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A NOTE ON CONNECTED GRAPHS

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In this paper, we show that every connected graph G of order $p \gg 3$ has two distinct vertices u and v, such that $G - u$, $G - v$ and $G - u - v$ are connected.

Introduction.

The terminology and notation, used in this paper, are those of [2]. A *digraph* D is a finite nonempty set V together with an irreflexive relation E on V. We call each element of V a *vertex*, and each ordered pair of E an *arc*. Furthermore, the cardinality of V is termed the *order* of D.

A finite sequence of vertices and arcs $u_0, e_0, u_1, \ldots, e_{n-1}, u_n$ is called an $u_0 - u_n$ *semiwalk* in D, if for each $i, 0 \ll i \ll n-1$, either $e_i = (u_i, u_{i+1})$ or $e_i = (u_{i+1}, u_i)$. If for each pair of vertices u and v there exists an $u - v$ semiwalk in D, we say that D is *weakly connected* or *weak*. In addition, if w is a vertex of D for which $D - w$ is not weak, then w is called a *cut-vertex* of D.

An $u_0 - u_n$ *walk* in D is a semiwalk $u_0, e_0, u_1, \ldots, e_{n-1}, u_n$, in which $e_i = (u_i, u_{i+1})$ for $i = 0, 1, \ldots, n-1$. An $u_0 - u_n$ walk is *closed*, if $u_0 = u_n$, and *spanning*, if it contains every vertex of D. Furthermore, if for each pair of vertices u and v, D contains either an $u - v$ walk or a $v - u$ walk, then D is called *unilateral*, while if D contains both an $u - v$ walk and a $v - u$ walk, D is *strong*.

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Main result.

We shall need the following known result which characterizes unilateral and strong digraphs [2]. Since our proof is instructive and believed to be new, we include it here.

Lemma 1. *A digraph is unilateral if and only if it has a spanning walk, and strong if and only if it has a closed spanning walk.*

Proof. Clearly, if a digraph has a spanning walk, then it is unilateral. It is equally clear that a digraph with a closed spanning walk is strong.
Suppose that D is an unilateral digraph and let W be an walk in D containing the maximum number of distinct vertices. If W is not spanning, then let v be a vertex not belonging to W. By our choice of W, there can be neither an walk from v to the first vertex of W nor an walk from the last vertex of W to v. Let u be the last vertex of W from which there exists an walk to v. Let W_1 be an $u - v$ walk. By our choice of u, since D is unilateral, there must be an walk, say W_2, from v to the vertex, say w, which follows u on W. But now, we can insert $W_1 U W_2$ between u and w on W to obtain an walk using at least one vertex, namely v, not belonging to W. This contradicts our choice of W. Thus, W must be spanning.
If D is a strong digraph, then it is also unilateral. Hence, it contains a spanning walk, say W, beginning and ending at say u and v, respectively. Since D is strong, there exists a $v - u$ walk, say W', in D. Then, W followed by W' is a closed spanning walk. □

Lemma 2. *If D is an unilateral digraph of order $p \gg 3$, then D has two distinct vertices u and v, such that $D - u$, $D - v$ and $D - u - v$ are unilateral.*

Proof. Let D be an unilateral digraph of order $p \gg 3$, and let W be a shortest spanning walk of D. Further, let u and v be the first and last vertices, respectively, of W. We note that u appears exactly once on W (otherwise, D is spanned by the shorter walk which starts at the second vertex of W and follows the remainder of W to v). Also, v appears but once on W (otherwise, D is spanned by the shorter walk which follows W from u and ends at the next to last vertex of W). Clearly then, $D - u$, $D - v$ and $D - u - v$ have spanning walks which are subwalks of W. Hence, by Lemma 1, each is unilateral. □

It is well-known that every connected graph has a spanning walk [2]. Thus, the proof of Lemma 2, when slightly modified, establishes the well-known result that every nontrivial connected graph has at least two non-cut-vertices. In fact, it gives us the following

Theorem. *Every connected graph G of order $p \gg 3$ has two distinct vertices u and v, such that $G - u$, $G - v$ and $G - u - v$ are connected.* □

Let us conclude with a result due to Geller [1], pag. 15, which is an immediate corollary of Lemma 2.

Corollary. *If D is a strong digraph of order $p \gg 4$ which has no cut-vertex, then D has at least two distinct vertices u and v, such that $D - u$ and $D - v$ are unilateral.* □

REFERENCES

[1] D.P. Geller, *Minimally strong digraphs,* Proc. Edinburgh Math. Soc., 17 (1970), p. 15.

[2] F. Harary, *Graph Theory,* Addison Wesley, 1969.

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