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APPROXIMATION OF FUNCTIONS OF TWO VARIABLES BY SOME LINEAR POSITIVE OPERATORS

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We introduce some linear positive operators $A_{m,n}$ and $B_{m,n}$ of the Szasz - Mirakjan type in the weighted spaces of continuous functions of two variables. We study the degree of approximation of functions by these operators. The similar results for functions of one variable are given in [5].
Some operators of the Szasz - Mirakjan type are examined also in [3], [4].

1. Preliminaries.

1.1. Let $R_0 := [0, +\infty)$, $N := \{1, 2, ...\}$, $N_0 := N \cup \{0\}$ and let for a fixed $p \in N_0$ and for all $x \in R_0$

$$(1) \qquad w_0(x) := 1 , \qquad w_p(x) := (1 + x^p)^{-1} \quad \text{if} \quad p \geq 1 .$$

Let for a fixed $p_1, p_2 \in N_0$

$$(2) \qquad w_{p_1,p_2}(x, y) := w_{p_1}(x)\, w_{p_2}(y) , \qquad (x, y) \in R_0^2 ,$$

where $R_0^2 := \{(x, y) : x, y \in R_0\}$.

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Denote by $C_{1;p_1,p_2}$, $p_1, p_2 \in N_0$, the space of real - valued functions f defined on R_0^2 such that $w_{p_1,p_2}(\cdot,\cdot)f(\cdot,\cdot)$ is uniformly continuous and bounded on R_0^2. The norm in $C_{1;p_1,p_2}$ is defined by

$$\|f\|_{1;p_1,p_2} := \sup_{(x,y)\in R_0^2} w_{p_1,p_2}(x,y)\,|f(x,y)|\,. \tag{3}$$

For a given $f \in C_{1;p_1,p_2}$ we define the modulus of continuity

$$\omega(f, C_{1;p_1,p_2}; t, s) := \sup_{0\le h\le t\,;\,0\le\delta\le s} \|\Delta_{h,\delta} f(\cdot,\cdot)\|_{1;p_1,p_2} \qquad t, s \ge 0\,, \tag{4}$$

where $\Delta_{h,\delta} f(x,y) = f(x+h, y+\delta) - f(x,y)$. Let for a given $p_1, p_2 \in N_0$ and $0 < \alpha, \beta \le 1$

$$\mathrm{Lip}(C_{1;p_1,p_2}; \alpha, \beta) := \Big\{ f \in C_{1;p_1,p_2} \,:\, \omega(f, C_{1;p_1,p_2}; t, s) =$$

$$= \mathrm{O}(t^\alpha + s^\beta) \text{ as } t, s \to 0_+ \Big\}\,,$$

$$C^1_{1;p_1,p_2} := \left\{ f \in C_{1,p_1,p_2} \,:\, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \in C_{1;p_1,p_2} \right\}.$$

1.2. For a fixed $q > 0$ let

$$v_q(x) := e^{-qx}\,, \qquad x \in R_0\,, \tag{5}$$

and for a fixed $q_1, q_2 > 0$ let

$$v_{q_1,q_2}(x,y) := v_{q_1}(x)\cdot v_{q_2}(y)\,, \qquad (x,y) \in R_0^2\,. \tag{6}$$

Similarly as above we denote by $C_{2;q_1,q_2}$, $q_1, q_2 > 0$, the space of real - valued functions f defined on R_0^2 for which $v_{q_1,q_2}(\cdot,\cdot)f(\cdot,\cdot)$ is uniformly continuous and bounded on R_0^2. Let

$$\|f\|_{2;q_1,q_2} := \sup_{(x,y)\in R_0^2} v_{q_1,q_2}(x,y)\,|f(x,y)|\,. \tag{7}$$

Analogously as above we define the modulus of continuity $\omega(f, C_{2;q_1,q_2}; \cdot,\cdot)$ for $f \in C_{2;q_1,q_2}$ and the classes $\mathrm{Lip}(C_{2;q_1,q_2}, \alpha, \beta)$ and $C^1_{2;q_1,q_2}$.

1.3. In spaces $C_{1;p_1,p_2}$ and $C_{2;q_1,q_2}$ we define the following operators $A_{m,n}$ and $B_{m,n}$

$$(8)\qquad \begin{aligned} A_{m,n}(f;x,y) &:= a_{m,n}(x,y)f(0,0) + \\ &+ a_{m,n}(x,0)\sum_{k=0}^{\infty} b_{n,k}(y)\, f\left(0, \frac{2k+1}{n}\right) + \\ &+ a_{m,n}(0,y)\sum_{j=0}^{\infty} b_{m,j}(x)\, f\left(\frac{2j+1}{m}, 0\right) + \\ &+ \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{m,j}(x)\, b_{n,k}(y)\, f\left(\frac{2j+1}{m}, \frac{2k+1}{n}\right), \end{aligned}$$

$$(9)\qquad \begin{aligned} B_{m,n}(f;x,y) &:= a_{m,n}(x,y)f(0,0) + \\ &+ a_{m,n}(x,0)\sum_{k=0}^{\infty} b_{n,k}(y)\,\frac{n}{2}\int_{\frac{2k+1}{n}}^{\frac{2k+3}{n}} f(0,v)\,dv + \\ &+ a_{m,n}(0,y)\sum_{j=0}^{\infty} b_{m,j}(x)\,\frac{m}{2}\int_{\frac{2j+1}{m}}^{\frac{2j+3}{m}} f(u,0)\,du + \\ &+ \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{m,j}(x)\, b_{n,k}(y)\,\frac{mn}{4}\int_{\frac{2j+1}{m}}^{\frac{2j+3}{m}}\int_{\frac{2k+1}{n}}^{\frac{2k+3}{n}} f(u,v)\,du dv, \end{aligned}$$

$(x, y) \in R_0^2$, $m, n \in N$, where

$$(10)\qquad a_{m,n}(x,y) := \frac{1}{(1+\sinh mx)(1+\sinh ny)},$$

$$(11)\qquad b_{m,j}(x) := \frac{1}{(1+\sinh mx)}\frac{(mx)^{2j+1}}{(2j+1)!}, \qquad j \in N_0,$$

and $\sinh x$, $\cosh x$, $\tanh x$ are the elementary hyperbolic functions.

From (8) - (11) follows

$$(12)\qquad A_{m,n}(1;x,y) = 1 = B_{m,n}(1;x,y),$$

for all $(x, y) \in R_0^2$ and $m, n \in N$. Moreover we remark that $A_{m,n}$ and $B_{m,n}$ are a linear positive operators well - defined in every $C_{1;p_1,p_2}$ and $C_{2;p_1,p_2}$.

In Section 2 we shall give some properties of these operators. In Section 3 we shall prove the main approximation theorems.

1.4 The operators (8) and (9) are some analogues of the operators A_m and B_m of the Szasz - Mirakjan type introduced for functions f of one variable in the papers [5], [6], i.e.

$$A_m(f;x) := \frac{f(0)}{1+\sinh mx} + \sum_{j=0}^{\infty} b_{m,j}(x)\, f\left(\frac{2j+1}{m}\right), \tag{13}$$

$$B_m(f;x) := \frac{f(0)}{1+\sinh mx} + \sum_{j=0}^{\infty} b_{m,j}(x)\, \frac{m}{2}\int_{\frac{2j+1}{m}}^{\frac{2j+3}{m}} f(t)\,dt, \tag{14}$$

$x \in R_0$, $m \in N$, where $b_{m,j}(x)$ is defined by (11).

The Szasz - Mirakjan operators were examined in the polynomial and exponential weight spaces in the papers [1], [2].

In our paper by $M_{a,b}$ we shall denote the suitable positive constants depending only on indicated parameters a, b.

2. Auxiliary results.

From (8) - (11), (13) and (14) we deduce that if $f(x,y) = f_1(x)\, f_2(y)$, $(x,y) \in R_0^2$, and f belonging to some $C_{1;p_1,p_2}$ or $C_2; q_1, q_2$, then for all $m, n \in N$ and $(x,y) \in R_0^2$ holds

$$A_{m,n}\left(f_1(t) f_2(z); x, y\right) = A_m\left(f_1(t); x\right) A_n\left(f_2(z); y\right), \tag{15}$$

$$B_{m,n}\left(f_1(t) f_2(z); x, y\right) = B_m\left(f_1(t); x\right) B_n\left(f_2(z); y\right). \tag{16}$$

In [5] and [6] were proved the following two lemmas

Lemma 1. ([5]). *For every fixed $p \in N_0$ there exists a positive constant M_p such that for all $m \in N$ holds*

$$\sup_{x \in R_0} w_p(x)\, A_m\left(\frac{1}{w_p(t)}; x\right) \le M_p, \tag{17}$$

$$\sup_{x \in R_0} w_p(x)\, B_m\left(\frac{1}{w_p(t)}; x\right) \le M_p. \tag{18}$$

Moreover, for all $m \in N$ and $x \in R_0$ we have

$$w_p(x)\, A_m\left(\frac{(t-x)^2}{w_p(t)}; x\right) \le M_p \frac{x+1}{m}, \tag{19}$$

$$w_p(x)\, B_m\left(\frac{(t-x)^2}{w_p(t)}; x\right) \le M_p \frac{x+1}{m}.$$

Lemma 2. ([6]). *For every fixed $q > 0$ and $r > q$ there exists a positive constant $M_{q,r}$ and a natural number m_0, $m_0 > q\left(ln\frac{r}{q}\right)^{-1}$, such that*

$$\sup_{x\in R_0} v_r(x)\, A_m\left(\frac{1}{v_q(t)}; x\right) \le M_{q,r}$$

$$\sup_{x\in R_0} v_r(x)\, B_m\left(\frac{1}{v_q(t)}; x\right) \le M_{q,r}$$

for all $m > m_0$.

Moreover for all $m > m_0$ and $x \in R_0$ we have

$$v_r(x)\, A_m\left(\frac{(t-x)^2}{v_q(t)}; x\right) \le M_{q,r} \frac{x+1}{m},$$

$$v_r(x)\, B_m\left(\frac{(t-x)^2}{v_q(t)}; x\right) \le M_{q,r} \frac{x+1}{m}.$$

Using these lemmas, we shall prove some basic properties of the operators $A_{m,n}$ and $B_{m,n}$.

Lemma 3. *For a given $p_1, p_2 \in N_0$ there exists a positive constant M_{p_1,p_2} such that for all $m, n \in N$ holds*

$$\left\| A_{m,n}\left(\frac{1}{w_{p_1,p_2}(t,z)}; \cdot, \cdot\right)\right\|_{1;p_1,p_2} \le M_{p_1,p_2},$$

$$\left\| B_{m,n}\left(\frac{1}{w_{p_1,p_2}(t,z)}; \cdot, \cdot\right)\right\|_{1;p_1,p_2} \le M_{p_1,p_2}.$$

Proof. From (1) - (3), (15) and (16) we get

$$\left\| A_{m,n}\left(\frac{1}{w_{p_1,p_2}(t,z)};\cdot,\cdot\right)\right\|_{1;p_1,p_2} =$$
$$= \sup_{(x,y)\in R_0^2}\left[w_{p_1}(x)\,A_m\left(\frac{1}{w_{p_1}(t)};x\right)\right]\left[w_{p_2}(y)\,A_n\left(\frac{1}{w_{p_2}(z)};y\right)\right],$$

and analogously

$$\left\| B_{m,n}\left(\frac{1}{w_{p_1,p_2}(t,z)};\cdot,\cdot\right)\right\|_{1;p_1,p_2} =$$
$$= \sup_{(x,y)\in R_0^2}\left[w_{p_1}(x)\,B_m\left(\frac{1}{w_{p_1}(t)};x\right)\right]\left[w_{p_2}(y)\,B_n\left(\frac{1}{w_{p_2}(z)};y\right)\right],$$

for all $m,n\in N$. Now, using (17) and (18), we obtain the desired inequalities.

Similarly, using Lemma 2, we can prove the following

Lemma 4. *For a given* $q_1,q_2>0$ *and* $r_1>q_1$, $r_2>q_2$ *there exists a positive constant* M_{q_1,q_2,r_1,r_2} *and two natural numbers* m_0 *and* n_0

$$m_0 > q_1\left(ln\frac{r_1}{q_1}\right)^{-1}, \qquad n_0 > q_2\left(ln\frac{r_2}{q_2}\right)^{-1}. \tag{20}$$

such that

$$\left.\begin{array}{l}\left\| A_{m,n}\left(\dfrac{1}{v_{q_1,q_2}(t,z)}\cdot,\cdot\right)\right\|_{2,r_1,r_2}\\ \left\| B_{m,n}\left(\dfrac{1}{v_{q_1,q_2}(t,z)}\cdot,\cdot\right)\right\|_{2,r_1,r_2}\end{array}\right\} \le M_{q_1,q_2,r_1,r_2},$$

for all $m>m_0$ *and* $n>n_0$.

Applying Lemmas 3 and 4, we easily derive the next two lemmas.

Lemma 5. *Suppose that* $f\in C_{1;p_1,p_2}$ *with some* $p_1,p_2\in N_0$. *Then there exists a positive constant* M_{p_1,p_2} *such that for all* $m,n\in N$ *holds*

$$\left.\begin{array}{l}\|A_{m,n}(f;\cdot,\cdot)\|_{1;p_1,p_2}\\ \|B_{m,n}(f;\cdot,\cdot)\|_{1;p_1,p_2}\end{array}\right\} \le M_{p_1,p_2}\|f\|_{1;p_1,p_2}.$$

These inequalities and (8) - (11) show that $A_{m,n}$ and $B_{m,n}$, $m, n \in N$, are a linear positive operators from the space $C_{1;p_1,p_2}$ into $C_{1;p_1,p_2}$.

Lemma 6. *Suppose that $f \in C_{2;q_1,q_2}$ with some $q_1, q_2 > 0$ and let $r_1 > q_1$, $r_2 > q_2$. Then there exists a positive constant $M^* \equiv M_{q_1,q_2,r_1,r_2}$ and two natural numbers m_0 and n_0 satisfying the conditions (20) such that*

$$\left.\begin{array}{l} \left\| A_{m,n}(f;\cdot,\cdot)\right\|_{2;r_1,r_2} \\ \left\| B_{m,n}(f;\cdot,\cdot)\right\|_{2;r_1,r_2} \end{array}\right\} \leq M^* \|f\|_{2;q_1,q_2},$$

for all $m > m_0$, $n > n_0$.

The above inequalities and (8) - (11) show that $A_{m,n}$ and $B_{m,n}$ with $m > m_0$, $n > n_0$ are a linear positive operators from the space $C_{2;q_1,q_2}$ into $C_{2;r_1,r_2}$.

3. Approximation theorems.

In this part we shall estimate the degree of approximation of functions belonging to $C_{1;p_1,p_2}$ or $C_{2;q_1,q_2}$ by the operators $A_{m,n}$ and $B_{m,n}$.

Theorem 1. *Suppose that $g \in C^1_{1;p_1,p_2}$ with some $p_1, p_2 \in N_0$. Then there exists a positive constant M_{p_1,p_2} such that for all $(x, y) \in R_0^2$ and $m, n \in N$ holds*

$$\left.\begin{array}{l} w_{p_1,p_2}(x,y)|A_{m,n}(g;x,y) - g(x,y)| \\ w_{p_1,p_2}(x,y)|B_{m,n}(g;x,y) - g(x,y)| \end{array}\right\} \leq \tag{21}$$

$$\leq M_{p_1,p_2}\left\{\left\|\frac{\partial g}{\partial x}\right\|_{1;p_1,p_2}\sqrt{\frac{x+1}{m}} + \left\|\frac{\partial g}{\partial y}\right\|_{1;p_1,p_2}\sqrt{\frac{y+1}{n}}\right\}.$$

Proof. We shall prove (21) for $A_{m,n}$ only, because the proof for $B_{m,n}$ is analogous. Let $(x, y) \in R_0^2$ be a fixed point. By the assumption on the function g we can write for every $(t, z) \in R_0^2$

$$g(t,z) - g(x,y) = \int_x^t \frac{\partial}{\partial u} g(u,z)\,du + \int_y^z \frac{\partial}{\partial v} g(x,v)\,dv.$$

Form this and by (12) we get for $m, n \in N$

$$A_{m,n}(g;x,y) - g(x,y) = A_{m,n}\left(\int_x^t \frac{\partial}{\partial u} g(u,z)\,du; x, y\right) +$$

$$+ A_{m,n}\left(\int_y^z \frac{\partial}{\partial v} g(x,v)\,dv; x, y\right),$$

(22)
$$w_{p_1,p_2}(x,y)|A_{m,n}(g;x,y)-g(x,y)|\le$$
$$\le w_{p_1,p_2}(x,y)\left\{A_{m,n}\left(\left|\int_x^t \frac{\partial}{\partial u}g(u,z)\,du\right|;x,y\right)+\right.$$
$$\left.+A_{m,n}\left(\left|\int_y^z \frac{\partial}{\partial v}g(x,v)dv\right|;x,y\right)\right\}.$$

Arguing as in [3] or [4], we have

$$\left|\int_x^t \frac{\partial}{\partial u}g(u,z)\,du\right|\le\left\|\frac{\partial g}{\partial x}\right\|_{1;p_1,p_2}\left|\int_x^t\frac{du}{w_{p_1,p_2}(u,z)}\right|\le$$
$$\le\left\|\frac{\partial g}{\partial x}\right\|_{1;p_1,p_2}\left\{\frac{1}{w_{p_1,p_2}(t,z)}+\frac{1}{w_{p_1,p_2}(x,z)}\right\}|t-x|,$$

$$\left|\int_y^z \frac{\partial}{\partial v}g(x,v)\,dv\right|\le\left\|\frac{\partial g}{\partial y}\right\|_{1;p_1,p_2}\left|\int_y^z\frac{dv}{w_{p_1,p_2}(x,v)}\right|\le$$
$$\le\left\|\frac{\partial g}{\partial y}\right\|_{1;p_1,p_2}\left\{\frac{1}{w_{p_1,p_2}(x,z)}+\frac{1}{w_{p_1,p_2}(x,y)}\right\}|z-y|,$$

which by (1), (2), (8), (15) and (13) yield

$$w_{p_1,p_2}(x,y)A_{m,n}\left(\left|\int_x^t\frac{\partial}{\partial u}g(u,z)\,du\right|;x,y\right)\le$$
$$\le\left\|\frac{\partial g}{\partial x}\right\|_{1;p_1,p_2}w_{p_1}(x)w_{p_2}(y)\left\{A_{m,n}\left(\frac{|t-x|}{w_{p_1}(t)w_{p_2}(z)};x,y\right)+\right.$$
$$\left.+A_{m,n}\left(\frac{|t-x|}{w_{p_1}(x)w_{p_2}(z)};x,y\right)\right\}=$$
$$=\left\|\frac{\partial g}{\partial x}\right\|_{1;p_1,p_2}w_{p_2}(y)A_n\left(\frac{1}{w_{p_2}(z)};y\right)\left\{w_{p_1}(x)A_m\left(\frac{|t-x|}{w_{p_1}(t)};x\right)+\right.$$
$$\left.+A_m(|t-x|;x)\right\},$$

$$w_{p_1,p_2}(x,y)A_{m,n}\left(\left|\int_y^z\frac{\partial}{\partial v}g(x,v)dv\right|;x,y\right)\le$$
$$\le\left\|\frac{\partial g}{\partial y}\right\|_{1;p_1,p_2}\left\{w_{p_2}(y)A_n\left(\frac{|z-y|}{w_{p_2}(z)};y\right)+A_n(|z-y|;y)\right\}.$$

But, by the Hölder inequality and (17) and (19) follows from (13)

$$w_{p_1}(x)A_m\left(\frac{|t-x|}{w_{p_1}(t)};x\right) \le$$

$$\le \left\{w_{p_1}(x)A_m\left(\frac{(t-x)^2}{w_{p_1}(t)};x\right)\right\}^{\frac{1}{2}} \left\{w_{p_1}(x)A_m\left(\frac{1}{w_{p_1}(t)};x\right)\right\}^{\frac{1}{2}} \le$$

$$\le M_{p_1}\sqrt{\frac{x+1}{m}},$$

$$A_m(|t-x|;x) \le \left\{A_m((t-x)^2;x)\right\}^{\frac{1}{2}} \{A_m(1;x)\}^{\frac{1}{2}} \le 4\sqrt{\frac{x+1}{m}},$$

and analogously

$$w_{p_2}(y)A_n\left(\frac{|z-y|}{w_{p_2}(z)};y\right) \le M_{p_2}\sqrt{\frac{y+1}{n}},$$

$$A_n(|z-y|;y) \le 4\sqrt{\frac{y+1}{n}}.$$

Consequently,

$$w_{p_1,p_2}(x,y)A_{m,n}\left(\left|\int_x^t \frac{\partial}{\partial u}g(u,z)\,du\right|;x,y\right) \le M_{p_1,p_2}\left\|\frac{\partial g}{\partial x}\right\|_{1;p_1,p_2}\sqrt{\frac{x+1}{m}},$$

$$w_{p_1,p_2}(x,y)A_{m,n}\left(\left|\int_y^z \frac{\partial}{\partial v}g(x,v)\,dv\right|;x,y\right) \le M_{p_1,p_2}\left\|\frac{\partial g}{\partial y}\right\|_{1;p_1,p_2}\sqrt{\frac{y+1}{n}},$$

for all $m, n \in N$. Using these to (22), we obtain (21) for $A_{m,n}$.

Arguing as in proof of Theorem 1 and applying Lemma 2, we can prove the following.

Theorem 2. *Suppose that $g \in C^1_{2;q_1,q_2}$ with some $q_1, q_2 > 0$ and let $r_1 > q_1$, $r_2 > q_2$. Then there exists a positive constant $M^* \equiv M_{q_1,q_2,r_1,r_2}$ and two natural numbers m_0 and n_0 satisfying the conditions (20) such that*

$$\left.\begin{array}{l} v_{r_1,r_2}(x,y)|A_{m,n}(g;x,y)-g(x,y)| \\ v_{r_1,r_2}(x,y)|B_{m,n}(g;x,y)-g(x,y)| \end{array}\right\} \le$$

$$\le M^*\left\{\left\|\frac{\partial g}{\partial x}\right\|_{2;q_1,q_2}\sqrt{\frac{x+1}{m}} + \left\|\frac{\partial g}{\partial y}\right\|_{2;q_1,q_2}\sqrt{\frac{y+1}{n}}\right\},$$

for all $(x,y) \in R_0^2$ and $m > m_0$, $n > n_0$.

Theorem 3. *Let $f \in C_{1;p_1,p_2}$ with some $p_1, p_2 \in N_0$. Then there exists a positive constant M_{p_1,p_2} such that for all $(x,y) \in R_0^2$ and $m,n \in N$ we have*

$$\left.\begin{array}{l} w_{p_1,p_2}(x,y)|A_{m,n}(f;x,y)-f(x,y)| \\ w_{p_1,p_2}(x,y)|B_{m,n}(f;x,y)-f(x,y)| \end{array}\right\} \le \tag{23}$$

$$\le M_{p_1,p_2}\,\omega\left(f, C_{1;p_1,p_2}; \sqrt{\frac{x+1}{m}}, \sqrt{\frac{y+1}{n}}\right),$$

where $\omega(f, C_{1;p_1,p_2}; \cdot, \cdot)$ is defined by (4).

Proof. We shall prove only (23) for $A_{m,n}$, because the proof of (23) for $B_{m,n}$ is analogous. We shall use the Stieklov means of $f \in C_{1;p_1,p_2}$ defined by the formula

$$f_{h,\delta}(x,y) := \frac{1}{h\delta}\int_0^h\int_0^\delta f(x+u, y+v)\,dudv\,, \quad (x,y)\in R_0^2 \quad h,\delta > 0\,.$$

We have

$$f_{h,\delta}(x,y) - f(x,y) = \frac{1}{h\delta}\int_0^h\int_0^\delta \Delta_{u,y} f(x,y)\,dudv\,,$$

$$\frac{\partial}{\partial x} f_{h,\delta}(x,y) = \frac{1}{h\delta}\int_0^\delta\int_0^\delta \Delta_{h,0} f(x, y+v)\,dv\,,$$

$$\frac{\partial}{\partial y} f_{h,\delta}(x,y) = \frac{1}{h\delta}\int_0^h\int_0^h \Delta_{0,\delta} f(x+u, y)\,du\,,$$

which imply $f_{h,\delta} \in C^1_{1;p_1,p_2}$ for every fixed $h, \delta > 0$. Moreover from this follows

$$\|f_{h,\delta} - f\|_{1;p_1,p_2} \le \omega(f, C_{1;p_1,p_2}; h, \delta)\,, \tag{24}$$

$$\left\|\frac{\partial f_{h,\delta}}{\partial x}\right\|_{1;p_1,p_2} \le 2h^{-1}\omega(f, C_{1;p_1,p_2}; h, \delta)\,, \tag{25}$$

$$\left\|\frac{\partial f_{h,\delta}}{\partial y}\right\|_{1;p_1,p_2} \le 2\delta^{-1}\omega(f, C_{1;p_1,p_2}; h, \delta)\,, \tag{26}$$

for all $h, \delta > 0$.

By (8) and (12) is clearly that for every fixed $(x, y) \in R_0^2$, $h, \delta > 0$ and $m, n \in N$ holds

$$(27) \qquad w_{p_1,p_2}(x,y)|A_{m,n}(f;x,y) - f(x,y)| \le$$
$$\le w_{p_1,p_2}(x,y)\Big\{|A_{m,n}(f - f_{h,\delta};x,y)| +$$
$$+ |A_{m,n}(f_{h,\delta};x,y) - f_{h,\delta}(x,y)| +$$
$$+ |f_{h,\delta}(x,y)| + |f_{h,\delta}(x,y) - f(x,y)|\Big\}.$$

Using Lemma 5 and (24), we get

$$w_{p_1,p_2}(x,y)|A_{m,n}(f - f_{h,\delta};x,y)| \le$$
$$\le M_{p_1,p_2}\|f - f_{h,\delta}\|_{1;p_1,p_2} \le M_{p_1,p_2}\,\omega(f; C_{1;p_1,p_2}; h, \delta).$$

Applying Theorem 1 and (25) and (26), we get

$$w_{p_1,p_2}|A_{m,n}(f_{h,\delta};x,y) - f_{h,\delta}(x,y)| \le$$
$$\le M_{p_1,p_2}\left\{\left\|\frac{\partial f_{h,\delta}}{\partial x}\right\|_{1;p_1,p_2}\sqrt{\frac{x+1}{m}} + \left\|\frac{\partial f_{h,\delta}}{\partial y}\right\|_{1;p_1,p_2}\sqrt{\frac{y+1}{n}}\right\} \le$$
$$\le M_{p_1,p_2}\,\omega(f, C_{1;p_1,p_2}; h, \delta)\left\{h^{-1}\sqrt{\frac{x+1}{m}} + \delta^{-1}\sqrt{\frac{y+1}{n}}\right\}.$$

Summing up, we obtain

$$(28) \quad w_{p_1,p_2}(x,y)|A_{m,n}(f;x,y) - f(x,y)| \le$$
$$\le M_{p_1,p_2}\omega(f, C_{1;p_1,p_2}; h, \delta)\left\{1 + h^{-1}\sqrt{\frac{x+1}{m}} + \delta^{-1}\sqrt{\frac{y+1}{n}}\right\},$$

for all $(x, y) \in R_0^2$, $m, n \in N$ and $h, \delta > 0$.

For a given $(x, y) \in R_0^2$ and $m, n \in N$ setting $h = \sqrt{\frac{x+1}{m}}$ and $\delta = \sqrt{\frac{y+1}{n}}$ to (28), we obtain (23) for $A_{m,n}$ and we complete the proof.

Similarly we shall prove

Theorem 4. *Suppose that* $f \in C_{2;q_1,q_2}$ *with some* $q_1, q_2 > 0$ *and* $r_1 > q_1$, $r_2 > q_2$. *Then there a positive constant* $M^* = M_{q_1,q_2,r_1,r_2}$ *and two natural numbers* m_0 *and* n_0 *satisfying the conditions (20) such that*

$$\left.\begin{array}{l} v_{r_1,r_2}|A_{m,n}(f;x,y)-f(x,y)| \\ v_{r_1,r_2}|B_{m,n}(f;x,y)-f(x,y)| \end{array}\right\} \leq \tag{29}$$

$$\leq M^* \omega\left(f, C_{2;q_1,q_2}; \sqrt{\frac{x+1}{m}}; \sqrt{\frac{y+1}{n}}\right),$$

for all $(x, y) \in R_0^2$ *and* $m > m_0$, $n > n_0$.

Proof. Similarly as in the proof of Theorem 3 we apply the Stieklov means $f_{h,\delta}$ of $f \in C_{2;q_1,q_2}$. Now we get by (5) - (7)

$$\|f_{h,\delta} - f\|_{2;r_1,r_2} \leq \|f_{h,\delta} - f\|_{2;q_1,q_2} \leq \omega(f, C_{2;q_1,q_2}; h, \delta) \tag{30}$$

and analogously

$$\left\|\frac{\partial f_{h,\delta}}{\partial x}\right\|_{2;r_1,r_2} \leq 2h^{-1}\, \omega(f, C_{2;q_1,q_2}; h, \delta), \tag{31}$$

$$\left\|\frac{\partial f_{h,\delta}}{\partial y}\right\|_{2;r_1,r_2} \leq 2\delta^{-1}\, \omega(f, C_{2;q_1,q_2}; h, \delta), \tag{32}$$

for all $h, \delta > 0$. Moreover, we can write the analogy of (27) for $A_{m,n}$

$$w_{r_1,r_2}(x,y)|A_{m,n}(f;x,y)-f(x,y)| \leq$$

$$\leq v_{r_1,r_2}(x,y)\Big\{|A_{m,n}(f - f_{h,\delta}; x, y)| +$$

$$+ |A_{m,n}(f_{h,\delta}; x, y) - f_{h,\delta}(x,y)| + |f_{h,\delta}(x,y) - f(x,y)|\Big\},$$

$(x, y) \in R_0^2$, $m, n \in N$, $h, \delta > 0$.

Now, arguing as in the proof of Theorem 3 and using Lemma 6, Theorem 2 and (30) - (32), we obtain the estimation of the type (28), i.e.

$$v_{r_1,r_2}(x,y)|A_{m,n}(f;x,y)-f(x,y)| \leq \tag{33}$$

$$\leq M_{q_1,q_2,r_1,r_2}\, \omega\left(f, C_{2;q_1,q_2}; h, \delta\right)\left\{1 + h^{-1}\sqrt{\frac{x+1}{m}} + \delta^{-1}\sqrt{\frac{y+1}{n}}\right\},$$

for all $(x, y) \in R_0^2$, $h, \delta > 0$ and $m > m_0$, $n > n_0$. Analogously as in the proof of Theorem 3 we immediately obtain the desired estimation (29) for $A_{m,n}$ from (33).

The proof (29) for $B_{m,n}$ is identical.

Theorems 3, 4 imply the following corollaries.

Corollary 1. *If* $f \in C_{1;p_1,p_2}$, $p_1, p_2 \in N_0$, *or* $f \in C_{2;q_1,q_2}$, $q_1, q_2 > 0$, *then*

$$\lim_{m,n\to\infty} A_{m,n}(f; x, y) = f(x, y), \tag{34}$$

$$\lim_{m,n\to\infty} B_{m,n}(f; x, y) = f(x, y), \tag{35}$$

for every fixed $(x, y) \in R_0^2$. *Moreover, (34) and (35) hold uniformly on every rectangle* $0 \le x \le a$, $0 \le y \le b$.

Corollary 2. *If* $f \in Lip(C_{1;p_1,p_2}, \alpha, \beta)$ *with some* $p_1, p_2 \in N_0$ *and* $0 < \alpha$, $\beta \le 1$, *then for every fixed* $(x, y) \in R_0^2$ *holds*

$$w_{p_1,p_2}(x, y)|A_{m,n}(f; x, y) - f(x, y)| = O\left(\left(\frac{x+1}{m}\right)^{\frac{\alpha}{2}} + \left(\frac{y+1}{n}\right)^{\frac{\beta}{2}}\right)$$

$$as \quad m, n \to \infty.$$

The similar assertion we can write from Theorem 4.

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