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STUDY OF THE MATHEMATICAL MODEL FOR ABSORPTION AND DIFFUSION IN ULTRA-NAPKINS

PAOLA MANNUCCI

We analyse a mathematical model for absorption and diffusion of a fluid in ultra-napkins. We consider a diffusion equation coupled with an ordinary differential equation, subjected to a discontinuous Neumann boundary condition.
We prove the existence and the uniqueness of a regular solution which is continuous up to the boundary.

1. Introduction and preliminaries.

In this paper we consider a nonlinear diffusion equation coupled with an ordinary differential equation modelling the liquid transport in a napkin. We study the diffusive flow of a fluid in a mixture of a discrete material embedded in a continuous medium. An ultra-napkin consists of cellulose with granules of superabsorbent able to absorb a large quantity of liquid and hence able to keep it dry.
The mathematical model was derived by J. Weickert in [6]. Let u and v denote the concentrations of the fluid respectively in the cellulose and in the granules. The liquid is transported in the cellulose but not in the granules where it can be only absorbed. This means that u is governed by a nonlinear parabolic equation while the equation describing the behaviour of the fluid concentration in the granulate, v, does not have any diffusion term.

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Another important feature of this model is that the fluid can penetrate into the napkin only through a part of its boundary so that the problem involve a discontinuous Neumann boundary condition.
Let $Q \in \mathbb{R}^3$ denote an open domain of real variables $x = (x_1, x_2, x_3)$ and Q_T the product space $\{(x,t) : x \in Q, 0 < t < T\}$ where $T > 0$. $\overline{Q}$ and $\overline{Q_T}$ are the closures of Q and Q_T, ∂Q is the boundary of Q and S_T is the cylinder $\{(x,t) : x \in \partial Q, 0 \leq t \leq T\}$.
Let $u = u(x,t)$, $v = v(x,t)$. We will discuss the existence and uniqueness of a solution of the quasilinear parabolic system:

$$\frac{\partial u}{\partial t} = \frac{1}{\theta(x)} \nabla \cdot (\theta(x) d(u) \nabla u) - A(u,v), \quad \text{on } Q_T, \tag{1}$$

$$\frac{\partial v}{\partial t} = A(u,v), \quad \text{on } Q_T. \tag{2}$$

The function $A(u,v)$ describing the exchange of liquid between granules and cellulose is given, according to [6], by :

$$A(u,v) = \gamma(u\, v_\infty - v\, u_\infty), \tag{3}$$

where γ is a suitable positive constant, depending on the fluid and on the material and u_∞, v_∞ are two positive constants denoting the saturation concentrations in the two media that is the maximal liquid absorbable respectively by the cellulose and the granules.
The experiments ([6]) suggest, as conduction coefficient, the following function:

$$d(u) = a e^{bu} \tag{4}$$

a, b are positive constants depending on the material.
The function $\theta(x)$ express the fact that the fluid diffuses only in a part of the cellulose, called intermediate space. More precisely $u(x,t)$ represents the concentration of the fluid in the intermediate space. The function $\theta(x)$ is such that

$$\theta(x) \in C^1(\overline{Q}), \quad \theta(x) \geq \theta_0 > 0, \quad \forall x \in \overline{Q}. \tag{5}$$

Let $\partial Q = S_1 \cup S_2$ where S_2 is a convex domain on ∂Q. $S_{1T} = S_1 \times [0,T]$, $S_{2T} = S_2 \times [0,T]$.
We consider the following initial and boundary conditions

$$\begin{aligned} u(x,0) &= u_0(x), \quad x \in \overline{Q}, \\ v(x,0) &= v_0(x), \quad x \in \overline{Q}, \end{aligned} \tag{6}$$

(7)
$$\frac{\partial u}{\partial n} = 0, \qquad \text{on} \quad S_{1T},$$
$$\theta(x)d(u)\frac{\partial u}{\partial n} = k(t)(u_\infty - u), \quad \text{on} \quad \overline{S}_{2T}.$$

Here n is the unit outward normal to ∂Q and

(8)
$$k(t) = 0, \quad \text{for} \quad t > t_0\,, \quad k(t) = k_0\,, \quad \text{for} \quad 0 \le t \le t_0,$$

where k_0 is a positive constant and t_0 is the time for complete liquid absorption. The boundary conditions (7) show that there is a discontinuity of the normal derivative on ∂S_2 and on $t = t_0$. The boundary conditions take into account that the fluid enters the napkin only through S_2 and only for $t \le t_0$, where t_0 represents the instant when all the liquid has been absorbed into the diaper. Let $\partial S_{2t_0} = \partial S_2 \times [0, t_0]$, $\mathcal{F} = \partial S_{2t_0} \cup \{S_2 \times (t = t_0)\}$.
We assume that the initial conditions (6) are such that

(9)
$$0 \le u_0(x) \le u_\infty,$$
$$0 \le v_0(x) \le v_\infty.$$

We prove existence and uniqueness of the solution of the problem (1), (2) with initial and boundary conditions (6), (7), such that,

$$0 \le u(x,t) \le u_\infty,$$
$$0 \le v(x,t) \le v_\infty.$$

Although similar problems have been already studied (e.g. [3], [4], [5]), they are however quasilinear parabolic systems with continuous Dirichlet conditions or homogeneous Neumann conditions on the boundary. In our case we consider a coupling of a diffusion equation with an ODE, with discontinuous Neumann boundary conditions.
In the paper we use the notations of [2] where not differently specified.
The boundary conditions (7) can be rewritten in a compact form in the following way

(10)
$$\theta(x)d(u)\frac{\partial u}{\partial n} = K(x,t)(u_\infty - u) \quad \text{on} \quad S_T,$$

(11)
$$K(x,t) = 0\,, \quad \forall (x,t) \in S_{1T}\,, \quad K(x,t) = k(t)\,, \quad \forall (x,t) \in \overline{S}_{2T}\,,$$

where the function $k(t)$ is defined by (8).

Definition 1.1. *A classical solution of the problem* (1) - (2) *with initial and boundary conditions* (6), (10) *in* Q_T, *with* $T > 0$, *is a pair of functions* $u(x,t)$, $v(x,t)$ *such that*

1) $u(x,t) \in H^{2+\alpha, 1+\alpha/2}(Q_T) \cap C^0(\overline{Q}_T) \cap C^1(\overline{Q}_T \setminus \mathcal{F})$,
   $v(x,t) \in C^1(Q_T) \cap C^0(\overline{Q}_T)$;
2) $u(x,t)$, $v(x,t)$ *satisfy equations* (1), (2) *with* (6), (10) *in* Q_T.

In Section 2 we will approximate the function $K(x,t)$ in (11) by a sequence of "smooth" functions. For the related problem we find some comparison principles and the existence and uniqueness of a classical solution.

In Section 3 we prove the existence and uniqueness of a classical solution for the problem (1), (2), (6), (10) according to Definition 1.1.

2. The approximating problem.

Let $K_n(x,t) \in C^\infty(S_T)$ be a sequence of smooth functions such that

$$\begin{aligned} &\lim_n |K_n(x,t) - K(x,t)|_{L^2} = 0, && \forall (x,t) \in S_T, \\ &K_n(x,t) \to K(x,t) \text{ in } C^1\text{-norm}, && \text{in } S_T \setminus \mathcal{F}. \end{aligned} \tag{12}$$

Let us consider the following approximating problem

$$\frac{\partial u_n}{\partial t} = \frac{1}{\theta(x)} \nabla \cdot (\theta(x) d(u_n) \nabla u_n) - A(u_n, v_n), \qquad \text{on } Q_T, \tag{13}$$

$$\frac{\partial v_n}{\partial t} = A(u_n, v_n), \qquad \text{on } Q_T, \tag{14}$$

$$\begin{aligned} u_n(x,0) &= u_0(x), \quad x \in \overline{Q}, \\ v_n(x,0) &= v_0(x), \quad x \in \overline{Q}, \end{aligned} \tag{15}$$

$$\theta(x) d(u_n) \frac{\partial u_n}{\partial n} = K_n(x,t)(u_\infty - u_n) \text{ on } S_T. \tag{16}$$

In this paragraph we prove existence and uniqueness of the solution for the problem (13) – (16) having regular Neumann conditions on the boundary. We start with some comparison principles leading immediately to the boundedness and uniqueness of a classical solution of the regularized problem.

Lemma 2.1. *Let $\theta(x)$, $d(u)$, $K_n(x,t)$, $A(u,v)$ be defined by* (5), (4), (12), (3). *Let*

$$F(u) = \frac{1}{\theta(x)}\nabla \cdot (\theta(x)d(u)\nabla u) - \frac{\partial u}{\partial t}\,. \tag{17}$$

Suppose that

a) *$u_i(x,t)$, $v_i(x,t)$ exist and $u_i(x,t) \in H^{2+\alpha,1+\alpha/2}(Q_T)\cap C^1(\overline{Q}_T)$, $v_i(x,t) \in C^1(Q_T)\cap C^0(\overline{Q}_T)$, $i = 1,2$;*

b) *$u_i(x,t)$ and $v_i(x,t)$ with their derivatives satisfy in Q_T the differential inequalities:*

$$F(u_1) - A(u_1,v_1) > F(u_2) - A(u_2,v_2)\,,$$
$$\frac{\partial v_1}{\partial t} - A(u_1,v_1) < \frac{\partial v_2}{\partial t} - A(u_2,v_2)\,;$$

c) *the following inequalities hold:*

$$u_1 < u_2, \quad v_1 < v_2, \quad at\ t = 0,$$
$$\theta(x)d(u_1)\frac{\partial u_1}{\partial n} + K_n(x,t)u_1 < \theta(x)d(u_2)\frac{\partial u_2}{\partial n} + K_n(x,t)u_2\,, \quad on\ S_T\,,$$

then $u_1 < u_2$ and $v_1 < v_2$ in $\overline{Q}_T$.

Proof. Let us suppose that a point $P' = (x',t') \in \overline{Q}_T$ exists such that $u_1 < u_2$ and $v_1 < v_2$ in $Q_{t'}$ and $u_1(x',t') = u_2(x',t')$ or $v_1(x',t') = v_2(x',t')$.
Suppose that in P' we have $u_1 = u_2$ and $v_1 < v_2$.

i) if $x' \notin S$, the function $u_1(x,t) - u_2(x,t)$ has an internal maximum in P', hence

$$F(u_1) - A(u_1,v_1) < F(u_2) - A(u_2,v_2),$$

which contradicts assumption b).

ii) if $x' \in S$, from assumption c)

$$K_n(x',t')(u_1(x',t') - u_2(x',t')) < \theta(x')d(u_2(x',t'))\Big(\frac{\partial u_2}{\partial n}(x',t') - \frac{\partial u_1}{\partial n}(x',t')\Big),$$

hence $\frac{\partial u_2}{\partial n}(x',t') > \frac{\partial u_1}{\partial n}(x',t')$ which is a contradiction. (n is the unit outward normal).
If in P' we have $v_1 = v_2$ and $u_1 < u_2$, the function $u_1(x,t) - u_2(x,t)$ has a maximum in P', hence a contradiction with assumption b). □

Lemma 2.2. *Let $\theta(x)$, $d(u)$, $K_n(x,t)$, $A(u,v)$ be defined by* (5), (4), (12),(3). *Let $F(u)$ be defined by* (17). *Suppose that*

a) *$u_i(x,t)$, $v_i(x,t)$ exist and $u_i(x,t) \in H^{2+\alpha,1+\alpha/2}(Q_T) \cap C^1(\overline{Q}_T)$, $v_i(x,t) \in C^1(Q_T) \cap C^0(\overline{Q}_T)$, $i = 1, 2$;*

b) *$u_i(x,t)$ and $v_i(x,t)$ with their derivatives satisfy in Q_T the differential inequalities:*

$$F(u_1) - A(u_1, v_1) \geq F(u_2) - A(u_2, v_2)\,,$$
$$\frac{\partial v_1}{\partial t} - A(u_1, v_1) \leq \frac{\partial v_2}{\partial t} - A(u_2, v_2)\,;$$

c) *the following inequalities hold:*

$$u_1 \leq u_2, \quad v_1 \leq v_2, \quad \text{at } t = 0,$$
$$\theta(x) d(u_1) \frac{\partial u_1}{\partial n} + K_n(x,t) u_1 \leq \theta(x) d(u_2) \frac{\partial u_2}{\partial n} + K_n(x,t) u_2 \quad \text{on } S_T\,,$$

then $u_1 \leq u_2$ and $v_1 \leq v_2$ in $\overline{Q}_T$.

Proof. We prove the theorem for a linear operator of the following type

$$L(u) = a(x,t)\Delta u + b(x,t)\nabla u + c(x,t)u - \frac{\partial u}{\partial t}$$

and next we pass to the nonlinear case.
Let us introduce the barrier function $\lambda\, e^{Mt}$, where λ and M are constants to be determined. Let L be the Lipschitz constant of $A(u, v)$, then

$$\begin{aligned}
&L(u_2 + \lambda\, e^{Mt}) - A(u_2 + \lambda\, e^{Mt}, v_2 + \lambda\, e^{Mt}) = \\
&= L(u_2) + c(x,t)\lambda\, e^{Mt} - \lambda M\, e^{Mt} - A(u_2 + \lambda\, e^{Mt}, v_2 + \lambda\, e^{Mt}) \leq \\
&\leq L(u_1) - A(u_1, v_1) + A(u_2, v_2) + c(x,t)\lambda\, e^{Mt} - \lambda\, M e^{Mt} - \\
&- A(u_2 + \lambda\, e^{Mt}, v_2 + \lambda\, e^{Mt}) \leq L(u_1) - A(u_1, v_1) + 2L\lambda\, e^{Mt} + \\
&+ c(x,t)\lambda\, e^{Mt} - \lambda\, M e^{Mt} = L(u_1) - A(u_1, v_1) + \lambda\, e^{Mt}(c(x,t) + 2L - M).
\end{aligned}$$

Moreover

$$\begin{aligned}
&\frac{\partial (v_2 + \lambda\, e^{Mt})}{\partial t} - A(u_2 + \lambda\, e^{Mt}, v_2 + \lambda\, e^{Mt}) = \\
&= \frac{\partial v_2}{\partial t} - \lambda\, M e^{Mt} - A(u_2 + \lambda\, e^{Mt}, v_2 + \lambda\, e^{Mt}) \leq \\
&\leq \frac{\partial v_1}{\partial t} - A(u_1, v_1) + A(u_2, v_2) - \lambda\, M\, e^{Mt} - \\
&- A(u_2 + \lambda\, e^{Mt}, v_2 + \lambda\, e^{Mt}) \leq \\
&\leq \frac{\partial v_1}{\partial t} - A(u_1, v_1) + 2L\lambda\, e^{Mt} - \lambda M\, e^{Mt} = \\
&= \frac{\partial v_1}{\partial t} - A(u_1, v_1) + \lambda\, e^{Mt}(2L - M).
\end{aligned}$$

If we take $M > \max(|c|_0 + 2L, 2L)$ we may apply Lemma 2.1 to u_1, v_1 and $u_2 + \lambda e^{Mt}$, $v_2 + \lambda e^{Mt}$ and we obtain $u_1 < u_2 + \lambda e^{Mt}$ and $v_1 < v_2 + \lambda e^{Mt}$ in $\overline{Q}_T$. If $\lambda \to 0$ we get the result.
Now we pass to the case of a nonlinear operator like $F(u)$.
From assumption b) we have

$$F(u_1) - F(u_2) \geq A(u_1, v_1) - A(u_2, v_2).$$

We may apply the mean-value theorem for multidimensional calculus. Evaluating the derivatives of $F(u)$ at the arguments $\tau u_1+(1-\tau)u_2$, $\tau\nabla u_1+(1-\tau)\nabla u_2$, $\tau\Delta u_1+(1-\tau)\Delta u_2$, where $\tau \in (0, 1)$, we obtain a linear parabolic operator $L(u)$ hence the proof follows. □

Corollary 2.3. *If we take $u_n(x, t)$, $v_n(x, t)$ a classical solution of the problem* (13), (14), (15), (16) *such that* (9) *hold, we immediately obtain from Lemma 2.2 that*

$$0 \leq u_n(x, t) \leq u_\infty, \quad \forall (x, t) \in \overline{Q}_T, \tag{18}$$

$$0 \leq v_n(x, t) \leq v_\infty, \quad \forall (x, t) \in \overline{Q}_T. \tag{19}$$

Remark 2.4. *If we take u_∞ = constant and $v_\infty = v_\infty(x)$, where $v_\infty(x)$ is a positive regular function as in* [6], *we obtain analogous results.*

Remark 2.5. *We note that Lemma 2.1 and Lemma 2.2 hold also for the classical solution of the problem* (1), (2), (6), (10). *We have only to replace assumptions* c) *with the corresponding assumptions for the boundary conditions* (10), (11).

Theorem 2.6. *Let $u_0(x) \in H^{2+\alpha}(\overline{Q})$, $v_0(x) \in C^1(\overline{Q})$, satisfying* (9). *Then there exists a unique solution $u_n(x, t)$, $v_n(x, t)$ of the problem* (13), (14), (15), (16) *such that $u_n(x, t) \in H^{2+\alpha, 1+\alpha/2}(\overline{Q}_T)$, $v_n(x, t) \in C^1(\overline{Q}_T)$, $\forall T > 0$.*

Proof. The uniqueness follows from Lemma 2.2.
We prove the existence of a classical solution by means of a fixed point theorem. Let us define

$$\mathcal{B} = \left\{u_n \in C^1(Q_T) : 0 \leq u_n \leq u_\infty, \mid u_n \mid_1 \leq M\right\}.$$

If $\bar{u}_n \in \mathcal{B}$, let us consider the solution v_n of the problem

$$\begin{aligned} \frac{\partial v_n}{\partial t} &= A(\bar{u}_n, v_n), \\ v_n(x, 0) &= v_0(x). \end{aligned} \tag{20}$$

Then $v_n(x,t) \in C^1(\overline{Q}_T)$, $0 \le v_n(x,t) \le v_\infty$, $\forall (x,t) \in \overline{Q}_T$. Let $F(u)$ be the parabolic operator defined by (17) and $u_n(x,t)$ the solution of the problem

$$
\begin{aligned}
&F(u_n) = A(u_n, v_n), && \text{on } Q_T, \\
&u_n(x,0) = u_0(x), && \text{on } \overline{Q}, \\
&\theta(x) d(u_n) \frac{\partial u_n}{\partial n} = K_n(x,t)(u_\infty - u_n) && \text{on } S_T.
\end{aligned} \tag{21}
$$

From the smoothness of $A(u,v)$ (see (3)) and classical results on parabolic problems (see Theorem 7.2, Theorem 7.4, p. 486, 491 of [2]) a solution u_n of (21), exists such that $u_n(x,t) \in H^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$,

$$0 \le u_n(x,t) \le u_\infty\,, \quad \forall (x,t) \in \overline{Q}_T\,, \tag{22}$$

$$|u_n|_{Q_T}^{1+\delta} \le C_n\,, \tag{23}$$

where C_n is a constant independent of M.
Let us define the operator $\mathcal{T}$ such that

$$\text{if } \bar{u}_n \in \mathcal{B}\,, \quad \mathcal{T}\bar{u}_n = u_n$$

where u_n is the solution of problem (21). From (22) and (23) we have that $u_n \in \mathcal{B}$, thus $\mathcal{T} : \mathcal{B} \to \mathcal{B}$.
Moreover, from the maximum principle,

$$
\begin{gathered}
\mathcal{T}\bar{u}_{1n} - \mathcal{T}\bar{u}_{2n} = |u_{1n} - u_{2n}| \le c(|\bar{u}_{1n} - \bar{u}_{2n}| + |v_{1n} - v_{2n}|), \\
|v_{1n} - v_{2n}| \le c(|\bar{u}_{1n} - \bar{u}_{2n}|),
\end{gathered}
$$

then $\mathcal{T}$ is a continuous operator. Furthermore from estimate (23), $\mathcal{T}$ is a compact operator.
Then there exists a fixed point $u_n(x,t) \in \mathcal{B}$ of the operator $\mathcal{T}$. The couple u_n, v_n is the classical solution of the problem (13), (14), (15), (16). □

3. Convergence of the approximating problem.

Theorem 3.1. *Let $u_0(x) \in H^{2+\alpha}(\overline{Q})$, $v_0(x) \in C^1(\overline{Q})$, satisfying* (9). *Suppose that the compatibility conditions of the zero order between boundary and initial data hold. Then there exists a unique solution $u(x,t)$, $v(x,t)$ of the problem* (1), (2), (6), (10) *such that $u(x,t) \in H^{2+\alpha,1+\alpha/2}(Q_T) \cap C^0(\overline{Q}_T)$, $v(x,t) \in C^1(Q_T) \cap C^0(\overline{Q}_T) \cap C^1(\overline{Q}_T \setminus \mathcal{F})$.*

Proof. From estimates (18) and (19), the sequences $\{u_n\}$ and $\{v_n\}$, classical solutions of the problems (13), (14), (15), (16) are equibounded.
We show that they are also equicontinuous. In fact, by means of (18) and (19), considering $d(u_n)$, $A(u_n, v_n)$, $K_n(x,t)(u_\infty - u_n)$ as a known terms and denoting by $\Gamma_n(x,t,\xi,\tau)$ the fundamental solution of the parabolic operator in (13), we represent the solution u_n in the following way

$$u_n(x,t) = \int_Q \Gamma_n(x,t,\xi,0)u_0(x)dx + \tag{24}$$

$$+\int_0^t\int_S \Gamma_n(x,t,\xi,\tau)\phi_n(\xi,\tau)d\sigma_\xi d\tau - \int_0^t\int_Q \Gamma_n(x,t,\xi,\tau)A(u_n,v_n)(\xi,\tau)d\xi d\tau,$$

where $\phi_n(x,t)$ is an unknown functions which can be determined imposing the boundary conditions (16).
Following a standard procedure (see [1]), we obtain an integral equation of the second kind for the unknown function $\phi_n(x,t)$

$$\phi_n(x,t) = 2\int_0^t\int_S \frac{\partial \Gamma_n(x,t,\xi,\tau)}{\partial \nu}\phi_n(\xi,\tau)d\sigma_\xi d\tau + 2F_n(x,t). \tag{25}$$

The kernel of this integral equation

$$N_n(x,t) = \frac{\partial \Gamma_n(x,t,\xi,\tau)}{\partial \nu}$$

has a weak singularity. In fact from standard estimates on the fundamental solution, (see [1]), and (23)

$$|N_n| \le \frac{|K_n|}{(t-\tau)^\mu |x-\xi|^\lambda}, \tag{26}$$

where $\mu \in (0,1)$, $\lambda \in (0,2)$ and $t > \tau$.
The nonhomogeneous term in (25) is

$$F_n(x,t) = \int_Q \Gamma_n(x,t,\xi,0)u_0(x)dx - \tag{27}$$

$$-\int_0^t\int_Q \frac{\partial \Gamma_n(x,t,\xi,\tau)}{\partial \nu}A(u_n,v_n)(\xi,\tau)d\xi d\tau - K_n(x,t)(u_\infty - u_n).$$

From assumptions (3), (12) and estimates (18), (19) follows that

$$\max_{\overline{Q}_T} |F_n(x,t)| \le C, \quad \forall n \in \mathbb{N}. \tag{28}$$

Hence, taking into account of (26), from expression (25) we deduce that

$$\max_{\overline{Q}_T} |\phi_n(x,t)| \leq C, \quad \forall n \in \mathbb{N}. \tag{29}$$

Then from expression (24) and the properties of the fundamental solution (see p. 395 of [2]) we have that the sequence $\{u_n\}$ is equicontinuous in $\overline{Q}_T$.
Moreover we may represent $v_n(x,t)$ by

$$v_n(x,t) = v_0(x) + \int_0^t A(u_n, v_n)(x,\tau)d\tau. \tag{30}$$

From Gronwall's Lemma and the equicontinuity of $\{u_n\}$, follows the equicontinuity of the sequence $\{v_n\}$ in $\overline{Q}_T$.
Then we may extract two subsequences $\{u_{nk}\}$, $\{v_{nk}\}$ converging to two functions $\bar{u}$ and $\bar{v}$. We have that

$$\begin{aligned} 0 \leq \bar{u}(x,t) \leq u_\infty, \quad \forall (x,t) \in \overline{Q}_T, \\ 0 \leq \bar{v}(x,t) \leq v_\infty, \quad \forall (x,t) \in \overline{Q}_T, \end{aligned} \tag{31}$$

and the functions $\bar{u}(x,t)$ and $\bar{v}(x,t)$ are continuous in $\overline{Q}_T$.
Let us consider now the solution $u'(x,t)$, $v'(x,t)$ of the problem

$$\begin{aligned} &\frac{\partial u'}{\partial t} = \frac{1}{\theta(x)} \nabla \cdot (\theta(x) d(\bar{u}) \nabla u') - A(\bar{u}, \bar{v}), && (x,t) \in Q_T, \\ &\frac{\partial v'}{\partial t} = A(\bar{u}, \bar{v}), && (x,t) \in Q_T, \\ &u'(x,0) = u_0(x), && x \in \overline{Q}, \\ &v'(x,0) = v_0(x), && x \in \overline{Q}, \\ &\theta(x) d(\bar{u}) \frac{\partial u'}{\partial n} = K(x,t)(u_\infty - \bar{u}), && \text{on } S_T. \end{aligned} \tag{32}$$

From (31), the continuity of $\bar{u}(x,t)$ and $\bar{v}(x,t)$ and classical results (see Theorem 6.3, p. 459 of [2]), follows that the problem (32) has a classical solution u', v' in Q_T such that $u' \in C^0(\overline{Q}_T)$.
Moreover if we consider u_n, v_n solution of (13), (14), (15), (16) and we apply the maximum principle to $u' - u_n$ and $v' - v_n$, taking into account of the continuity of $d(u)$, $A(u,v)$ and of (12), we obtain that the subsequences $\{u_{nk}\}$, $\{v_{nk}\}$ converge to u', v' in $L_2(Q_T)$ then $u' \equiv \bar{u}$, $v' \equiv \bar{v}$. Hence $\bar{u}(x,t)$, $\bar{v}(x,t)$ is a

solution of the problem (1), (2), (6), (10) such that $\bar{u}(x,t) \in H^{2+\alpha,1+\alpha/2}(Q_T) \cap C^1(\overline{Q}_T \setminus \mathcal{F}) \cap C^0(\overline{Q}_T)$, $\bar{v}(x,t) \in C^1(Q_T) \cap C^0(\overline{Q}_T)$.
Moreover this solution is unique. In fact, from Remark 2.5, we may apply Lemma 2.2 to a solution $u(x,t)$, $v(x,t)$. The boundedness of such a solution, assumption (3) and the regularity of the coefficient of the problem give the uniqueness of a weak solution and hence the uniqueness of the solution $u(x,t) \in H^{2+\alpha,1+\alpha/2}(Q_T) \cap C^1(\overline{Q}_T \setminus \mathcal{F}) \cap C^0(\overline{Q}_T)$, $v(x,t) \in C^1(Q_T) \cap C^0(\overline{Q}_T)$ of the problem. □

Remark 3.2. *We note that the discontinuity of the function $k(t)$ in* (8) *for $t = t_0$ does not affect the continuity of the solution $u(x,t)$ up to the boundary S_T.*

Remark 3.3. *If we substitute the boundary condition* (7) *with (see* [6])

$$\begin{aligned} &\frac{\partial u}{\partial n} = 0, \quad \text{on } S_{1t}, \\ &u = M, \quad \text{on } \overline{S}_{2t}, \text{ with } t \le t_O, \\ &\frac{\partial u}{\partial n} = 0, \quad \text{on } \overline{S}_{2t}, \text{ with } t \ge t_0, \end{aligned} \tag{33}$$

with M a suitable constant, we may apply an analogous approximation for the condition on S_{2T} as done previously and we obtain the same results as in Theorem 3.1.

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Dipartimento di Matematica, "U. Dini",
Università di Firenze,
Viale Morgagni 67/a,
50134 Firenze (ITALY)