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A CONTRIBUTE TO THEORY OF FRACTIONAL INTEGRAL OPERATORS FOR CERTAIN CLASSES OF GENERALIZED FUNCTIONS

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Lowndes has defined the new operators of fractional integration which generalize the operators due to Erdelyi and Kober. The purpose of this paper, is to develop a theory of these fractional integral operators given by Lowndes for a certain classes of generalized functions.

1. Introduction.

In [6] Lowndes has defined the new operators of fractional integration which generalize the operators due to Erdelyi and Kober [2], [5]. The purpose of this paper, is to develope a theory of these fractional integral operators given by Lowndes for a certain classes of generalized functions. It is expected that the solution of certain dual integral equations involving special functions of applied mathematics, applied physics and engineering sciences can be obtained by the application of such operators to generalized functions which have applications in electromagnetic theory. The resulting fractional integrals also have applications to the Hankel transformation, to some singular differential operators and to certain integral equations of the first kind.

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2. Fractional integral operators.

The Riemann-Liouville integral of order $\alpha > 0$ of a function $f \in L_{\mathrm{loc}}[0, \infty]$ is defined by

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - y)^{\alpha-1} f(y)\, dy, \quad x > 0. \tag{2.1}$$

If α is an integer, this is simply the α-times repeated integral of f with lower limit 0.

The Weyl integral of order α which, for $\alpha > 0$ and $f \in L_{\mathrm{loc}}(0, \infty)$ with suitable behaviour at infinity, is defined by

$$K^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (y - x)^{\alpha-1} f(y)\, dy. \tag{2.2}$$

Riemann-Liouville integrals (of fractional order) of distributions can be obtained from the convolution theory of distributions whose support is bounded on the left [4]. It is more difficult to define Weyl integrals of distributions; and multiplication by powers of the variables or integration of fractional order with respect to a power of the variable, both of which occur in applications are not feasible in distribution theory. An alternative approach is based on the remark that the operator of Riemann-Liouville integrals and that of Weyl integrals are adjoint to each other and which is based on the formula for fractional integration by parts.

By formal computation (which can be justified under appropriate conditions by Fubini's theorem) we have

$$\int_0^\infty I^\alpha f(x) g(x)\, dx = \int_0^\infty f(x) K^\alpha g(x)\, dx.$$

This can be written as

$$\langle I^\alpha f, g\rangle = \langle f, K^\alpha g\rangle \tag{2.3}$$

and exhibits I^α and K^α as adjoint operators in some sense.

We shall adhere to this adjoint operator approach in our present work.

Erdelyi and Kober investigated the properties of the fractional integral

$$\frac{x^{-\eta-\alpha+1}}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} t^{\eta-1} f(t)\, dt, \quad (\alpha > 0, \eta > 0) \tag{2.4}$$

which is obviously a generalization on the Riemann fractional integral (2.1) and the integral

$$\frac{x^{\eta}}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\eta}f(t)\,dt, \quad (\alpha>0,\eta>0) \tag{2.5}$$

which is generalization of Weyl integral (2.2).

We define the Erdelyi-Kober operators [8] for $m>0$, $\operatorname{Re}\eta>1/m-1$ and $\operatorname{Re}\alpha>0$ as under

$$I^m_{\eta,\alpha}f(x)=\frac{mx^{-m^{\eta}-m\alpha}}{\Gamma(\alpha)}\int_0^x(x^m-u^m)^{\alpha-1}u^{m^{\eta}+m-1}f(u)\,du\,, \tag{2.6}$$

$$K^m_{\eta,\alpha}f(x)=\frac{mx^{m^{\eta}}}{\Gamma(\alpha)}\int_x^{\infty}(u^m-x^m)^{\alpha-1}u^{-m^{\eta}-m\alpha+m-1}f(u)\,du. \tag{2.7}$$

These operators are simple modifications of (2.1) and (2.2). If we let α tend to zero in equations (2.6) and (2.7) and make use of the results $I^0=I$, $K^0=I$ [8], it is easy to see that

$$I^m_{\eta,0}=I, \quad K^m_{\eta,0}=I \tag{2.8}$$

where I is the identity operator.

In a similar fashion, we shall define slight extension of the generalized Erdelyi-Kober operators defined by Lowndes [6] for $m>0$, $\operatorname{Re}\alpha>0$, $\operatorname{Re}\eta>1/m-1$ and $\operatorname{Re}\xi>1/m-1$ in the forms

$$\begin{aligned} I^m_k(\eta,\alpha)f(x)&=m2^{\alpha-1}x^{-m^{\eta}-m\alpha}\int_0^x u^{m^{\eta}+m-1}\,\cdot \\ &\cdot(x^m-u^m)^{\alpha/2-1/2}J_{\alpha-1}\left[k\sqrt{(x^m-u^m)}\right]f(u)\,du \\ &=2^{\alpha-1}x^{-m\alpha/2+m/2}k^{1-\alpha}\int_0^1 y^{m^{\eta}}(1-y^m)^{\alpha/2-1/2}\,\cdot \\ &\cdot J_{\alpha-1}\left[kx^{m/2}\sqrt{(1-y^m)}\right]f(xy)\,dy^m, \end{aligned} \tag{2.9}$$

$$K^m_k(\xi,\alpha)f(x)=m2^{\alpha-1}x^{m\xi}k^{1-\alpha}\int_x^{\infty}u^{-m\xi-m\alpha+m-1}(u^m-x^m)^{\alpha/2-1/2}\,. \tag{2.10}$$

$$\cdot J_{\alpha-1}\left[k\sqrt{(u^m - x^m)}\right] f(u)\,du = 2^{\alpha-1}x^{-m\alpha/2+m/2}k^{1-\alpha}\,.$$

$$\cdot \int_1^\infty y^{-m\xi-m\alpha}(y^m-1)^{\alpha/2-1/2}J_{\alpha-1}\left[kx^{m/2}\sqrt{(y^m-1)}\right] f(xy)\,dy^m$$

and the operators $I_{ik}^m(\eta,\alpha)$ and $K_{ik}^m(\xi,\alpha)$ by the above equations when $J_{\alpha-1}$ is replaced by $I_{\alpha-1}$ i.e.

(2.11) $$I_{ik}^m(\eta,\alpha) f(x) = 2^{\alpha-1}x^{-m\alpha/2+m/2}k^{1-\alpha}\,.$$

$$\cdot \int_0^1 y^{m^\eta}(1-y^m)^{\alpha/2-1/2}I_{\alpha-1}\left[kx^{m/2}\sqrt{(1-y^m)}\right] f(xy)\,dy^m$$

(2.12) $$K_{ik}^m(\xi,\alpha) f(x) = 2^{\alpha-1}x^{-m\alpha/2+m/2}k^{1-\alpha}\,.$$

$$\cdot \int_1^\infty y^{-m\xi-m\alpha}(y^m-1)^{\alpha/2-1/2}I_{\alpha-1}\left[kx^{m/2}\sqrt{(y^m-1)}\right] f(xy)\,dy^m$$

where $J_\nu(z)$ and $I_\nu(z)$ are Bessel function and modified Bessel function of order ν, respectively.

If we let k tend to zero we see that these operators are related to the Erdelyi-Kober operators $I^m_{\eta,\alpha}$ and $K^m_{\eta,\alpha}$ through the formulae

(2.13) $$I_0^m(\eta,\alpha) = I^m_{\eta,\alpha}, \quad K_0^m(\xi,\alpha) = K^m_{\xi,\alpha}\,.$$

Letting α tend to zero and using (2.8) we have the equations

(2.14) $$I_0^m(\eta,0) = I, \quad K_0^m(\xi,0) = I$$

where I is the identity operator.

It follows from (2.3) that $I_k^m(\eta,\alpha)$ and $K_k^m(\xi,\alpha)$ are adjoint if

(2.15) $$\xi = \eta + 1 - m^{-1}.$$

We shall now introduce the spaces of generalized functions for which I_k^m, K_k^m and I_{ik}^m, K_{ik}^m can be defined, using freely the conventions, many of the notations, and some of results developed in [9], [3].

3. The spaces.

For $a \in \mathbb{R}$, $\phi \in C^{\infty}(0, \infty)$ we define the seminorms

$$\lambda_{a,r}(\phi) \triangleq \sup\left\{x^{1-a+r}\left|\phi^{(r)}(x)\right|: x > 0\right\}, \quad r = 0, 1, 2, \dots \tag{3.1}$$

where $\lambda_{a,0}$ is a norm; and for $l > 0$ we define a space of testing functions,

$$g_{a,l} \triangleq \Big\{\phi: \phi \in C^{\infty}(0, \infty), \phi(x) = 0 \quad \text{for} \quad x > l, \tag{3.2}$$

$$\lambda_{a,r}(\phi) < \infty,\ r = 0, 1, 2, \dots\Big\}.$$

With the topology by $\{\lambda_{a,r}: r = 0, 1, 2, \dots\}$, $g_{a,l}$ is a complete countably multinormed space, and

$$g_a \triangleq \bigcup_{l=1}^{\infty} J_{a+1/l,l} \tag{3.3}$$

is a complete countable union space which contains $D(0, \infty)$ as a dense subspace. For details of the spaces see Erdelyi [3].

4. Fractional integrals.

In order to define $I_k^m(\eta, \alpha)$ on g', we must consider first $K_k^m(\xi, \alpha)$ and $K_{ik}^m(\xi, \alpha)$ on g. For $\phi \in g_a$ we have, for some $c > a$, and $l > 0$,

$$\left|\phi^{(r)}(x)\right| \le \lambda_{c,r}(\phi)x^{c-r-1}, \ x > 0; \ \phi(x) = 0, \ x > l. \tag{4.1}$$

It follows from (2.10) and (2.12) that $K_k^m(\xi, \alpha)$ and $K_{ik}^m(\xi, \alpha)$ exist for $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\xi > 1/m - 1$ and are smooth functions since the integrals are finite integrals, and differentiations under the integrals signs are permissible in the second form of (2.10) and (2.12).

For simplifications we make use of the results

$$x^{-m\alpha+m/2} J_{\alpha-1}\left[x^{m/2}k\sqrt{(y^m - 1)}\right] = k^{\alpha-1}2^{1-\alpha}(y^m - 1)^{\alpha/2-1/2} \cdot$$

$$\cdot\, H_{0,2}^{1,0}\left[\frac{x^m k^2(y^m - 1)}{4}\middle|(0, 1)(1 - \alpha, 1)\right],$$

$$I_n(z) = (i)^{-n} J_n(iz) = \frac{(z/2)^n}{\Gamma(1+n)} \, {}_0F_1\left[-; 1+n; z^{2/4}\right]$$

where n is not a negative integer, and some differentiation formulae of H-function [7] which is defined as a Mellin-Barnes type integral and is discussed in details by Braaksma [1]. It is obvious that

$$\left(\frac{d}{dx}\right)^r K_k^m(\xi,\alpha)\phi(x) = \sum_{n=0}^{r} \binom{r}{n} x^{n-r} \int_1^\infty y^{n-m\xi-m\alpha}(y^m-1)^{\alpha-1} \cdot$$

$$\cdot\, H_{1,3}^{1,1}\left[\frac{x^m k^2(y^m-1)}{4}\,\middle|\, \begin{matrix}(0,m)\\(0,1)(1-\alpha,1)(r-n,m)\end{matrix}\right]\phi^{(n)}(xy)\,dy^m.$$

Then assuming

$$a < c < m\operatorname{Re}\xi + 1 \tag{4.2}$$

we have

$$\left|\left(\frac{d}{dx}\right)^r K_k^m(\xi,\alpha)\phi(x)\right| \le x^{c-r-1}\sum_{n=0}^{r}\binom{r}{n}\lambda_{c,n}(\phi)\,\cdot$$

$$\cdot \int_1^\infty \left| y^{-m\xi-m\alpha+c-1}(y^m-1)^{\alpha-1} \cdot\right.$$

$$\left.\cdot\, H_{1,3}^{1,1}\left[\frac{x^m k^2(y^m-1)}{4}\,\middle|\, \begin{matrix}(0,m)\\(0,1)(1-\alpha,1)(r-n,m)\end{matrix}\right]\right| dy^m,$$

and as the integral is convergent under the condition (4.2) we have

$$\lambda_{c,r}\left[K_k^m(\xi,\alpha)\phi\right] \le C\sum_{n=0}^{r}\binom{r}{n}\lambda_{c,n}(\phi) \tag{4.3}$$

for some $c > 0$. Also $\left[K_k^m(\xi,\alpha)\phi(x)\right] = 0$ if $x > l$ showing that, under (4.2), $K_k^m(\xi,\alpha)$ maps each $g_{c,l}$ and hence maps also g_a into itself. The map is clearly linear and by (4.3) it is also continuous.

Similarly we can show that

$$\lambda_{c,r}\left[K_{ik}^m(\xi,\alpha)\phi\right] \le C\sum_{n=0}^{r}\binom{r}{n}\lambda_{c,n}(\phi)$$

for some $c > 0$ under the condition $a < c < m\operatorname{Re}\xi + 1$, showing that $K_{ik}^m(\xi,\alpha)$ maps each $g_{c,l}$ and hence maps also g_a into itself. Hence the map is linear and continuous.

$K_k^m(\xi,\alpha)$ is analytic function of α for $\operatorname{Re}\alpha > 0$, in the sense that

$$\frac{1}{h}\left[K_k^m(\xi,\alpha+h)\phi - K_k^m(\xi,\alpha)\phi\right] \to \frac{\partial}{\partial\alpha}K_k^m(\xi,\alpha)\phi, \text{ as } h \to 0$$

in the topology of g_a. Similarly, $K_{ik}^m(\xi,\alpha)$ is analytic function of α for $\operatorname{Re}\alpha > 0$. We shall now to show how $K_k^m(\xi,\alpha)\phi$ can be continued analytically.

For $\operatorname{Re}\alpha > 0$, we have from the second form of (2.10)

$$\delta K_k^m(\xi,\alpha+1)\phi = K_k^m(\xi,\alpha+1)\delta\phi$$

where $\delta = x\dfrac{d}{dx}$, and by integration by parts this becomes

$$(m\xi + m\alpha)K_k^m(\xi,\alpha+1)\phi - m\,K_k^m(\xi,\alpha)\phi,$$

so that

$$K_k^m(\xi,\alpha) = \left(\xi+\alpha-\frac{1}{m}\delta\right)K_k^m(\xi,\alpha+1)$$

and more generally

$$K_k^m(\xi,\alpha) = \prod_{j=0}^{l-1}\left(\xi+\alpha+j-\frac{1}{m}\delta\right)K_k^m(\xi,\alpha+l), \quad l = 1,2,\dots . \tag{4.4}$$

Similarly for $K_{ik}^m(\xi,\alpha)$ in which case k is to be replaced by ik in above. This makes it possible to extend the definition of $K_k^m(\xi,\alpha)$ and $K_{ik}^m(\xi,\alpha)$ to $\operatorname{Re}\alpha > -l$ and since l is arbitrary positive integer, ultimately to all complex numbers α. For fixed α, $K_k^m(\xi,\alpha)$ and $K_{ik}^m(\xi,\alpha)$ are continuous linear maps of g_a, with a condition satisfying (4.2), into itself. $K_k^m(\xi,\alpha)$, $K_{ik}^m(\xi,\alpha)$ are an entire functions of α in the sense that its derivatives with respect to α exist for all α as a limit in the weak topology of continuous linear operators on g_a.

In particular,

$$K_0^m(\xi,0)\phi(x) = K_0^m(\xi,1)\left(\xi-\frac{1}{m}\delta\right)\phi(x) = K_{\xi,1}^m\left(\xi-\frac{1}{m}\delta\right)\phi(x)$$

$$= x^{m\xi}\int_x^\infty y^{-m\xi-1}\left(m\xi\phi(y) - y\frac{\partial}{\partial y}\phi(y)\right)dy = -x^{m\xi}\int_x^\infty \frac{d}{dy}\left\{y^{-m\xi}\phi(y)\right\}dy,$$

so that

$$K_0^m(\xi,0)\phi = \phi \tag{4.5}$$

and similarly

$$K_k^m(\xi, -n)\phi = \prod_{j=1}^{n}\left(\xi - j - \frac{1}{m}\delta\right) K_k^m(\xi + n, 0)\phi. \tag{4.6}$$

Similar results are true for K_{ik}^m.

The addition theorem

$$K_{ik}^m(\xi + \alpha, \zeta) K_k^m(\xi, \alpha)\phi = K_k^m(\xi + \alpha, \zeta) K_{ik}^m(\xi, \alpha)\phi \tag{4.7}$$

$$= K_{\xi, \alpha+\zeta}^m \phi, \quad \phi \in g_a$$

can be proved under the condition

$$a < 1 + \min\{m \operatorname{Re}\xi, m \operatorname{Re}(\xi + \alpha)\}. \tag{4.8}$$

It follows from (4.7) in particular that under (4.8) $K_k^m(\xi, \alpha)$ and $K_{ik}^m(\xi, \alpha)$ are automorphisms of g_a with inverses $K_{ik}^m(\xi + \alpha, -\alpha)$ and $K_k^m(\xi + \alpha, -\alpha)$ respectively since $K_{\xi,\alpha+\zeta}^m \phi \in g_a$ for $\phi \in g_a$ [3].

We are ready to define $I_k^m(\eta, \alpha) f$ and $I_{ik}^m(\eta, \alpha) f$ for $f \in g_a'$ with

$$a < m \operatorname{Re}\eta + m \tag{4.9}$$

by

$$\langle I_k^m(\eta, \alpha) f, \phi\rangle = \langle f, K_k^m(\xi, \alpha)\phi\rangle, \ \phi \in g_a \tag{4.10}$$

and

$$\langle I_{ik}^m(\eta, \alpha) f, \phi\rangle = \langle f, K_{ik}^m(\xi, \alpha)\phi\rangle, \ \phi \in g_a$$

where η and ξ are connected, as always by (2.15) so that (4.2) holds. For fixed α, $I_k^m(\eta, \alpha)$ is a continuous linear operator on g_a' as a function of α it is an entire function. For regular elements of g_a' generated by conventional functions of the kind as described in [3], and for $\operatorname{Re}\alpha > 0$ fractional integration by parts shows that (4.10) and (2.9) are in agreement.

$$I_0^m(\eta, 0) f = f, \tag{4.11}$$

$$I_{ik}^m(\eta + \alpha, \zeta) I_k^m(\eta, \alpha) f = I_k^m(\eta + \alpha, \zeta) I_{ik}^m(\eta, \alpha) f = I_{\eta, \alpha+\zeta}^m f, \tag{4.12}$$

provided

$$a < m + \min\{m \operatorname{Re}\eta, m \operatorname{Re}(\eta+\alpha)\}; \tag{4.13}$$

and under this latter condition $I_k^m(\eta, \alpha)$ is an automorphism of g'_a with inverse $I_{ik}^m(\eta+\alpha, -\alpha)$ since $I_{\eta,\alpha+\zeta}^m f \in g'_a$ for $f \in g'_a$ [3].

Also

$$I_k^m(\eta, \alpha) = \prod_{j=1}^{l}\left(\eta+\alpha+j+\frac{1}{m}\delta\right) I_k^m(\eta, \alpha+l),\ l = 1, 2, \dots . \tag{4.14}$$

Here we have used the fact that the adjoint of $-\delta$ is $\delta' = 1+\delta$ and in particular

$$I_k^m(\eta, -n) = \prod_{j=0}^{n-1}\left(\eta - j + \frac{1}{m}\delta\right) I_k^m(\eta, 0),\ n = 1, 2. \tag{4.15}$$

All of these results follow directly from (4.10) and the properties of $K_k^m(\xi, \alpha)$ on g_a by the known properties of the adjoint operators. Similar statements are true for $I_{ik}^m(\eta, \alpha)$.

5. The Spaces k and fractional integrals on k'.

For $b \in \mathbb{R}$ and $l > 0$, we consider the space of testing functions

$$k_{b,l} = \left\{\phi\colon \phi \in C^\infty(0,\infty), \phi(x) = 0 \quad \text{for} \quad x < \frac{1}{l}, \right. \\ \left. \lambda_{b,r}(\phi) < \infty,\ r = 0, 1, 2, \dots\right\}. \tag{5.1}$$

With topology determined by $\{\lambda_{b,r} : r = 0, 1, 2, \dots\}$, $k_{b,l}$ is a complete countably multinormed space, and

$$k_b = \bigcup_{l=1}^{\infty} k_{b-1/l,l} \tag{5.2}$$

is a complete countable union space which contains $D(0, \infty)$ as dense subspace. For details of the spaces k and k' see Erdelyi [3].

For $\phi \in k_b$ and $\operatorname{Re}\alpha > 0$, (2.10) can be used to define $I_k^m(\eta, \alpha)\phi$. With this definition (4.15) is valid and make it possible to extend the definition of $I_k^m(\eta, \alpha)$ to all complex values of α. For $\phi \in k_b$ we have for some $c < b$ and $l > 0$,

$$\left|\phi^{(r)}(x)\right| \le \lambda_{c,r}(\phi) x^{c-r-1},\ x > 0;\ \phi(x) = 0,\ x < 1/l. \tag{5.3}$$

It follows from (2.9) and (2.11) that $I_k^m(\eta,\alpha)$ and $I_{ik}^m(\eta,\alpha)$ exist for $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\eta > 1/m - 1$ and are smooth functions since the integrals are finite and differentiation under the integral signs are permissible in the second form of (2.9) and (2.11).

It is easy to see that

$$\left(\frac{d}{dx}\right)^r I_k^m(\eta,\alpha)\phi(x) = \sum_{n=0}^{r}\binom{r}{n} x^{n-r}\int_0^1 y^{n+m^\eta}(1-y^m)^{\alpha-1}\cdot$$

$$\cdot H_{1,3}^{1,1}\left[\frac{x^m k^2(1-y^m)}{4}\middle|\begin{matrix}(0,m)\\(0,1)(1-\alpha,1)(r-n,m)\end{matrix}\right]\phi^{(n)}(xy)\,dy^m.$$

Therefore

$$\left|\left(\frac{d}{dx}\right)^r I_k^m(\eta,\alpha)\phi(x)\right| \le x^{c-r-1}\sum_{n=0}^{r}\binom{r}{n}\lambda_{c,n}(\phi)\cdot$$

$$\cdot\int_0^1\left|y^{c+m^\eta-1}(1-y^m)^{\alpha-1}H_{1,3}^{1,1}\left[\frac{x^m k^2(1-y^m)}{4}\middle|\begin{matrix}(0,m)\\(0,1)(1-\alpha,1)(r-n,m)\end{matrix}\right]\right|dy^m$$

under the condition

$$b > c > 1 - m - m\operatorname{Re}\eta. \tag{5.4}$$

Hence

$$\lambda_{c,r}\left[I_k^m(\eta,\alpha)\phi\right] \le C\sum_{n=0}^{r}\binom{r}{n}\lambda_{c,n}(\phi) \tag{5.5}$$

for some $c > 0$.

Also, $I_k^m(\eta,\alpha)\phi = 0$ if $x < 1/l$, showing that under (5.4), $I_k^m(\eta,\alpha)$ maps each $k_{b,l}$ and hence maps also k_b, into itself. The map is clearly linear and by (5.5) it is also continuous, and it is an entire function of α. The equations (4.11) (with f replaced by $\phi\in k_b$) and (4.15) hold and if

$$b > 1 - m - \min\{m\operatorname{Re}\eta,\, m\operatorname{Re}(\eta+\alpha)\} \tag{5.6}$$

then also (4.12) (with $\phi\in k_b$ in place of f) holds. Under the condition (5.6), $I_k^m(\eta,\alpha)$ and $I_{ik}^m(\eta,\alpha)$ are automorphisms of k_b with inverses $I_{ik}^m(\eta+\alpha,-\alpha)$ and $I_k^m(\eta+\alpha,-\alpha)$ respectively since $I_{ik}^m(\eta,\alpha)$ also maps k_b, into itself which can be proved in the similar manner.

For $\phi \in k_b$, $f \in k_b'$ and $b > -m \operatorname{Re} \xi$, $K_k^m(\xi, \alpha)$ and $K_{ik}^m(\xi, \alpha)$ can be defined by

$$(5.7) \qquad \langle K_k^m(\xi, \alpha) f, \phi \rangle = \langle f, I_k^m(\eta, \alpha)\phi \rangle, \quad \phi \in k_b\,,$$

and

$$(5.8) \qquad \langle K_{ik}^m(\xi, \alpha) f, \phi \rangle = \langle f, I_{ik}^m(\eta, \alpha)\phi \rangle, \quad \phi \in k_b\,,$$

with η and ξ connected as in (2.15). This definition is in agreement with (2.10) in case both apply. $K_k^m(\xi, \alpha)$ is continuous injection of k_b' into itself, it is an entire function of α, it satisfies (4.4), (4.5) and (4.6) (with $f \in k_{b'}$ replacing ϕ), if

$$(5.9) \qquad b > -\min\{m \operatorname{Re} \xi, m \operatorname{Re}(\xi + \alpha)\}$$

then it also satisfies (4.7) and under (5.9) it is an automorphism of $k_{b'}$ with inverses $K_{ik}^m(\xi + \alpha, -\alpha)$. Similar statements are also true for $K_{ik}^m(\xi, \alpha) f$, $f \in k_{b'}$.

6. The spaces m, m' and fractional integrals on them.

We shall consider certain testing function space [9] which can be envisaged as vector sums of $g_{a,l}$ and $k_{b,l}$. The operators $I_k^m(\eta, \alpha)$, $I_{ik}^m(\eta, \alpha)$ and $K_k^m(\xi, \alpha)$, $K_{ik}^m(\xi, \alpha)$ can be defined under appropriate conditions, on the dual spaces.

For $a \in \mathbb{R}$, $b \in \mathbb{R}$, $r = 0, 1, 2, \ldots$

$$(6.1) \qquad \mu_{a,b,r}(\phi) \triangleq \sup\left\{x^{1-a+r}(1+x)^{a-b}|\phi^{(r)}(x)| : x > 0\right\}$$

defines seminorms, with $\mu_{a,b,0}$ a norm, and

$$(6.2) \qquad \eta_{a,b} = \left\{\phi : \phi \in C^\infty(0, \infty), \mu_{a,b,r}(\phi) < \infty, r = 0, 1, 2 \ldots\right\}.$$

With the topology induced by the seminorms, $\mu_{a,b,r}$ is space of testing functions which is a complete countable multinormed space. If $c \geq a$ and $d \leq b$, then $\eta_{c,d} \subset \eta_{a,b}$ and the topology of $\eta_{c,d}$ is stronger than that induced on it by $\eta_{a,b}$.

For $-\infty \leq a < +\infty$ and $-\infty < b \leq +\infty$ let $a_n \downarrow a$ and $b_n \uparrow b$ as $n \to \infty$. Then

$$(6.3) \qquad \eta(a, b) = \bigcup_{n=1}^{\infty} \eta_{a_n, b_n}$$

defines a countable union space of testing functions which is independent of the sequences a_n and b_n. For details see Erdelyi [3].

Letting $\operatorname{Re}\alpha > 0$,

$$\max(a, b) < m \operatorname{Re}\xi + 1. \tag{6.4}$$

We can form $K_k^m(\xi, \alpha)\phi$ for $\phi \in \eta_{a,b}$.
Also $(1 + xy)/(1 + x)$ is between 1 and y for all $x > 0$,

$$\left(\frac{1+xy}{1+x}\right)^{b-a} \leq \max(1, y^{b-a}),$$

and by a computation similar to that leading to (4.3)

$$\mu_{a,b,r}\left[K_k^m(\eta, \alpha)\phi\right] \leq \sum_{n=0}^{r} \binom{r}{n} \mu_{a,b,n}(\phi) \int_1^\infty \Big| y^{-m\xi - m\alpha + \alpha - 1} \max(1, y^{b-a}) \cdot$$
$$\cdot H_{1,3}^{1,1}\left[\frac{x^m k^2(y^m - 1)}{4} \middle| \begin{matrix} (0, m) \\ (0, 1)(1-\alpha, 1)(r-n, m) \end{matrix}\right]\Big| \, dy^m.$$

The integral is convergent by virtue of (6.4), and it follows that $K_k^m(\xi, \alpha)$ is a continuous map of $\eta_{a,b}$ into itself. The extension to all α, and the formal relationships (4.3) to (4.8) follow as in Section 4, (4.8) being replaced by

$$\max(a, b) < 1 + \min\{\operatorname{Re}\xi, \operatorname{Re}(\xi + \alpha)\}. \tag{6.5}$$

The results are the same for $K_{ik}^m(\xi, \alpha)$ (k is to be replaced by ik) also. Similar statements hold for $\phi \in \eta(a, b)$.

With $\phi \in \eta(a, b)$, (4.10) now defines $I_k^m(\eta, \alpha)$ and $I_{ik}^m(\eta, \alpha)$ on $\eta'(a, b)$ provided that

$$\max(a, b) < m \operatorname{Re}\eta + m, \tag{6.6}$$

and the results of Section 4 will hold except that (4.13) must be replaced by

$$\max(a, b) < m + \min\{m \operatorname{Re}\eta, m \operatorname{Re}(\eta + \alpha)\}. \tag{6.7}$$

We can similarly consider $I_k^m(\eta, \alpha)$ and $I_{ik}^m(\eta, \alpha)$ on $\eta'_{a,b}$.

In the same way, (5.7) and (5.8) with $\phi \in \eta(a, b)$ define $K_k^m(\xi, \alpha)$ and $K_{ik}^m(\xi, \alpha)$ on $\eta'(a, b)$ provided that

$$\min(a, b) > -m \operatorname{Re}\xi \tag{6.8}$$

and the results of Section 5 will hold that (5.9) must replaced by

$$\min(a, b) > -\min\{m \operatorname{Re}\xi, m \operatorname{Re}(\xi + \alpha)\}; \tag{6.9}$$

and analogous results hold for $K_k^m(\xi, \alpha)$ and $K_{ik}^m(\xi, \alpha)$ on $\eta'_{a,b}$.

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