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DIFFERENTIABILITY OF WEAK SOLUTIONS OF NONLINEAR PARABOLIC SYSTEMS WITH QUADRATIC GROWTH

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We show, making use of the interpolation theory in Besov spaces, an imbedding theorem of Gagliardo-Nirenberg type for functions u belonging to $W^{m,r} \cap C^{s,\lambda}$, from which we deduce the local differentiability result:

$$u \in L^2\Big(-a, 0, H^2(B(\sigma), \mathbb{R}^N)\Big) \cap H^1\Big(-a, 0, L^2(B(\sigma), \mathbb{R}^N)\Big),$$

for the solutions u of class $L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \cap C^{0,\lambda}(Q, \mathbb{R}^N)$ (N integer ≥ 1, $Q = \Omega \times (-T, 0)$, $0 < \lambda < 1$) to the nonlinear parabolic system:

$$-\sum_{i=1}^{n} D_i a^i(X, u, Du) + \frac{\partial u}{\partial t} = B^0(X, u, Du), \; X = (x, t) \in Q,$$

with quadratic growth:

$$\|B^0(X, u, Du)\| \leq M\left(1 + \|Du\|^2\right).$$

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1. Introduction.

Let Ω be an open bounded subset of $\mathbb{R}^n (n > 2)$ of generic point $x = (x_1, x_2, \dots, x_n)$, Q the cylinder $\Omega \times (-T, 0)$ $(0 < T < +\infty)$ and $u \in L^2\left(-T, 0, H^1(\Omega, \mathbb{R}^N)\right) \cap C^{0,\lambda}(Q, \mathbb{R}^N)$ (N integer positive, $0 < \lambda < 1$) (1) a solution in Q to the second order nonlinear parabolic system of variational type (2)

$$(1.1) \qquad -\sum_{i=1}^{n} D_i a^i (X, u, Du) + \frac{\partial u}{\partial t} = B^0 (X, u, Du),$$

in the sense that

$$\int_Q \left\{ \sum_{i=1}^{n} (a^i(X, u, Du) \mid D_i\varphi) - \left(u \middle| \frac{\partial \varphi}{\partial t}\right) \right\} dX =$$

$$= \int_Q (B^0(X, u, Du) \mid \varphi) \, dX, \qquad \forall \varphi \in C_0^\infty(Q, \mathbb{R}^N),$$

(1) By $W^{m,r}(\Omega, \mathbb{R}^N)$, $m = 0, 1, 2, \dots, 1 < r < \infty$, we will denote the usual Sobolev spaces.
$W^{\vartheta,r}(\Omega, \mathbb{R}^N)$, $0 < \vartheta < 1$, $1 < r < \infty$, will denote the Slobodeckij space of those vectors $u \in L^r(\Omega, \mathbb{R}^N)$ such that

$$|u|^r_{\vartheta,r,\Omega} = \int_\Omega dx \int_\Omega \frac{\|u(x) - u(y)\|^r}{\|x - y\|^{n+\vartheta r}} dy < +\infty.$$

$W^{m+\vartheta,r}(\Omega, \mathbb{R}^N)$, $m = 1, 2, \dots, 0 < \vartheta < 1, 1 < r < \infty$, will denote the space of those vectors $u \in W^{m,r}(\Omega, \mathbb{R}^N)$ such that $D^\alpha u \in W^{\vartheta,r}(\Omega, \mathbb{R}^N)$, $\forall |\alpha| = m$.
If $r = 2$ we shall use the notation $H^{m+\vartheta}$, $m = 0, 1, 2, \dots, 0 \leq \vartheta < 1$, instead of $W^{m+\vartheta,2}$.
By $C^{s,\lambda}(\Omega, \mathbb{R}^N)$, $s = 0, 1, 2, \dots, 0 < \lambda < 1$, we shall denote the space of those vectors $u \in C^s(\overline{\Omega}, \mathbb{R}^N)$ for which

$$[D^\alpha u]_{\lambda,\Omega} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|D^\alpha u(x) - D^\alpha u(y)\|}{\|x - y\|^\lambda} < +\infty, \forall |\alpha| = s.$$

In Q the Hölder continuity is considered with respect to the parabolic metric

$$d(X, Y) = \max\{\|x - y\|, |t - \tau|^{\frac{1}{2}}\}, X = (x, t), Y = (y, \tau).$$

(2) For the notations and the symbols we refer to [3]. In particular if $u : Q \to \mathbb{R}^N$ we shall write $Du = (D_1 u \mid \dots \mid D_n u)$, $D_i = \frac{\partial}{\partial x_i}$.

where $X = (x, t)$ and $a^i(X, u, p)$, $i = 1, 2, \ldots, n$, and $B^0(X, u, p)$ are vectors of $\mathbb{R}^N$ defined on $\Lambda = Q \times \mathbb{R}^N \times \mathbb{R}^{nN}$, satisfying the following conditions:

(1.2) *the vector $B^0(X, u, p)$ is measurable in X, continuous in (u, p) and, for each $(X, u, p) \in \Lambda$ with $\|u\| \leq K$, such that*

$$\|B^0(X, u, p)\| \leq M(K)\left\{1 + \|p\|^2\right\};$$

(1.3) *the vectors $a^i(X, u, p)$, $i = 1, 2, \ldots, n$, are of class C^1 in $\overline{Q} \times \mathbb{R}^N \times \mathbb{R}^{nN}$ and, for each $(X, u, p) \in \Lambda$ with $\|u\| \leq K$,*

$$\|a^i\| + \sum_{r=1}^{n}\left\|\frac{\partial a^i}{\partial x_r}\right\| + \sum_{k=1}^{N}\left\|\frac{\partial a^i}{\partial u_k}\right\| \leq M(K)\Big\{1 + \|p\|\Big\}, \quad i = 1, 2, \ldots, n,$$

$$\sum_{k=1}^{N}\sum_{j=1}^{n}\left\|\frac{\partial a^i}{\partial p_k^j}\right\| \leq M(K),\ i = 1, 2, \ldots, n;$$

(1.4) *there exists $\nu(K) > 0$ such that*

$$\sum_{i,j=1}^{n}\sum_{h,k=1}^{N}\frac{\partial a_h^i(X, u, p)}{\partial p_k^j}\xi_h^i\xi_k^j \geq \nu(K)\|\xi\|^2$$

for each $\xi = (\xi^1|\xi^2|\ldots|\xi^n) \in \mathbb{R}^{nN}$ and for each $(X, u, p) \in \Lambda$ with $\|u\| \leq K$.

In the work [3] L. Fattorusso is concerned with the local differentiability, with respect to the spatial derivatives, of the solutions

$$u \in L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \cap C^{0,\lambda}(Q, \mathbb{R}^N)\,, \quad 0 < \lambda < 1\,,$$

to the system (1.1), proving that, if the assumptions (1.2), (1.3) and (1.4) are fulfilled, then, for each cube $B(2\sigma) = B(x^0, 2\sigma) = \{x \in \mathbb{R}^n : |x_i - x_i^0| < 2\sigma, i = 1, 2, \ldots, n\} \subset\subset \Omega$ and $\forall 2a \in (0, T)$, it results

(1.5) $$u \in L^2(-a, 0, H^{1+\vartheta}(B(\sigma), \mathbb{R}^N)), \forall \vartheta \in (0, 1),$$

and the following estimate [3]

$$\int_{-a}^{0} |Du|^2_{\vartheta,B(\sigma)} dt \leq c\left\{1 + \int_{-2a}^{0} |u|^2_{1,B(2\sigma)} dt\right\}$$

holds.
In the same work is pointed out that it is not possible to improve this result, in such a way to achieve for each solution u to the system (1.1) the differentiability

$$u \in L^2\left(-a, 0, H^2(B(\sigma), \mathbb{R}^N)\right), \tag{1.6}$$

if, preliminarily, the regularity

$$D_i u \in L^4\left(-a, 0, L^4(B(\rho), \mathbb{R}^N)\right), \; i = 1, 2, \ldots, n, \tag{1.7}$$

$\forall B(\rho) \subset\subset \Omega$ and $\forall a \in (0, T)$, is not ensured.

The technique used in [3] allows us to achieve, instead of (1.7), the condition

$$D_i u \in L^{2(1+\vartheta)}\left(-a, 0, L^4(B(\rho), \mathbb{R}^N)\right), \quad i = 1, 2, \ldots, n,$$

$\forall B(\rho) \subset\subset \Omega$, $\forall a \in (0, T)$ and $\forall \vartheta \in \left(\frac{n}{n+4\lambda}, 1\right)$, which is not enough to ensure that

$$u \in L^2\left(-a, 0, H^2(B(\sigma), \mathbb{R}^N)\right).$$

In the papers [4], [6] and [5] the problem of differentiability has been considered again, under assumptions of monotony and of nonlinearity $q \neq 2$, always achieving results of the type (1.5).

In the paper [10], the author obtains the desidered result (1.6) under assumptions stronger than (1.2), (1.3), (1.4) and requiring that the solutions u to

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(3) If $1 < r < \infty$, $0 < \vartheta < 1$ and $m = 0, 1, 2, \ldots$, we shall set

$$|u|_{m,r,\Omega} = \left(\int_\Omega \sum_{|\alpha|=m} \|D^\alpha u\|^r dx\right)^{\frac{1}{r}}, \; \|u\|_{m,r,\Omega} = \left(\sum_{k=0}^{m} |u|^r_{k,r,\Omega}\right)^{\frac{1}{r}},$$

$$\|u\|_{m+\vartheta,r,\Omega} = \left(\|u\|^r_{m,r,\Omega} + \sum_{|\alpha|=m} |D^\alpha u|^r_{\vartheta,r,\Omega}\right)^{\frac{1}{r}}.$$

For the sake of simplicity we shall write: $|\cdot|_{m,\Omega}$, $|\cdot|_{\vartheta,\Omega}$, $\|\cdot\|_{m,\Omega}$, $\|\cdot\|_{m+\vartheta,\Omega}$ instead of $|\cdot|_{m,2,\Omega}$, $|\cdot|_{\vartheta,2,\Omega}$, $\|\cdot\|_{m,2,\Omega}$, $\|\cdot\|_{m+\vartheta,2,\Omega}$, respectively, and $|Du|^2_{\vartheta,\Omega} = \sum_{i=1}^{n} |D_i u|^2_{\vartheta,\Omega}$.

the system (1.1) belong to the space $L^2\big(-T, 0, H^1(\Omega, \mathbb{R}^N)\big) \cap C^{0,\lambda}(Q, \mathbb{R}^N)$ with $\frac{1}{2} < \lambda < 1$.

The aim of this paper is to obtain, under the assumptions of [3], the differentiability result (1.6) for the solutions $u \in L^2\big(-T, 0, H^1(\Omega, \mathbb{R}^N)\big) \cap C^{0,\lambda}(Q, \mathbb{R}^N)$, $0 < \lambda < 1$, to the system (1.1). The technique followed is to reach, preliminarily, the regularity result (1.7), by using the condition (1.5) proved in [3], the assumption $u \in C^{0,\lambda}\left(Q, \mathbb{R}^N\right)$ and a suitable imbedding theorem of Gagliardo-Nirenberg type for functions $u \in W^{m,r} \cap C^{s,\lambda}$ (see Section 2).

2. An interpolation result.

In this section we establish, making use of the interpolation theory in Besov spaces (cf.: [12] and [13]), an interpolation result of Gagliardo-Nirenberg type for functions $u \in W^{m,r} \cap C^{s,\lambda}$ with m non integer > 1, $1 < r < \infty$, s integer ≥ 0, $0 < \lambda < 1$, that we will use in Sect. 3 to show the differentiability result (1.6) for the solutions $u \in L^2\big(-T, 0, H^1(\Omega, \mathbb{R}^N)\big) \cap C^{0,\lambda}(Q, \mathbb{R}^N)$ to the system (1.1).

If $u \in W^{m,r} \cap C^{s,\lambda}$, with m integer ≥ 2, the following result due to Nirenberg [11] (see also C. Miranda [9]) is well known :

Theorem 2.1. *Let N be a positive integer and Ω a cube of $\mathbb{R}^n$. If*

$$u \in W^{m,r}(\Omega, \mathbb{R}^N) \cap C^{s,\lambda}(\Omega, \mathbb{R}^N),$$

with m integer ≥ 2, $1 < r < \infty$, s integer ≥ 0, $0 < \lambda < 1$, $s < m - 1$, then, for each integer j with $s + \lambda < j < m$, there exist two constants c_1 and c_2 (depending on $\Omega, m, r, s, \lambda, j$) such that:

$$\max_{|\alpha|=j} |D^\alpha u|_{0,p,\Omega} \leq c_1 \left(\max_{|\alpha|=m} |D^\alpha u|_{0,r,\Omega} \right)^a \left(\max_{|\alpha|=s} [D^\alpha u]_{\lambda,\Omega} \right)^{1-a} +$$

$$+ c_2 \max_{|\alpha|=s} [D^\alpha u]_{\lambda,\Omega},$$

where $\dfrac{1}{p} = \dfrac{j}{n} + a\left(\dfrac{1}{r} - \dfrac{m}{n}\right) - (1-a)\dfrac{s+\lambda}{n}$, $\forall a \in \left[\dfrac{j-s-\lambda}{m-s-\lambda}, 1\right[$ (4).

(4) In the work [11], Theorem 2.1 is stated in a slightly more general form.

Making use of this theorem, in [8] we proved that the solutions

$$u \in L^2\left(-T, 0, H^2(\Omega, \mathbb{R}^N)\right) \cap C^{0,\lambda}(Q, \mathbb{R}^N)$$

to the system (1.1) belong to $H^1\big(-T, 0, L^2(\Omega, \mathbb{R}^N)\big)$ and, hence, the partial Hölder continuity in Q of $D_i u$, $i = 1, 2, \ldots, n$.

Now let us show an analogous result for functions $u \in W^{m,r} \cap C^{s,\lambda}$, with m non integer > 1.

Theorem 2.2. *Let N be a positive integer and Ω a cube of $\mathbb{R}^n$. If*

$$u \in W^{m+\vartheta,r}(\Omega, \mathbb{R}^N) \cap C^{s,\lambda}(\Omega, \mathbb{R}^N),$$

with m integer ≥ 1, $0 < \vartheta < 1$, $1 < r < \infty$, s integer ≥ 0, $0 < \lambda < 1$, $s < m$, then, for each j with $\max(s + \lambda, m + \vartheta - \frac{n}{r}) < j < m + \vartheta$, it results

$$u \in W^{j,p}(\Omega, \mathbb{R}^N) \tag{2.1}$$

and there exists a constant c (depending on $\Omega, m, \vartheta, r, s, \lambda, j, n, a$) such that:

$$\|u\|_{j,p,\Omega} \leq c\|u\|^a_{m+\vartheta,r,\Omega}\|u\|^{1-a}_{C^{s,\lambda}(\Omega,\mathbb{R}^N)} \text{ (}^5\text{)}, \tag{2.2}$$

where $\dfrac{1}{p} = \dfrac{j}{n} + a\left(\dfrac{1}{r} - \dfrac{m+\vartheta}{n}\right) - (1-a)\dfrac{s+\lambda}{n}$, $\forall a \in \left[\dfrac{j-s-\lambda}{m+\vartheta-s-\lambda}, 1\right[$ *with $(1-a)(s+\lambda) + a(m+\vartheta)$ non integer.*

Proof. It is well known that (6)

$$W^{m+\vartheta,r}(\Omega, \mathbb{R}^N) = B^{m+\vartheta}_{r,r}(\Omega, \mathbb{R}^N) \tag{2.3}$$

and that (7)

$$C^{s,\lambda}(\Omega, \mathbb{R}^N) = B^{s+\lambda}_{\infty,\infty}(\Omega, \mathbb{R}^N). \tag{2.4}$$

Then it results $u \in B^{m+\vartheta}_{r,r}(\Omega, \mathbb{R}^N) \cap B^{s+\lambda}_{\infty,\infty}(\Omega, \mathbb{R}^N)$ and, hence, by means of Theorem 3.3.6 in [13] (8), we get, for each $a \in]\,0, 1[$:

$$u \in \left(B^{s+\lambda}_{\infty,\infty}(\Omega, \mathbb{R}^N), B^{m+\vartheta}_{r,r}(\Omega, \mathbb{R}^N)\right)_a = B^h_{q,q}(\Omega, \mathbb{R}^N) \tag{2.5}$$

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(5) $\|u\|_{C^{s,\lambda}(\Omega,\mathbb{R}^N)} = \sum_{|\alpha|\leq s} \sup_{x\in\overline{\Omega}} \|D^\alpha u(x)\| + \sum_{|\alpha|=s}[D^\alpha u]_{\lambda,\Omega}$.

(6) For the scalar case ($N = 1$) see [12], Remark 4.4.2/2. The result can be extended to the vectorial case ($N > 1$).

(7) For the scalar case ($N = 1$) see [12], Remark 4.5.2/3 and (3.4.2/2) in [13]. The result can be extended to the vectorial case ($N > 1$).

(8) The result can be extended to the vectorial case.

and by (1.9.3/3) in [12]

$$\|u\|_{B^h_{q,q}(\Omega,\mathbb{R}^N)} \le c\|u\|^a_{B^{m+\vartheta}_{r,r}(\Omega,\mathbb{R}^N)}\|u\|^{1-a}_{B^{s+\lambda}_{\infty,\infty}(\Omega,\mathbb{R}^N)} \tag{2.6}$$

where $h = (1-a)(s+\lambda) + a(m+\vartheta)$, $\frac{1}{q} = \frac{a}{r}$.

Now fixed $j \in]\max(s+\lambda, m+\vartheta-\frac{n}{r}), m+\vartheta[$, let us consider (2.5) and (2.6) for $a \in \left[\frac{j-s-\lambda}{m+\vartheta-s-\lambda}, 1\right[$ with $(1-a)(s+\lambda)+a(m+\vartheta)$ non integer; for such values of a, setting

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m+\vartheta}{n}\right) - (1-a)\frac{s+\lambda}{n}$$

it results:

$$1 < q \le p < \infty, \quad 0 < j \le h < \infty, \quad h - \frac{n}{q} = j - \frac{n}{p},$$

from which, thanks to the Remark 4.4.2/2 in [12] ([8]) and to (4.6.1/8) in [12] ([8]) it follows

$$B^h_{q,q}(\Omega,\mathbb{R}^N) = W^{h,q}(\Omega,\mathbb{R}^N) \subset W^{j,p}(\Omega,\mathbb{R}^N), \tag{2.7}$$

with algebraic and topological inclusion.
Then we reach (2.1) and (2.2) from (2.5), (2.7), (2.6), (2.3) and (2.4).

From Theorem 2.2 (with $m = 1$, $r = 2$, $s = 0$, $j = 1$) we easily get the following

Corollary 2.1. *Let N be a positive integer and Ω a cube of $\mathbb{R}^n$. If*

$$u \in H^{1+\vartheta}(\Omega,\mathbb{R}^N) \cap C^{0,\lambda}(\Omega,\mathbb{R}^N),$$

with $0 < \vartheta < 1$ and $0 < \lambda < 1$, then $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ and there exists a constant c (depending on $\Omega, \vartheta, \lambda, n, a$) such that:

$$\|u\|_{1,p,\Omega} \le c\|u\|^a_{1+\vartheta,\Omega}\|u\|^{1-a}_{C^{0,\lambda}(\Omega,\mathbb{R}^N)},$$

where $\frac{1}{p} = \frac{1}{n} + a\left(\frac{1}{2} - \frac{1+\vartheta}{n}\right) - (1-a)\frac{\lambda}{n}$, $\forall a \in \left]\frac{1-\lambda}{1+\vartheta-\lambda}, 1\right[$.

In particular, if $1-\lambda < \vartheta < 1$, for $a = \frac{1}{2}$ we get:
$u \in W^{1,p}(\Omega,\mathbb{R}^N)$ and there exists a constant c (depending on $\Omega, \vartheta, \lambda, n$) such that:

$$\|u\|_{1,p,\Omega} \le c\|u\|^{\frac{1}{2}}_{1+\vartheta,\Omega}\|u\|^{\frac{1}{2}}_{C^{0,\lambda}(\Omega,\mathbb{R}^N)},$$

where $p = 4 + \frac{8(\vartheta+\lambda-1)}{n-2(\vartheta+\lambda-1)} (> 4)$.

3. Differentiability of the solutions to the system (1.1).

Let $u \in L^2\left(-T, 0, H^1(\Omega, \mathbb{R}^N)\right) \cap C^{0,\lambda}(Q, \mathbb{R}^N)$, $0 < \lambda < 1$, be a solution to the system (1.1) and let us suppose that the assumptions (1.2), (1.3) and (1.4) are fulfilled; in what follows we shall set

$$K = \sup_Q \|u\|, \quad U = [u]_{\lambda,Q} = \sup_{\substack{X,Y \in Q \\ X \neq Y}} \frac{\|u(X) - u(Y)\|}{d^\lambda(X, Y)},$$

where $d(X, Y)$ is the parabolic metric:

$$d(X, Y) = \max\left\{\|x - y\|, |t - \tau|^{\frac{1}{2}}\right\}, \quad X = (x, t), \quad Y = (y, \tau).$$

In this section we shall prove that

$$u \in L^2\big(-a, 0, H^2(B(\sigma), \mathbb{R}^N)\big) \cap H^1\big(-a, 0, L^2(B(\sigma), \mathbb{R}^N)\big),$$

$\forall\, B(3\sigma) = B(x^0, 3\sigma) \subset\subset \Omega$ and $\forall\, a \in (0, T)$.
First we recall the following result:

Lemma 3.1. *If $v \in L^2(-a, 0, L^2(B(2\sigma), \mathbb{R}^{nN}))$, $a, \sigma > 0$, and if there exists a real number $M > 0$ such that* (9)

$$\int_{-a}^0 dt \int_{B(\sigma)} \|\tau_{i,h} v\|^2\, dx \le |h|^2 M, \ \forall\, |h| < \sigma, \ i = 1, 2, \ldots, n,$$

then $v \in L^2\left(-a, 0, H^1(B(\sigma), \mathbb{R}^{nN})\right)$ and the following estimate holds:

$$\int_{-a}^0 dt \int_{B(\sigma)} \|D_i v\|^2\, dx \le M, \ i = 1, 2, \ldots, n.$$

The proof is the same of Theorem 3.X in [1], Chap. I.
Now we show the following

(9) Let τ, ρ and a be three positive real numbers, with $\tau < 1$ and let h be a real number such that $|h| < (1 - \tau)\rho$. If v is a function from $B(\rho) \times (-a, 0)$ in $\mathbb{R}^k$ and $X = (x, t) \in B(\tau\rho) \times (-a, 0)$, we shall set here and in what follows

$$\tau_{i,h} v(X) = v(x + h\, e^i, t) - v(X), \ i = 1, 2, \ldots, n,$$

where $\{e^i\}_{i=1,2,\ldots,n}$ is the canonic base of $\mathbb{R}^n$.

Theorem 3.1. *If* $u \in L^2\left(-T, 0, H^1(\Omega, \mathbb{R}^N)\right) \cap C^{0,\lambda}\left(Q, \mathbb{R}^N\right)$, $0 < \lambda < 1$, *is a solution to the system* (1.1), *if the assumptions* (1.2), (1.3) *and* (1.4) *hold, then,* $\forall B(3\sigma) = B(x^0, 3\sigma) \subset\subset \Omega, \forall a, b \in (0, T), a < b$, *it results:*

$$u \in L^2\left(-a, 0, H^2(B(\sigma), \mathbb{R}^N)\right) \cap H^1\left(-a, 0, L^2(B(\sigma), \mathbb{R}^N)\right) \tag{3.1}$$

and the following estimate holds:

$$\int_{-a}^{0} \left\{ |u|^2_{2,B(\sigma)} + \left|\frac{\partial u}{\partial t}\right|^2_{0,B(\sigma)} \right\} dt \leq \tag{3.2}$$

$$\leq c\,(\nu, K, U, \lambda, \sigma, a, b, n) \left\{ 1 + \int_{-b}^{0} |u|^2_{1,B(3\sigma)} dt \right\}.$$

Proof. Fixed $B(3\sigma) = B(x^0, 3\sigma) \subset\subset \Omega$, $a, b \in (0, T)$ with $a < b$ and $h \in \mathbb{R}$ such that $|h| < \frac{\sigma}{2}$, set $b^* = \frac{a+b}{2}$ and let $\psi(x)$, $\rho_m(t)$ be the real functions defined as in the proof of Theorem 3.I in [3]. Arguing as in the proof of Theorem 3.I in [3], we obtain, for each integer i, $1 \leq i \leq n$, and for each $\varepsilon > 0$ (see (3.31) of [3]):

$$\nu \int_{-b^*}^{-\frac{1}{m}} dt \int_{B(2\sigma)} \psi^2 \rho_m^2 \|\tau_{i,h} Du\|^2 dx \leq \tag{3.3}$$

$$\{\varepsilon + c(K, U)(|h|^\lambda + |h|^{2\lambda} + |h| + |h|^2)\} \int_{-b^*}^{-\frac{1}{m}} dt \int_{B(2\sigma)} \psi^2 \rho_m^2 \|\tau_{i,h} Du\|^2 dx +$$

$$+ c(K, \sigma, a, b, \varepsilon)|h|^2 \int_{-b^*}^{-\frac{1}{m}} dt \int_{B(3\sigma)} (1 + \|Du\|^2) dx +$$

$$+ c\,(K, \varepsilon) \int_{-b^*}^{-\frac{1}{m}} dt \int_{B(2\sigma)} \psi^2 \rho_m^2 \|\tau_{i,h} u\|^2 \|Du\|^2 dx +$$

$$+ c\,(K) \int_{-b^*}^{-\frac{1}{m}} \rho_m^2 dt \int_{B(\frac{5}{2}\sigma)} (1 + \|Du\|^2) \|\tau_{i,-h}(\psi^2 \tau_{i,h} u)\|\, dx.$$

Now it is obvious that there exists a real number $h_0(\nu, K, U, \lambda)$ with $0 < h_0 < \min\left\{1, \frac{\sigma}{2}\right\}$ such that, for each $|h| < h_0$, it results

$$c\,(K, U)\left(|h|^\lambda + |h|^{2\lambda} + |h| + |h|^2\right) < \frac{\nu}{4}$$

and, hence, from (3.3) (with $\varepsilon = \frac{\nu}{4}$) it follows, for each integer i, $1 \leq i \leq n$, and for each $|h| < h_0$:

$$\frac{\nu}{2} \int_{-b^*}^{-\frac{1}{m}} dt \int_{B(2\sigma)} \psi^2 \rho_m^2 \|\tau_{i,h} Du\|^2 dx \leq \tag{3.4}$$

$$\leq c(\nu, K, \sigma, a, b)|h|^2 \int_{-b^*}^{-\frac{1}{m}} dt \int_{B(3\sigma)} (1 + \|Du\|^2)\, dx +$$

$$+ c(\nu, K) \int_{-b^*}^{-\frac{1}{m}} dt \int_{B(2\sigma)} \psi^2 \rho_m^2 \|\tau_{i,h} u\|^2 \|Du\|^2\, dx +$$

$$+ c(K) \int_{-b^*}^{-\frac{1}{m}} \rho_m^2 dt \int_{B(\frac{5}{2}\sigma)} (1 + \|Du\|^2) \|\tau_{i,-h}(\psi^2 \tau_{i,h} u)\|\, dx.$$

Let us consider the last integral that appears at the right hand side of (3.4). Theorem 3.III in [3] (with $\sigma_0 = 3\sigma$, $a = b^*$) ensures that

$$u \in L^2\Big(-b^*, 0, H^{1+\vartheta}\big(B(\tfrac{5}{2}\sigma), \mathbb{R}^N\big)\Big), \forall \vartheta \in (0, 1), \tag{3.5}$$

and

$$\int_{-b^*}^{0} |Du|^2_{\vartheta, B(\frac{5}{2}\sigma)} dt \leq \tag{3.6}$$

$$\leq c(\nu, K, U, \vartheta, \lambda, \sigma, a, b, n) \left\{1 + \int_{-b}^{0} |u|^2_{1, B(3\sigma)} dt\right\};$$

hence, thanks also to the assumption $u \in C^{0,\lambda}(Q, \mathbb{R}^N)$, it results for a.e. $t \in (-b^*, 0)$

$$u(x,t) \in H^{1+\vartheta}\big(B(\tfrac{5}{2}\sigma), \mathbb{R}^N\big) \cap C^{0,\lambda}\big(B(\tfrac{5}{2}\sigma), \mathbb{R}^N\big),\ \forall \vartheta \in (0, 1).$$

From Corollary 2.1 (with $\Omega = B\left(\frac{5}{2}\sigma\right)$ and $\vartheta = 1 - \frac{\lambda}{2}$) we get for a.e. $t \in (-b^*, 0)$:

$$u(x,t) \in W^{1,p}\big(B(\tfrac{5}{2}\sigma), \mathbb{R}^N\big) \tag{3.7}$$

and

$$\|u\|_{1,p,B(\frac{5}{2}\sigma)} \leq c(\lambda, \sigma, n) \|u\|^{\frac{1}{2}}_{2-\frac{\lambda}{2}, B(\frac{5}{2}\sigma)} \|u\|^{\frac{1}{2}}_{C^{0,\lambda}(B(\frac{5}{2}\sigma), \mathbb{R}^N)}, \tag{3.8}$$

where $p = 4 + \frac{4\lambda}{n-\lambda}$.
Now, since $p > 4$, we obtain the algebraic and topological inclusion

$$W^{1,p}\big(B(\tfrac{5}{2}\sigma), \mathbb{R}^N\big) \subset W^{1,4}\big(B(\tfrac{5}{2}\sigma), \mathbb{R}^N\big),$$

from which, by virtue of (3.7) and (3.8), it follows for a.e. $t \in (-b^*, 0)$:

$$u(x,t) \in W^{1,4}\big(B(\tfrac{5}{2}\sigma), \mathbb{R}^N\big) \tag{3.9}$$

and

$$\|u\|^4_{1,4,B(\frac{5}{2}\sigma)} \leq c(K, U, \lambda, \sigma, n)\left\{1 + |u|^2_{1,B(\frac{5}{2}\sigma)} + |Du|^2_{1-\frac{\lambda}{2},B(\frac{5}{2}\sigma)}\right\}. \tag{3.10}$$

The estimate (3.9) holds in particular for a.e. $t \in (-b^*, -\frac{1}{m})$; for such values of t, therefore we have:

$$c(K)\int_{B(\frac{5}{2}\sigma)} (1 + \|Du\|^2)\|\tau_{i,-h}(\psi^2\tau_{i,h}u)\|\, dx \leq$$

$$\leq \left(\int_{B(\frac{5}{2}\sigma)} c^2(K)|h|^2(1 + \|Du\|^2)^2 dx\right)^{\frac{1}{2}}\left(\int_{B(\frac{5}{2}\sigma)} |h|^{-2}\|\tau_{i,-h}(\psi^2\tau_{i,h}u)\|^2 dx\right)^{\frac{1}{2}}$$

and $\forall \varepsilon > 0$

$$\begin{aligned} (3.11)\quad & c(K)\int_{B(\frac{5}{2}\sigma)} (1 + \|Du\|^2)\|\tau_{i,-h}(\psi^2\tau_{i,h}u)\|\, dx \leq \\ & \leq \frac{\varepsilon}{2}|h|^{-2}\int_{B(\frac{5}{2}\sigma)} \|\tau_{i,-h}(\psi^2\tau_{i,h}u)\|^2\, dx + c(K,\varepsilon)|h|^2\int_{B(\frac{5}{2}\sigma)} (1 + \|Du\|^2)^2\, dx \leq \\ & \leq \varepsilon\int_{B(2\sigma)} \psi^2\|\tau_{i,h}Du\|^2\, dx + c(\sigma,\varepsilon)|h|^2\int_{B(3\sigma)} \|Du\|^2\, dx + \\ & \quad + c(K,\sigma,n,\varepsilon)|h|^2\left\{1 + \int_{B(\frac{5}{2}\sigma)} \|Du\|^4\, dx\right\} \text{ (10)}. \end{aligned}$$

From (3.11) we deduce, for $\varepsilon = \frac{\nu}{4}$:

$$c(K)\int_{B(\frac{5}{2}\sigma)} (1 + \|Du\|^2)\|\tau_{i,-h}(\psi^2\tau_{i,h}u)\| dx \leq$$

$$\leq \frac{\nu}{4}\int_{B(2\sigma)} \psi^2\|\tau_{i,h}Du\|^2 dx + c(\nu, K, \sigma, n)|h|^2\{1 + |u|^2_{1,B(3\sigma)} + |u|^4_{1,4,B(\frac{5}{2}\sigma)}\}$$

(10) In the last estimate we made use of (3.38) in [3].

from which, by virtue of (3.10):

$$(3.12)\qquad c(K)\int_{B(\frac{5}{2}\sigma)}(1+\|Du\|^2)\|\tau_{i,-h}(\psi^2\tau_{i,h}u)\|dx \leq$$
$$\leq \frac{\nu}{4}\int_{B(2\sigma)}\psi^2\|\tau_{i,h}Du\|^2\,dx +$$
$$+c(\nu,K,U,\lambda,\sigma,n)|h|^2\left\{1+|u|^2_{1,B(3\sigma)}+|Du|^2_{1-\frac{\lambda}{2},B(\frac{5}{2}\sigma)}\right\}.$$

By multiplicating both members of (3.12) for ρ_m^2 and by integrating with respect to t in $(-b^*,-\frac{1}{m})$ we obtain:

$$(3.13)\qquad c(K)\int_{-b^*}^{-\frac{1}{m}}\rho_m^2dt\int_{B(\frac{5}{2}\sigma)}(1+\|Du\|^2)\|\tau_{i,-h}(\psi^2\tau_{i,h}u)\|dx \leq$$
$$\leq \frac{\nu}{4}\int_{-b^*}^{-\frac{1}{m}}dt\int_{B(2\sigma)}\psi^2\rho_m^2\|\tau_{i,h}Du\|^2dx +$$
$$+c(\nu,K,U,\lambda,\sigma,n)|h|^2\int_{-b^*}^{-\frac{1}{m}}(1+|u|^2_{1,B(3\sigma)}+|Du|^2_{1-\frac{\lambda}{2},B(\frac{5}{2}\sigma)})dt,$$

which is the desidered estimate of the last terms of the right hand side of (3.4). Using (3.13) and taking into account the meaning of ψ and of ρ_m, (3.4) becomes:

$$\int_{-a}^{-\frac{2}{m}}dt\int_{B(\sigma)}\|\tau_{i,h}Du\|^2dx \leq$$
$$\leq c(\nu,K,U,\lambda,\sigma,a,b,n)|h|^2\int_{-b^*}^{0}(1+|u|^2_{1,B(3\sigma)}+$$
$$+|Du|^2_{1-\frac{\lambda}{2},B(\frac{5}{2}\sigma)})dt+c(\nu,K)\int_{-b^*}^{0}dt\int_{B(2\sigma)}\|\tau_{i,h}u\|^2\|Du\|^2\,dx,$$

from which, taking the limit as $m\to\infty$, we get

$$(3.14)\qquad \int_{-a}^{0}dt\int_{B(\sigma)}\|\tau_{i,h}Du\|^2dx \leq$$
$$\leq c(\nu,K,U,\lambda,\sigma,a,b,n)|h|^2\Big\{1+\int_{-b^*}^{0}(|u|^2_{1,B(3\sigma)}+|Du|^2_{1-\frac{\lambda}{2},B(\frac{5}{2}\sigma)})dt\Big\}+$$
$$+c(\nu,K)\int_{-b^*}^{0}dt\int_{B(2\sigma)}\|\tau_{i,h}u\|^2\|Du\|^2dx.$$

Now let us consider the integral

$$\int_{-b^*}^{0} dt \int_{B(2\sigma)} \|\tau_{i,h}u\|^2 \|Du\|^2 dx$$

that appears in the right hand side of (3.14). For a.e. $t \in (-b^*, 0)$, we have, using the Hölder inequality and thanks to Lemma 2.II in [2] and to (3.10):

$$\int_{B(2\sigma)} \|\tau_{i,h}u\|^2 \|Du\|^2 dx \leq \left(\int_{B(2\sigma)} \|\tau_{i,h}u\|^4 dx \right)^{\frac{1}{2}} \left(\int_{B(2\sigma)} \|Du\|^4 dx \right)^{\frac{1}{2}} \leq$$

$$\leq |h|^2 \left(\int_{B(\frac{5}{2}\sigma)} \|Du\|^4 dx \right)^{\frac{1}{2}} \left(\int_{B(2\sigma)} \|Du\|^4 dx \right)^{\frac{1}{2}} \leq |h|^2 |u|^4_{1,4,B(\frac{5}{2}\sigma)} \leq$$

$$\leq c\,(K, U, \lambda, \sigma, n)\, |h|^2 \left\{ 1 + |u|^2_{1,B(\frac{5}{2}\sigma)} + |Du|^2_{1-\frac{\lambda}{2},B(\frac{5}{2}\sigma)} \right\},$$

from which, by integrating in $(-b^*, 0)$ it follows

$$\int_{-b^*}^{0} dt \int_{B(2\sigma)} \|\tau_{i,h}u\|^2 \|Du\|^2 dx \leq c(K, U, \lambda, \sigma, a, b, n)|h|^2 \cdot \left\{ 1 + \int_{-b^*}^{0} (|u|^2_{1,B(\frac{5}{2}\sigma)} + |Du|^2_{1-\frac{\lambda}{2},B(\frac{5}{2}\sigma)})dt \right\}. \tag{3.15}$$

From (3.14), (3.15) and (3.6) (with $\vartheta = 1 - \frac{\lambda}{2}$), we deduce, for each integer i, $1 \leq i \leq n$, and for each $|h| < h_0$:

$$\int_{-a}^{0} dt \int_{B(\sigma)} \|\tau_{i,h}Du\|^2 dx \leq \tag{3.16}$$

$$\leq c(\nu, K, U, \lambda, \sigma, a, b, n)|h|^2 \left\{ 1 + \int_{-b^*}^{0} \left(|u|^2_{1,B(3\sigma)} + |Du|^2_{1-\frac{\lambda}{2},B(\frac{5}{2}\sigma)} \right) dt \right\} \leq$$

$$\leq c(\nu, K, U, \lambda, \sigma, a, b, n)|h|^2 \left\{ 1 + \int_{-b}^{0} |u|^2_{1,B(3\sigma)} dt \right\}.$$

If $h_0 \leq |h| < \sigma$ the estimate (3.16) is trivial; hence (3.16) will be true for each integer i, $1 \leq i \leq n$, and for each $|h| < \sigma$.
From (3.16), by virtue of Lemma 3.1, it follows

$$u \in L^2\left(-a, 0, H^2(B(\sigma), \mathbb{R}^N)\right) \tag{3.17}$$

and

$$\text{(3.18)} \qquad \int_{-a}^{0} |u|^2_{2,B(\sigma)} dt \leq$$
$$\leq c\,(\nu, K, U, \lambda, \sigma, a, b, n) \left\{ 1 + \int_{-b}^{0} |u|^2_{1,B(3\sigma)} dt \right\}.$$

It remains to show that $u \in H^1\left(-a, 0, L^2(B(\sigma), \mathbb{R}^N)\right)$ and that the relative estimate holds. From (3.10) it follows, for a.e. $t \in (-a, 0)$:

$$\int_{B(\sigma)} \|D_i u\|^4 dx \leq c(K, U, \lambda, \sigma, n) \left\{ 1 + |u|^2_{1,B(\frac{5}{2}\sigma)} + |Du|^2_{1-\frac{\lambda}{2},B(\frac{5}{2}\sigma)} \right\},$$

$i = 1, 2, \ldots, n$, from which, by integrating with respect to t in $(-a, 0)$, we deduce:

$$D_i u \in L^4\left(B(\sigma) \times (-a, 0), \mathbb{R}^N\right), \quad i = 1, 2, \ldots, n,$$

and

$$\text{(3.19)} \qquad \int_{-a}^{0} dt \int_{B(\sigma)} \|Du\|^4 dx \leq$$
$$\leq c\,(K, U, \lambda, \sigma, n) \int_{-a}^{0} \left\{ 1 + |u|^2_{1,B(\frac{5}{2}\sigma)} + |Du|^2_{1-\frac{\lambda}{2},B(\frac{5}{2}\sigma)} \right\} dt \leq$$
$$\leq c(\nu, K, U, \lambda, \sigma, a, b, n) \left\{ 1 + \int_{-b}^{0} |u|^2_{1,B(3\sigma)} dt \right\} \ (^{11}).$$

Now, taking into account that $B^0(X, u, p)$ satisfies (1.2), we obtain

$$\text{(3.20)} \qquad B^0(X, u, Du) \in L^2\left(B(\sigma) \times (-a, 0), \mathbb{R}^N\right)$$

and

$$\text{(3.21)} \qquad \int_{-a}^{0} dt \int_{B(\sigma)} \|B^0(X, u, Du)\|^2 dx \leq$$
$$\leq c(K) \int_{-a}^{0} dt \int_{B(\sigma)} \left(1 + \|Du\|^4\right) dx.$$

(11) In the last estimate we made use of (3.6) (with $\vartheta = 1 - \frac{\lambda}{2}$).

On the other hand, from the assumption (1.3) on $a^i(X, u, p)$ we deduce that ([12])

$$D_i a^i(X, u, Du) \in L^2\left(B(\sigma) \times (-a, 0), \mathbb{R}^N\right), \quad i = 1, 2, \dots, n, \tag{3.22}$$

and that

$$\begin{aligned} \int_{-a}^{0} dt \int_{B(\sigma)} \sum_{i=1}^{n} \| D_i a^i\,(X, u, Du)\,\|^2 dx \leq \\ \leq c(K, n) \int_{-a}^{0} dt \int_{B(\sigma)} (1 + \|Du\|^4 + \sum_{i,j=1}^{n} \|D_{ij} u\|^2) dx\,. \end{aligned} \tag{3.23}$$

Now let us recall that u is a solution in Q (and hence in $B(\sigma) \times (-a, 0)$) to the system (1.1), and therefore, for each $\varphi \in C_0^\infty(B(\sigma) \times (-a, 0), \mathbb{R}^N)$, it results:

$$\begin{aligned} &\int_{-a}^{0} dt \int_{B(\sigma)} \left(u \Big| \frac{\partial \varphi}{\partial t} \right) dx = \\ &= - \int_{-a}^{0} dt \int_{B(\sigma)} \left(\sum_{i=1}^{n} D_i a^i(X, u, Du) + B^0(X, u, Du) \Big| \varphi \right) dx\,, \end{aligned}$$

from which, by virtue of (3.20) and (3.22), it follows that

$$\exists \; \frac{\partial u}{\partial t} \in L^2(B(\sigma) \times (-a, 0), \mathbb{R}^N). \tag{3.24}$$

From (3.21) and (3.23) we deduce

$$\begin{aligned} &\int_{-a}^{0} dt \int_{B(\sigma)} \left\| \frac{\partial u}{\partial t} \right\|^2 dx \leq \\ &\leq c\,(K, n) \int_{-a}^{0} dt \int_{B(\sigma)} \left(1 + \|Du\|^4 + \sum_{i,j=1}^{n} \|D_{ij} u\|^2 \right) dx\,, \end{aligned}$$

that, thanks to (3.18) and (3.19), provides us with

$$\int_{-a}^{0} dt \int_{B(\sigma)} \left\| \frac{\partial u}{\partial t} \right\|^2 dx \leq \tag{3.25}$$

([12]) See the proof of Theorem 2.1 in [8].

$$\leq c\,(\nu, K, U, \lambda, \sigma, a, b, n)\left\{1 + \int_{-b}^{0} |u|^2_{1,B(3\sigma)} dt\right\} \quad (^{13}).$$

Finally (3.1) and (3.2) follow from (3.17), (3.18), (3.24) and (3.25).

Remark 3.1. Theorem 3.1 can be proved by substituting (1.3) and (1.4) with the monotony assumptions (2.4) and (2.5) of [4]. It is enough to follow the technique in [4] instead of the one in [3].

Remark 3.2. Using Theorem 3.1 in this section and Theorem 4.1 in [7] it is easy to obtain a result of partial Hölder continuity for the spatial gradient of the solutions $u \in L^2\left(-T, 0, H^1(\Omega, \mathbb{R}^N)\right) \cap C^{0,\lambda}(Q, \mathbb{R}^N)$ to the system (1.1).

Remark 3.3. After the proof reading of this work, we got to the knowledge of a paper by J. Naumann and J. Wolf (see [14]), where a result similar to our one obtained in Sect. 3 was stated for differentiability of solutions to the system (1.1) by means of a different technique.

REFERENCES

[1] S. Campanato, *Sistemi ellittici in forma divergenza. Regolarità all'interno,* Quaderni Scuola Norm. Sup. Pisa, 1980.

[2] S. Campanato - P. Cannarsa, *Differentiability and Partial Hölder Continuity of the Solutions of Non-Linear Elliptic Systems of Order 2m with Quadratic Growth,* Ann. Scuola Norm. Sup. Pisa, (4) 8 (1981), pp. 285-309.

[3] L. Fattorusso, *Sulla differenziabilità delle soluzioni di sistemi parabolici non lineari del secondo ordine ad andamento quadratico,* Boll. U.M.I., (7) 1-B (1987), pp. 741-764.

[4] L. Fattorusso, *Un risultato di differenziabilità per sistemi parabolici non lineari in ipotesi di monotonia,* Rend. Circ. Matem. Palermo, (2) 39 (1990), pp. 412-426.

[5] L. Fattorusso, *Differenziabilità locale per sistemi parabolici non lineari del secondo ordine con non linearità $1 < q < 2$,* Le Matematiche, 48 (1993), pp. 331-347.

(^{13}) Taking into account that $u \in L^2\left(-a, 0, H^2(B(\sigma), \mathbb{R}^N)\right)$, the further result

$$u \in H^1\left(-a, 0, L^2(B(\sigma), \mathbb{R}^N)\right)$$

can be also obtained making use of Theorem 2.1 (with $\Omega = B(\sigma), m = r = 2, s = 0$, $j = 1, a = \frac{1-\lambda}{2-\lambda}$), as in the proof of Theorem 2.1 in [8].

[6] L. Fattorusso - M. Marino, *Differenziabilità locale per sistemi parabolici non lineari del secondo ordine con non linearità $q \geq 2$,* Ricerche di Matematica, 41 (1992), pp. 89-112.

[7] M. Marino - A. Maugeri, *Partial Hölder Continuity of the Spatial Derivatives of the Solutions to Nonlinear Parabolic Systems with Quadratic Growth,* Rend. Sem. Mat. Univ. Padova, 76 (1986), pp. 219-245.

[8] M. Marino - A. Maugeri, *A Remark on the Note : ≪Partial Hölder Continuity of the Spatial Derivatives of the Solutions to Nonlinear Parabolic Systems with Quadratic Growth≫,* Rend. Sem. Mat. Univ. Padova, 95 (1996).

[9] C. Miranda, *Su alcuni teoremi di inclusione,* Annales Polonici Math., 16 (1965), pp. 305-315.

[10] J. Naumann, *On the Interior Differentiability of Weak Solutions of Parabolic Systems with Quadratic Growth Nonlinearities,* Rend. Sem. Mat. Univ. Padova, 83 (1990), pp. 55-70.

[11] L. Nirenberg, *An extended interpolation inequality,* Ann. Scuola Norm. Sup. Pisa, (3) 20 (1966), pp. 733-737.

[12] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators,* North-Holland Publ. Comp., Amsterdam, New York, Oxford, 1978.

[13] H. Triebel, *Theory of Function Spaces,* Birkhäuser Verlag, Basel, Boston, Stuttgart, 1983.

[14] J. Naumann - J. Wolf, *Interior Differentiability of Weak Solutions to Parabolic Systems with Quadratic Growth Nonlinearities,* Rend. Sem. Mat. Univ. Padova, to appear.

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