LE MATEMATICHE
Vol. L (1995) – Fasc. II, pp. 355-360

REMARKS ON PRIMITIVE SERIES ON PLANE CURVES

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In this paper we relate primitive linear series on a smooth plane curve C with the following properties of zero-dimensional subschemes X of $\mathbb{P}^2$: 1) Cayley-Bacharach "at degree a", 2) the absence of non assigned base points on C for the linear system of curves of degree a through X.

This allows us to construct examples of primitive series on C.

Introduction.

A special complete linear series g_n^r on a smooth irreducible projective curve C is called primitive if it is base point free and if also its dual series $|K - g_n^r|$ is base point free.

The number $n - 2r$ is called the Clifford index of the g_n^r. The sequence obtained by ordering the Clifford indices of all primitive series of C, omitting the trivial linear series g_0^0 and $|K|$ and repetitions, is called Clifford sequence Σ of C. The length of Σ is called the primitive length of C and it is an invariant of curves of fixed genus and gonality (see [2] for details and motivations).

In this paper we begin a systematic study of primitive series on a smooth plane curve C in order to obtain further informations on primitive sequence and primitive length of C, a problem suggested by Coppens in [2].

We use the techniques introduced in [6] which are based on the study of the Hilbert function of zero-dimensional subschemes X of $\mathbb{P}^2$ and its geometric interpretation in terms of the linear systems of curves passing through X. For

Entrato in Redazione il 28 maggio 1996.

example, in Section 2, we relate primitive linear series on a smooth plane curve C with the properties of Cayley-Bacharach at degree a and a-free relative to C. In Section 3, we first show how to construct subschemes X which are CB(a) on an assigned curve C_a then we prove the existence of curves of degree d on which the linear system of curves of degree a cut out primitive series of dimension one or two.

1. Preliminaries and Notation.

We denote by C a smooth plane curve of degree d defined over an algebraic closed field k of arbitrary characteristic and by $|K|$ its canonical series. For general notions on curves and linear series we refer to [7].

Definition 1.1. A special complete linear series $g_n^r = |D|$ on C is called *primitive* if it is base point free and also its dual series $|K - D|$ is base point free(see [2]).

If $X \subset \mathbb{P}^2$ is a zero-dimensional closed subscheme of degree $\delta(X) = n$, we denote by $H(X, i)$ the Hilbert function of X and by $\Delta H(X, i) = H(X, i) - H(X, i-1)$, for every $i > 0$, its first difference, $H(X, 0) = \Delta H(X, 0) = 1$. Moreover we put $t = \max\{i \in \mathbb{N} \mid \Delta H(X, i) \neq 0\} = \min\{i \in \mathbb{N} \mid H(X, i) = n\}$ and $\alpha = \alpha(X)$ the least degree of a curve through X. In the following we will use freely the fact that if $X' \subset X$ are zero-dimensional closed subschemes of $\mathbb{P}^2$, then $\Delta H(X', i) \leq \Delta H(X, i)$ (see [8]). We refer to [6] for basic facts about $H(X, i)$ and $\Delta H(X, i)$.

Definition 1.2. We say that X is a Cayley-Bacharach scheme (CB-scheme for short) if, for any subscheme $X' \subset X$ with $\deg X' = n - 1$, the following equivalent conditions hold:

1) $H(X, i) = H(X', i)$ for $i < t$ and $H(X', i) = H(X, i) - 1 = n - 1$

for $i \geq t$

2) $\Delta H(X', t) = \Delta H(X, t) - 1$.

For further equivalent form of this definition see Definition 2.7 in [5].

Generalizing Definition 1.2 we give the following

Definition 1.3. We say that X is a CB(a)-scheme, $1 \leq a \leq t - 1$, if, for any subscheme $X' \subset X$, with $\deg X' = n - 1$, $H(X', i) = H(X, i)$ for $1 \leq i \leq a$, i.e. every curve of degree $\leq a$ which contains all but one point of X must contain all the points of X (see [3]).

Remark 1.4. Since $H(X', a) = H(X, a)$ implies $H(X', i) = H(X, i)$ for $i < a$ then Definition 1.3 is equivalent to $\Delta H(X', i) = \Delta H(X, i)$ for $1 \leq i \leq a$.

Remark 1.5. For any zero-dimensional subscheme $X \subset \mathbb{P}^2$ we have $\Delta H(X, i) = i + 1$ for $i < \alpha$, $\Delta H(X, \alpha) \leq \alpha$ and $\Delta H(X, i) \geq \Delta H(X, i + 1)$ for $i \geq \alpha$. Then there cannot exist any subscheme $X' \subset X$, $\deg X' = n - 1$, such that $\Delta H(X', i) = \Delta H(X, i) - 1$ for $i \leq \alpha - 2$ i.e. any X is a CB($\alpha - 2$)-scheme.

Remark 1.6. A zero-dimensional subscheme X of $\mathbb{P}^2$ is a CB-scheme if and only if it is a CB(a)-scheme for every $a \leq t - 1$.

In fact, if $\Delta H(X', t) = \Delta H(X, t) - 1$ then $\Delta H(X, i) = \Delta H(X', i)$ for $i \leq t - 1$ and conversely.

The scheme X consisting of five points on a line r and of three other points on a line $s \neq r$ is a CB(a)-scheme for $a \leq 2$ but it is not a CB(3)-scheme. Then it is not a CB-scheme.

Let X be a zero-dimensional closed subscheme contained in a curve C of degree d and $\Sigma_a(X)$ be the linear system of curves of degree a through X. Recalling that the base scheme of $\Sigma_a(X)$ is the subscheme of $\mathbb{P}^2$ defined by the homogeneous ideal generated by $\Sigma_a(X)$, we introduce the following

Definition 1.7. X is called a-free relative to C if the intersection of C with the base scheme of $\Sigma_a(X)$ is X.

In the following we will say that an effective divisor $D \in g^r_n$ is CB(a) or a-free relative to C if such is the corresponding closed subscheme X.

2. Primitive series and CB(a) plus a-free relative to C schemes.

In this section we relate primitive linear series with the properties CB(a) and a-free relative to C.

Proposition 2.1. *Let D be an effective special divisor on C and $K \geq D$ a canonical divisor. Then $|D|$ is primitive if and only if D and $K - D$ are CB($d - 3$).*

Proof. By Riemann-Roch a subscheme X is CB($d-3$) if and only if $H(X, d - 3) = H(X', d-3)$ for every subscheme $X' \subset X$ of degree $n-1$ (see Lemma 3.6 in [6] for details). Apply this to the closed subschemes corresponding to D and $K - D$.

Lemma 2.2. *Let g_n^r be a special complete linear series without base points on C of degree $d \geq 4$. Then there exist a unique integer a, $1 \leq a \leq d-1$, such that*

1) $ad - a^2 \leq n \leq ad$.
If moreover $(a+1)^2 < d$ then
2) g_n^r is cut out on C by the linear system $\Sigma_a(E)$ of curves of degree a through an effective divisor E on C of degree $ad - n$, outside of E.

Proof. 1) Write $n - 1 = (a-1)d + r$, $0 \leq r < d$, then $(a-1)d + 1 \leq n \leq ad$. Moreover Theorem 3.1 in [6] shows that if $(a-1)d + 1 \leq n \leq ad - a^2 - 1$ there is no g_n^r on C without base points.

2) By 1) $n \leq ad$ then it follows $n < (a+1)[d - (a+1)]$ then apply 1.4 in [1].

Remark 2.3. In [1], assuming the same hypothesis of Lemma 2.2, it has been proved that if $n \notin [ad - a^2, ad]$ then there exist on C free linear series of degree n.

Theorem 2.4. *Let a, s two integers such that $0 < s \leq a^2$, $(a+1)^2 < d - 2$ and let be E an effective divisor of degree s. Let g_{ad-s}^r the complete linear series cut out on C by $\Sigma_a(E)$ outside of the effective divisor E on C. If E is CB(a) and a-free relative to C then g_{ad-s}^r is primitive.*

Proof. $g_{ad-s}^r = |F|$ is base points free because $\Sigma_a(E)$ has no base point on C outside of E.
We have to show that $|K - F|$ is base point free i.e., that any $P \in C$ is a base point for $|K - (K - F) + P| = |F + P|$.
By Lemma 2.2, if $F \in g_{ad-s}^r$ then $F = D - E$ where D is a divisor of the complete linear series cut out on C by the linear system Σ_a of curves of degree a. If $P \not\leq E$, by Lemma 2.2, P is a base point for $|D + P|$ because $D + P$ has degree $ad + 1 < (a+1)d - [(a+1)^2 + 1]$ (see [6]) and then P is a base point for $|D + P - E| = |F + P|$. If $P \leq E$ the statement follows from the CB(a) property of E.

Remark 2.5. We don't know if the converse of Theorem 2.4 holds.

3. Existence of curves which carry some primitive series.

In this section we show how to construct on an assigned curve C_a subschemes X which are CB(a). Successively we prove the existence of curves of degree d on which $\Sigma_a(X)$ cuts out primitive series.

Theorem 3.1. *Let C_a be a non singular curve of degree a, $P_1, \dots, P_a$ be a distinct collinear points on C_a, C'_a be a curve of degree a containing $P_1, \dots, P_s$, $2 \le s \le a-1$ but no one of the points $P_{s+1}, \dots, P_a$. Let $X = C_a \cap C'_a$ and X' be the subscheme linked to $\{P_1, \dots, P_s\}$ in X. Then*

$$1) \quad \Delta H(X', i) = \begin{cases} i+1 & \text{for } 0 \le i \le a-1 \\ 2a-i-1 & \text{for } a \le i \le 2a-2-s \\ 2a-i-2 & \text{for } 2a-1-s \le i \le 2a-3 \\ 0 & \text{for } i \ge 2a-2 \end{cases}$$

2) X' *is* CB(a).

Proof. If Y and Y' are algebraically linked in a zero-dimensional complete intersection (a, b), by [4] we have

$$\Delta H(Y, i) + \Delta H(Y', j) = \Delta H(CI(a, b), i) \quad \text{if} \quad i + j = a + b - 2.$$

Furthermore:

$$\Delta H(CI(a,a), i) = \begin{cases} i+1 & \text{for } 0 \le i \le a-1 \\ 2a-i-1 & \text{for } a \le i \le 2a-2 \\ 0 & \text{for } i \ge 2a-1 \end{cases}$$

$$\Delta H(\{P_1, \dots, P_s\}, i) = \begin{cases} 1 & \text{for } 0 \le i \le s-1 \\ 0 & \text{for } i \ge s \end{cases}$$

and then 1) follows.

2) We have

$$\Delta H(\{P_1, \dots, P_s, P\}, -) = 1\ 1\ \dots\ 1\ 0\ \dots$$

if and only if P is collinear with $P_1, \dots, P_s$. Otherwise

$$\Delta H(\{P_1, \dots, P_s, P\}, -) = 1\ 2\ 1\ \dots\ 1\ 0\ \dots .$$

Then, for any point $P \in X'$, we have

$$\Delta H(X' - P, 2a-3) = \Delta H(X', 2a-3) - 1 = 0,$$

i.e. X' is CB and then CB(a).

Theorem 3.2. *If* char$(k) = 0$, $2 \leq s \leq a-1$ *and* $(a+1)^2 < d-2$ *then there exists a non singular curve C of degree d which carries a primitive* $g^r_{ad-(a^2-s)}$.

Proof. Let X be a subscheme of $\mathbb{P}^2$ such as in Theorem 3.1. If $s = a-1$, from the proof of Proposition 4.3 in [6] it follows that there exists a non singular curve of degree d which contains X' and no one other point of X.The curves of degree a through X', by Theorem 3.1, form a net and, by Theorem 2.4, cut out on C, outside of X', a primitive $g^2_{ad-(a^2-a+1)}$.

If $2 \leq s \leq a-2$ an analogous argument shows that $\Sigma_a(X')$ is a pencil and cuts out on C a primitive $g^1_{ad-(a^2-s)}$.

Remark 3.3. Theorem 3.2 implies that there exist curves of degree d such that the numbers $ad-(a^2-a+1)-4$ and $ad-(a^2-s)-2$ belong to their Clifford sequence.We believe that every curve of degree d carries primitive series such as in Theorem 3.2.

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