\infty}$ *are defined by conditions* 1^0 - 3^0 *of Corollary,*

3^0 *there is* $K > 0$ *such that*

$$\left|D_p f(x, p, w)\right| \leq K, \left\|D_w f(x, p, w)\right\| \leq K$$

for a.e. $x \in I$, $p \in \mathbb{R}$, $w \in C(I_0)$ *satisfying* $u_0(x) \leq p \leq v_0(x)$ *and* $(u_0)_x \leq w \leq (v_0)_x$, *where* K *does not depend on* x, p, w,

4^0 *there exists a constant* $H > 0$ *such that*

$$\left|D_p f(x, p, w) - D_p f(x, \bar{p}, \bar{w}\right| \leq \max\{|p - \bar{p}|, \|w - \bar{w}\|_0\},$$
$$\left\|D_w f(x, p, w) - D_w f(x, \bar{p}, \bar{w}\right\| \leq \max\{|p - \bar{p}| + \|w - \bar{w}\|_0\},$$

for almost every $x \in I$ *and for* $p, \bar{p} \in \mathbb{R}$, $w, \bar{w} \in C(I_0)$, *satisfying* $u_0(x) \leq p$, $\bar{p} \leq v_0(x)$ *and* $(u_0)_x \leq w, \bar{w} \leq (v_0)_x$,

5^0 *the initial estimations*

$$|\varphi_n(x) - \psi_n(x)| \leq [2^{2^n+3} a H e^{4aK}]^{-1}, \; x \in I_0, \; n = 1, 2, \ldots$$

are satisfied,

6^0 *the starting functions* v_0, u_0 *satisfy* $0 \leq v_0(x) - u_0(x) \leq (8aHe^{2aK})^{-1}$, $x \in I^*$.

Under the conditions stated above, the following estimations

$$|u_n(x) - v_n(x)| \leq [2^{2^n+2} a H e^{2aK}]^{-1}, \; x \in I^*, \; n = 0, 1, 2, \ldots \tag{19}$$

hold.

Proof. Set $B = (8aHe^{2aK})^{-1}$, $A_n = B(2^{2^n} \cdot e^{2aK})^{-1}$, $n = 1, 2, \ldots$ (19) is easily seen to be satisfied for $n = 0$. Assume (19) for n; we will prove it for $n + 1$. Since

$$v'_{n+1}(x) - u'_{n+1}(x) = H(x, v_{n+1}(x), (v_{n+1})_x; u_n, v_n) -$$
$$G(x, u_{n+1}(x), (u_{n+1})_x; u_n) = f(x, v_n(x), (v_n)_x) +$$
$$D_p f(x, u_n(x), (u_n)_x)(v_{n+1}(x) - v_n(x)) +$$
$$D_w f(x, u_n(x), (u_n)_x)((v_{n+1})_x - (v_n)_x) - f(x, u_n(x), (u_n)_x) -$$
$$D_p f(x, u_n(x), (u_n)_x)(u_{n+1}(x) - u_n(x)) -$$
$$D_w f(x, u_n(x), (u_n)_x)((u_{n+1})_x - (u_n)_x) =$$
$$D_p f(x, u_n(x), (u_n)_x)(v_{n+1}(x) - u_{n+1}(x)) -$$
$$\int_0^1 D_p f(x, u_n(x), (u_n)_x) ds (v_n(x) - u_n(x)) +$$
$$D_w f(x, u_n(x), (u_n)_x)((v_{n+1})_x - (u_{n+1})_x) -$$

$$\int_0^1 D_w f(x, u_n(x), (u_n)_x) ds((v_n)_x - (u_n)_x) +$$

$$\int_0^1 D_p f(P_n(x; u, v)) ds(v_n(x) - u_n(x)) +$$

$$\int_0^1 D_w f(P_n(x; u, v)) ds((v_n)_x - (u_n)_x) =$$

$$D_p f(x, u_n(x), (u_n)_x)(v_{n+1}(x) - u_{n+1}(x)) +$$

$$D_w f(x, u_n(x), (u_n)_x)((v_{n+1})_x - (u_{n+1})_x) +$$

$$\int_0^1 [D_p f(P_n(x; u, v)) - D_p f(x, u_n(x), (u_n)_x)] ds(v_n(x) - u_n(x)) +$$

$$\int_0^1 [D_w f(P_n(x; u, v)) - D_p f(x, u_n(x), (u_n)_x)] ds((v_n)_x - (u_n)_x),$$

where $P_n(x; u, v) = (x, u_n(x) + s(v_n(x) - u_n(x)), (u_n)_x + s((v_n)_x - (u_n)_x))$, it follows that

$$|v'_{n+1}(x) - u'_{n+1}(x)| \le 2K\|(v_{n+1} - u_{n+1})_x\|_0 + 2H\|(v_n - u_n)_x\|_0^2.$$

From the inductive assumption we have

$$|v'_{n+1}(x) - u'_{n+1}(x)| \le 2K\|(v_{n+1} - u_{n+1})_x\|_0 + 2H\left(\frac{2B}{2^{2^n}}\right)^2. \tag{20}$$

From condition 5^0 and (20), applying the Comparison Theorem, we get

$$|v_{n+1}(x) - u_{n+1}(x)| \le A_{n+1}e^{2Kx} + 2H\left(\frac{2B}{2^{2^n}}\right)^2 \int_0^x e^{2K(x-s)} ds$$

$$\le \frac{B}{2^{2^{n+1}}} + \frac{4B^2}{2^{2^{n+1}}} \cdot 2aHe^{2aK} = \frac{2B}{2^{2^{n+1}}}$$

and this is precisely the inductive assertion. This proves the theorem.

Remark 1. Condition 5^0 in Theorem 2 deals with the initial functions given on the interval I_0. The condition is not restrictive and it is only a generalization. If $\varphi_n \equiv \omega_0$, $\psi_n \equiv \omega_0$ on I_0, then the condition is easily satisfied.

Remark 2. If the assumptions of Theorem 2 are satisfied, then we have the following error estimation

$$\overline{y} - u_n \leq [2^{2^n+2} a H e^{2aK}]^{-1},$$
$$v_n - \overline{y} \leq [2^{2^n+2} a H e^{2aK}]^{-1},$$

where $\overline{y}$ is the solution of problem (1), (2).

Remark 3. The results of this paper can be extended for the ordinary differential-functional systems with the Cauchy conditions:

$$y'(x) = F(x, y(x), y_x) \text{ for a.e. } x \in I,$$
$$y(x) = \omega_0(x), \; x \in I_0\,,$$

where $F = (F_1, \ldots, F_m)$. The following quasi-monotone conditions are required in this case: for each i, $1 \leq i \leq m$, if $p \leq \bar{p}$ and $p_i = \bar{p}_i$, where $p = (p_1, \ldots, p_m)$, $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_m)$, then $F_i(x, p, w) \leq F_i(x, \bar{p}, w)$.

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