FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS OF TYPE (P) AND APPLICATIONS TO DYNAMIC PROGRAMMING

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In this paper, we prove some common fixed point theorems for compatible mappings of type (P). As applications, the existence and uniqueness of common solutions for a class of the functional equations in dynamic programming are discussed.

1. Introduction.

In [18], the concept of compatible mappings of type (P) was introduced and compared with compatible mappings ([9]–[16]) and compatible mappings of type (A) ([13], [17]). The purpose of this paper is to prove some common fixed point theorems for compatible mappings of type (P), which extend and improve some recent results of [5], [8], [10] and [13]. As applications, we use our main results to study the existence and uniqueness problems of common solutions for a class of functional equations arising in dynamic programming. The main results extend and improve the corresponding results of [2], [4] and [5].

Entrato in Redazione l’8 settembre 1994.

1991 AMS Mathematics Subject Classification : 54H25, 47H10.

Key words and Phrases: Common fixed point, Compatible mappings of types (A) and (P) and dynamic programming.
2. Compatible Mappings of Type (P).

Throughout this section, let \((X, d)\) denote a metric space. We recall the following definitions and properties of compatible mappings, compatible mappings of type (A) and compatible mappings of type (P) ([9], [13], [18]).

**Definition 2.1.** Let \(S, T : (X, d) \to (X, d)\) be mappings. The mappings \(S\) and \(T\) are said to be **compatible** if

\[
\lim_{n \to \infty} d(STx_n, TTx_n) = 0
\]

whenever \(\{x_n\}\) is a sequence in \(X\) such that \(\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z\) for some \(z\) in \(X\).

**Definition 2.2.** Let \(S, T : (X, d) \to (X, d)\) be mappings. The mappings \(S\) and \(T\) are said to be **compatible of type (A)** if

\[
\lim_{n \to \infty} d(TSx_n, SSx_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(STx_n, TTx_n) = 0
\]

whenever \(\{x_n\}\) is a sequence in \(X\) such that \(\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z\) for some \(z\) in \(X\).

**Definition 2.3.** Let \(S, T : (X, d) \to (X, d)\) be mappings. The mappings \(S\) and \(T\) are said to be **compatible of type (P)** if

\[
\lim_{n \to \infty} d(SSx_n, TTx_n) = 0
\]

whenever \(\{x_n\}\) is a sequence in \(X\) such that \(\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z\) for some \(z\) in \(X\).

The following propositions show that Definitions 2.1 and 2.2 are equivalent under some conditions:

**Proposition 2.1.** Let \(S, T : (X, d) \to (X, d)\) be continuous mappings. If \(S\) and \(T\) are compatible, then they are compatible of type (A).

**Proposition 2.2.** Let \(S, T : (X, d) \to (X, d)\) be compatible mappings of type (A). If one of \(S\) and \(T\) is continuous, then \(S\) and \(T\) are compatible.

The following is a direct consequence of Propositions 2.1 and 2.2:

**Proposition 2.3.** Let \(S, T : (X, d) \to (X, d)\) be continuous mappings. Then \(S\) and \(T\) are compatible if and only if they are compatible of type (A).
Remark 1. In [13], we can find two examples that Proposition 2.3 is not true if $S$ and $T$ are not continuous on a metric space.

We can show also that if $S$ and $T$ are continuous, then $S$ and $T$ are compatible if and only if they are compatible of type $(P)$ as follows:

**Proposition 2.4.** Let $S$, $T : (X, d) \to (X, d)$ be continuous mappings. Then $S$ and $T$ are compatible if and only if they are compatible of type $(P)$.

**Proof.** Let $\{x_n\}$ be a sequence in $X$ such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z$$

for some $z \in X$. Since $S$ and $T$ are continuous,

$$\lim_{n \to \infty} SSx_n = \lim_{n \to \infty} STx_n = Sz$$

and

$$\lim_{n \to \infty} TSSx_n = \lim_{n \to \infty} TTx_n = Tz.$$ 

Suppose that $S$ and $T$ are compatible. Then we have

$$\lim_{n \to \infty} d(STx_n, TSSx_n) = 0.$$ 

Now, since we have

$$d(SSx_n, TTx_n) \leq d(SSx_n, STx_n) + d(STx_n, TTx_n)$$

$$\leq d(SSx_n, STx_n) + d(STx_n, TSSx_n) + d(TSSx_n, TTx_n),$$

it follows that $\lim_{n \to \infty} d(SSx_n, TTx_n) = 0$. Thus, the mappings $S$ and $T$ are compatible of type $(P)$.

Conversely, suppose that $S$ and $T$ are compatible mappings of type $(P)$, that is,

$$\lim_{n \to \infty} d(SSx_n, TTx_n) = 0.$$ 

We then have

$$d(STx_n, TSSx_n) \leq d(STx_n, SSx_n) + d(SSx_n, TSSx_n)$$

$$\leq d(STx_n, SSx_n) + d(SSx_n, TTx_n) + d(TTx_n, TSSx_n).$$

Therefore, it follows that $\lim_{n \to \infty} d(STx_n, TSSx_n) = 0$. This completes the proof.
**Proposition 2.5.** Let \( S, T : (X, d) \to (X, d) \) be compatible mappings of type (A). If one of \( S \) and \( T \) is continuous, then \( S \) and \( T \) are compatible of type (P).

**Proof.** Let \( \{x_n\} \) be a sequence in \( X \) such that

\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z
\]

for some \( z \in X \). Suppose that \( S \) and \( T \) are compatible mappings of type (A).

Assume, without loss of generality, that \( S \) is continuous. We then have

\[
d(SSx_n, TTx_n) \leq d(SSx_n, STx_n) + d(STx_n, TTx_n)
\]

and so, since \( S \) and \( T \) are compatible of type (A), we have

\[
\lim_{n \to \infty} d(SSx_n, TSx_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(STx_n, TTx_n) = 0.
\]

Therefore, it follows that

\[
\lim_{n \to \infty} d(SSx_n, TTx_n) = 0.
\]

This completes the proof.

As a direct consequence of Propositions 2.3 – 2.5, we have the following:

**Proposition 2.6.** Let \( S, T : (X, d) \to (X, d) \) be continuous mappings. Then

1. \( S \) and \( T \) are compatible if and only if they are compatible of type (P).
2. \( S \) and \( T \) are compatible of type (A) if and only if they are compatible of type (P).

Next, we give several properties of compatible mappings of type (P) for our main theorems:

**Proposition 2.7.** Let \( S, T : (X, d) \to (X, d) \) be mappings. If \( S \) and \( T \) are compatible of type (P) and \( Sz = Tz \) for some \( z \in X \), then \( SSz = STz = TSz = TTz \).

**Proof.** Let \( \{x_n\} \) be a sequence in \( X \) defined by \( x_n = z \), \( n = 1, 2, \ldots \), and \( Sz = Tz \) for some \( z \in X \). Then we have \( Sx_n, Tx_n \to Sz \) as \( n \to \infty \). Since \( S \) and \( T \) are compatible of type (P), we have

\[
d(SSz, TTz) = \lim_{n \to \infty} d(SSx_n, TTx_n) = 0.
\]

Therefore, \( SSz = TTz \). But \( Sz = Tz \) implies \( SSz = STz = TSz = TTz \). This completes the proof.
Proposition 2.8. Let $S, T : (X, d) \to (X, d)$ be mappings. Let $S$ and $T$ are compatible mappings of type $(P)$ and let $Sx_n, Tx_n \to z$ as $n \to \infty$ for some $z \in X$. Then we have the following:

1. $\lim_{n \to \infty} TTx_n = Sz$ if $S$ is continuous at $z$.
2. $\lim_{n \to \infty} SSx_n = Tz$ if $T$ is continuous at $z$.
3. $STz = TSz$ and $Sz = Tz$ if $S$ and $T$ are continuous at $z$.

Proof. (1) Suppose that $S$ is continuous at $z$. Since

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z$$

for some $z \in X$, we have $SSx_n \to Sz$ as $n \to \infty$. Again, since $S$ and $T$ are compatible of type $(P)$, we have $\lim_{n \to \infty} d(TTx_n, SSx_n) = 0$ and so, since we have

$$d(TTx_n, Sz) \leq d(TTx_n, SSx_n) + d(SSx_n, Sz),$$

it follows that $TTx_n \to Sz$ as $n \to \infty$.

(2) The proof of $\lim_{n \to \infty} SSx_n = Tz$ follows on the similar lines as argued in (1).

(3) Suppose that $S$ and $T$ are continuous at $z$. Since $Tx_n \to z$ as $n \to \infty$ and $S$ is continuous at $z$, by (1), $TTx_n \to Sz$ as $n \to \infty$. On the other hand, since $Tx_n \to z$ as $n \to \infty$ and $T$ is also continuous at $z$, $TTx_n \to Tz$. Thus, we have $Sz = Tz$ by the uniqueness of the limit and so, by Proposition 2.7, $TSz = STz$. This completes the proof.

3. Common Fixed Point Theorems (I).

In this section, we prove some common fixed point theorems in metric spaces:

Theorem 3.1. Let $(X, d)$ be a complete metric space and $A, B, S$ and $T$ be mappings from $X$ into itself. Suppose that $S$ and $T$ are continuous mappings satisfying the following conditions:

1. $A(X) \subset T(X)$ and $B(X) \subset S(X)$,
2. the pairs $\{A, S\}$ and $\{B, T\}$ are compatible of type $(P)$,
3. $d(Ax, By) \leq \Phi(\max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By)\}$
$$\frac{1}{2}[d(Sx, By) + d(Ty, Ax)])$$. 


for all \( x, y \in X \), where \( \Phi : [0, \infty) \to [0, \infty) \) is a nondecreasing and upper semicontinuous function and \( \Phi(t) < t \) for all \( t > 0 \). Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**Proof.** Since \( A(X) \subset T(X) \) and \( B(X) \subset S(X) \), we can choose a sequence \( \{x_n\} \) in \( X \) such that \( Sx_{2n} = Bx_{2n-1} \) and \( Tx_{2n-1} = Ax_{2n-2} \) for \( n = 1, 2, 3, \ldots \). Suppose that

\[
(3.4) \quad y_{2n-1} = Tx_{2n-1} = Ax_{2n-2} \quad \text{and} \quad y_{2n} = Sx_{2n} = Bx_{2n-1}
\]

for \( n = 1, 2, 3, \ldots \). By using the technique of Chang [5], we can prove that \( \{y_n\} \) is a Cauchy sequence in \( X \) and so, since \( X \) is complete, it converges to a point \( z \) in \( X \). On the other hand, the subsequences \( \{Ax_{2n-2}\} \), \( \{Bx_{2n-1}\} \), \( \{Sx_{2n}\} \) and \( \{Tx_{2n-1}\} \) of \( \{y_n\} \) also converge to the point \( z \).

Since \( \{A, S\} \) and \( \{B, T\} \) are compatible of type \((P)\), it follows from the continuity of \( S \) and \( T \), (3.4) and Proposition 2.8 that

\[
(3.5) \quad Ty_{2n} \to Tz, \quad By_{2n} = BBx_{2n-1} \to Tz,
\]

\[
Sy_{2n-1} \to Sz, \quad Ay_{2n-1} = AAx_{2n-2} \to Sz
\]
as \( n \to \infty \). By (3.3) and (3.4), we have

\[
d(Ay_{2n-1}, By_{2n}) \\
\leq \Phi(\max\{d(Sy_{2n-1}, Ty_{2n}), d(Sy_{2n-1}, Ay_{2n-1}), d(Ty_{2n}, By_{2n}), \\
\frac{1}{2}[d(Sy_{2n-1}, By_{2n-2}) + d(Ty_{2n}, Ay_{2n-1})]\}).
\]

By the upper semicontinuity of \( \Phi(t) \), (3.4) and (3.5), if \( Sz \neq Tz \), then we have

\[
d(Sz, Tz) \leq \Phi(\max\{d(Sz, Tz), 0, 0, d(Sz, Tz)\}) \\
= \Phi(d(Sz, Tz)) < d(Sz, Tz),
\]

which is a contradiction. Thus it follows that \( Sz = Tz \).

Similarly, from (3.3), (3.4), (3.5) and the upper semicontinuity of \( \Phi \), we can obtain \( Sz = Bz \) and \( Tz = Az \). Hence we have

\[
(3.6) \quad Az = Bz = Sz = Tz.
\]

From (3.3) and (3.4), we have also

\[
d(Ax_{2n}, Bz) \leq \Phi(\max\{d(Sx_{2n}, Tz), d(Sx_{2n}, Ax_{2n}), d(TzBz), \\
\frac{1}{2}[d(Sx_{2n}, Bz) + d(Tz, Ax_{2n})]\})).
\]
This implies that, if $Bz \neq z$, then

$$d(z, Bz) \leq \Phi(d(z, Bz)) < d(z, Bz),$$

which is a contradiction. Therefore, we have $z = Az = Bz = Sz = Tz$. The uniqueness of the fixed point $z$ is obvious from (3.2). This completes the proof.

From Theorem 3.1, we have the following:

**Theorem 3.2.** Let $(X, d)$ be a complete metric space and $A$ and $B$ be mappings from $X$ into itself satisfying the following condition:

$$d(Ax, By) \leq \Phi(\max\{d(x, y), d(x, Ax), d(y, By),$$

$$\frac{1}{2}[d(x, By) + d(y, Ax)]\})$$

for all $x, y$ in $X$, where $\Phi(t)$ is the same as in Theorem 3.1. Then $A$ and $B$ have a unique common fixed point in $X$.

**Proof.** Define a sequence $\{x_n\}$ in $X$ by

$$x_{2n-1} = Ax_{2n-2} \quad \text{and} \quad x_{2n} = Bx_{2n-2}$$

for $n = 1, 2, 3, \ldots$. Then it is easy to show that $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, letting $x_n \rightarrow z \in X$ as $n \rightarrow \infty$, we know that $\{x_{2n-1}\}$ and $\{x_{2n}\}$ converge to $z$, too. By (3.7) and (3.8), we have

$$d(Az, x_{2n}) \leq d(Az, Bx_{2n-2})$$

$$\leq \Phi(\max\{d(z, x_{2n-2}), d(z, Az), d(x_{2n-2}, x_{2n}),$$

$$\frac{1}{2}[d(z, x_{2n}) + d(x_{2n-2}, Az)]\}).$$

By the upper semicontinuity of $\Phi(t)$, if $Az \neq z$, then we have

$$d(Az, z) \leq \Phi(d(z, Az)) < d(z, Az),$$

which is contradiction and so $z = Az$. Similarly, we have $z = Bz$. This completes the proof.

The following result is an immediate consequence of Theorem 3.1:
Theorem 3.3. Let \((X, d)\) be a complete metric space and \(S, T\) and \(A_n\) be mappings from \(X\) into itself, \(n = 1, 2, \ldots\). Suppose further that \(S\) and \(T\) are continuous and, for every \(n \in \mathbb{N}\), the pairs \(\{A_{2n-1}, S\}\) and \(\{A_{2n}, T\}\) are compatible of type \((P)\), \(A_{2n-1}(X) \subset T(X)\) and \(A_{2n}(X) \subset S(X)\) and, for any \(n \in \mathbb{N}\), the set of positive integers, the following condition is satisfied:

\[
(3.9) \quad d(A_nx, A_{n+1}y) \leq \Phi(\max\{d(Sx, Ty), d(Sx, A_nx), d(Ty, A_{n+1}y),
\frac{1}{2}[d(Sx, A_{n+1}y) + d(Ty, A_nx)])
\]

for all \(x, y \in X\), where \(\Phi(t)\) is the same as in Theorem 3.1. Then \(S\), \(T\) and \(\{A_n\}, n \in \mathbb{N}\), have a unique common fixed point in \(X\).

Remark 2. Theorem 3.3 extends Theorem 3.1 in [10], Theorem 1 in [7] and the main results in [5] and [19].


In this section, we give some common fixed point theorems in convex metric spaces.

Definition 4.1. A metric space \((X, d)\) is convex if for \(x, y \in X\) with \(x \neq y\), there exists a point \(z \in X\) such that

\[d(x, z) + d(z, y) = d(x, y).\]

Lemma 4.1. ([1]) Let \(K\) be a closed subset of a complete convex metric space \((X, d)\). If \(x \in K\) and \(y \in K\), then there exists a point \(z \in K\) such that

\[d(x, z) + d(z, y) = d(x, y).\]

Definition 4.2. Let \((X, d)\) be a metric space, \(K\) be a subset of \(X\) and \(A, S : K \to X\) be mappings. The mappings \(A\) and \(S\) are said to be relatively compatible of type \((P)\) if

\[
\lim_{n \to \infty} d(AAx_n, SSx_n) = 0
\]

whenever \(\{x_n\}\) is a sequence in \(K\) such that \(Ax_n, Sx_n \in K\) and

\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t \in K.
\]
Lemma 4.2. Let $(X, d)$ be a metric space, $K$ be a subset of $X$ and $A, S : K \to X$ be mappings. If the pair $(A, S)$ is relatively compatible of type (P), $Ax_n, Sx_n \in K$ and

$$
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t
$$

for some $t \in K$, then $\lim_{n \to \infty} AAx_n = St$ if $S$ is continuous at $t$.

Proof. From Definition 4.2, we have this lemma.

Theorem 4.3. Let $(X, d)$ be a complete convex metric space and $K$ be a non-empty closed subset of $X$. Suppose that $S$ and $T$ are continuous mappings from $X$ into itself with $\partial K \subset S(K) \cap T(K)$, where $\partial K$ denotes the boundary of $K$, and $A, B : K \to X$ are continuous mappings with $A(K) \cap K \subset S(K)$, $B(K) \cap K \subset T(K)$. Suppose further that the pairs $(A, T)$ and $(B, S)$ are relatively compatible of type (P) satisfying

\begin{equation}
(4.1) \quad d(Ax, By) \leq \Phi(d(Tx, Sy))
\end{equation}

for all $x, y \in K$, where $\Phi : [0, \infty) \to [0, \infty)$ is a nondecreasing and upper semicontinuous function such that $\Phi(t) < t$ and $\sum \Phi^n(t) < \infty$ for all $t > 0$.

If for $x \in K$, $Tx \in \partial K \Rightarrow Ax, Bx \in K$ and $Sx \in \partial K \Rightarrow Ax, Bx \in K$, then there exists a point $z \in K$ such that $z = Az = Bz = Sz = Tz$. Further, if $Tv = Sv = Av = Bv$, then $Tz = Tv$.

Proof. Let $x \in \partial K$ and $p_0 \in K$ be such that $x = Tp_0$. Then $Ap_0 \in K$ and so $Ap_0 \in A(K) \cap K \subset S(K)$, which implies that there exists a point $p_1 \in K$ such that $Sp_1 = Ap_0 \in K$. Let $p'_1 = Ap_0$ and $p'_2 = Bp_1$. If $p'_2 \notin K$, then $p'_2 \in B(K) \cap K \subset T(K)$ and so there exists a point $p_2 \in K$ such that $Tp_2 = Bp_1$, and if $p'_2 \notin K$, since $(M, d)$ is a convex metric space, by Lemma 4.1, there exists a point $p_2 \in K$ such that $Tp_2 \in \partial K$ and

$$
d(Sp_1, Tp_2) + d(Tp_2, Bp_1) = d(Sp_1, Bp_1).
$$

If we continue this process, we obtain two sequences $(p_n)_{n \in \mathbb{N}}$ and $(p'_n)_{n \in \mathbb{N}}$ in $K$ such that, for every $n \in N$, $p_n \in K$, $p'_{2n-1} = Ap_{2n}$, $p'_{2n} = Bp_{2n-1}$ and the following implications hold:

1. $p'_{2n} \in K \Rightarrow p'_{2n} = Tp_{2n},$
2. $p'_{2n} \notin K \Rightarrow p_{2n} \in \partial K$ and

$$
d(Sp_{2n-1}, Tp_{2n}) + d(Tp_{2n}, Bp_{2n-1}) = d(Sp_{2n-1}, Bp_{2n-1}),
$$

$(2) p'_{2n+1} \in K \Rightarrow p'_{2n+1} = Tp_{2n+1},$
\( p_{2n+1}^' \notin K \Rightarrow p_{2n+1} \in \partial K \) and
\[
d(Sp_{2n}, Tp_{2n+1}) + d(Tp_{2n+1}, Bp_{2n}) = d(Sp_{2n}, Bp_{2n}).
\]

Now, we prove that there exists a point \( z \in K \) such that
\[
\lim_{n \to \infty} Tp_{2n} = \lim_{n \to \infty} Sp_{2n+1} = z.
\]

In fact, we define the sets \( P_0, \ P_1, \ Q_0, \ Q_1 \) as follows:

\[
P_0 = \{ p_{2n} \in K : p_{2n}^' = Tp_{2n}, \ n \in N \},
\]
\[
P_1 = \{ p_{2n} \in K : p_{2n}^' \neq Tp_{2n}, \ n \in N \},
\]
\[
Q_0 = \{ p_{2n+1} \in K : p_{2n+1}^' = Sp_{2n+1}, \ n \in N \},
\]
\[
Q_1 = \{ p_{2n+1} \in K : p_{2n+1}^' \neq Sp_{2n+1}, \ n \in N \}.
\]

Then it is easy to show that
\[
(p_{2n}, p_{2n+1}) \notin P_1 \times Q_1 \quad \text{and} \quad (p_{2n-1}, p_{2n}) \notin Q_1 \times P_1.
\]

Thus we have
\[
(p_{2n}, p_{2n+1}) \in P_0 \times Q_0, \quad (p_{2n}, p_{2n+1}) \in P_0 \times Q_1, \quad (p_{2n}, p_{2n+1}) \in P_1 \times Q_0,
\]
and
\[
(p_{2n-1}, p_{2n}) \in Q_0 \times P_0, \quad (p_{2n-1}, p_{2n}) \in Q_0 \times P_1, \quad (p_{2n-1}, p_{2n}) \in Q_1 \times P_0.
\]

(i) \( (p_{2n}, p_{2n+1}) \in P_0 \times Q_0 \):
\[
d(Tp_{2n}, Sp_{2n+1}) = d(Bp_{2n-1}, Ap_{2n})
\]  
\[
\leq \Phi(d(Tp_{2n}, Sp_{2n-1})).
\]

(ii) \( (p_{2n}, p_{2n+1}) \in P_0 \times Q_1 \):
\[
d(Tp_{2n}, Sp_{2n+1}) = d(Tp_{2n}, Ap_{2n}) - d(Sp_{2n+1}, Ap_{2n})
\]  
\[
\leq d(Tp_{2n}, Ap_{2n})
\]  
\[
= d(Bp_{2n-1}, Ap_{2n})
\]  
\[
\leq \Phi(d(Sp_{2n-1}, Tp_{2n})).
\]
(iii) \((p_{2n}, p_{2n+1}) \in P_1 \times Q_0:\)
\[
d(T_{p_{2n}}, S_{p_{2n+1}}) \leq d(T_{p_{2n}}, B_{p_{2n-1}}) + d(B_{p_{2n-1}}, S_{p_{2n+1}})
= d(T_{p_{2n}}, B_{p_{2n-1}}) + d(B_{p_{2n-1}}, A_{p_{2n}})
\leq d(T_{p_{2n}}, B_{p_{2n-1}}) + \Phi(d(S_{p_{2n-1}}, T_{p_{2n}}))
\leq d(S_{p_{2n-1}}, T_{p_{2n}}) + d(T_{p_{2n}}, B_{p_{2n-1}})
= d(S_{p_{2n-1}}, B_{p_{2n-1}}).
\]

Since \(p_{2n} \in P_1\) implies that \(p_{2n-1} \in Q_0\), we have \(S_{p_{2n}} = A_{p_{2n-2}}\) and so
\[
d(T_{p_{2n}}, S_{p_{2n+1}}) \leq d(S_{p_{2n-1}}, B_{p_{2n-1}})
= d(A_{p_{2n-2}}, B_{p_{2n-1}})
\leq \Phi(d(T_{p_{2n-2}}, S_{p_{2n-1}})).
\]

Similarly, we have
(iv) \((p_{2n+1}, p_{2n}) \in Q_0 \times P_0:\)
\[
d(S_{p_{2n-1}}, T_{p_{2n}}) \leq \Phi(d(T_{p_{2n-2}}, S_{p_{2n-1}})),
\]
(v) \((p_{2n+1}, p_{2n}) \in Q_0 \times P_1:\)
\[
d(S_{p_{2n-1}}, T_{p_{2n}}) \leq \Phi(d(T_{p_{2n-2}}, S_{p_{2n-1}})),
\]
(vi) \((p_{2n+1}, p_{2n}) \in Q_1 \times P_0:\)
\[
d(S_{p_{2n-1}}, T_{p_{2n}}) \leq \Phi(d(T_{p_{2n-2}}, S_{p_{2n-3}})).
\]

Therefore, it follows that
\[
(4.2) \quad d(T_{p_{2n}}, S_{p_{2n+1}}) \leq \Phi^n(r), \quad d(S_{p_{2n+1}}, T_{p_{2n+2}}) \leq \Phi^n(r)
\]
for every \(n \in N\), where \(r = \max\{d(T_{p_2}, S_{p_3}, d(T_{p_2}, S_{p_1})\}. This implies that for every \(n \in N\),
\[
d(T_{p_{2n}}, T_{p_{2n+2}}) \leq \Phi^{n-1}(r) + \Phi^n(r).
\]

Hence \(\sum \Phi^n(r)\) is finite, the sequence \(\{T_{p_{2n}}\}_{n \in N}\) is a Cauchy sequence in \(K\).
Since \(X\) is complete and \(K\) is closed, it follows that there exists a point \(z \in K\) such that \(z = \lim_{n \to \infty} T_{p_{2n}}\). Then from (4.2), we have
\[
z = \lim_{n \to \infty} T_{p_{2n}} = \lim_{n \to \infty} S_{p_{2n+1}}.
\]
By hypothesis, there exists a sequence \( \{n_k\} \) in \( N \) such that \( Tp_{2n_k} = Bp_{2n_k - 1} \) for all \( k \in N \) or \( Sp_{2n_k - 1} = Ap_{2n_k - 2} \) for all \( k \in N \). Without loss of generality, we can suppose that \( Tp_{2n_k} = Bp_{2n_k - 1} \) for all \( k \in N \). From (4.1), we have

\[
d(SSp_{2n_k - 1}, Az) \leq d(SSp_{2n_k - 1}, BBp_{2n_k - 1}) + d(BBp_{2n_k - 1}, Az) \\
\leq d(SSp_{2n_k - 1}, BBp_{2n_k - 1}) + \Phi(d(SBp_{2n_k - 1}, Tz)).
\]

Since the pair \( \{B, S\} \) is relatively compatible of type \( (P) \) and \( S \) is continuous, we have

\[
(4.3) \quad d(Sz, Az) \leq \Phi(d(Sz, Tz)).
\]

From (4.1), we have

\[
d(Ap_{2n_k}, Tp_{2n_k}) = d(Ap_{2n_k}, Bp_{2n_k - 1}) \leq \Phi(d(Sp_{2n_k - 1}, Tp_{2n_k})).
\]

By the upper semi-continuity of \( \Phi(t) \), it follows that

\[
(4.4) \quad \lim_{k \to \infty} Ap_{2n_k} = z.
\]

Again, using (4.1), we have

\[
d(Ap_{2n_k}, BBp_{2n_k - 1}) \leq \Phi(d(Tp_{2n_k}, SBp_{2n_k - 1})).
\]

Since the pair \( \{B, S\} \) are relatively compatible of type \( (P) \) and \( S \) is continuous, it follows from (4.4) and Lemma 4.2 that

\[
d(z, Sz) \leq \Phi(d(z, Sz)).
\]

This implies that \( d(z, Sz) = 0 \), i.e., \( z = Sz \).

Since the pair \( \{A, T\} \) is relatively compatible of type \( (P) \) and \( A \) and \( T \) are continuous, from (4.4) and Lemma 4.2, we have

\[
Az = \lim_{k \to \infty} AAp_{2n_k} = Tz.
\]

In view of (4.3), we have \( d(Sz, Tz) \leq \Phi(d(Sz, Tz)). \) Hence \( z = Sz = Tz = Az \). Besides, from (4.1), we have

\[
d(Az, Bz) \leq \Phi(d(Sz, Tz)) = \Phi(0) = 0.
\]

Thus \( z \in K \) and \( z = Az = Bz = Sz = Tz \).

Finally, if \( Tv = Sv = Av = Bv \), then \( d(Tv, Sz) = d(Av, Bz) \leq \Phi(d(Tv, Sz)) \). Therefore, \( Tv = Sz = Tz \). This completes the proof.

The following result is an immediate consequence of Theorem 4.3:
Theorem 4.4. Let \((X, d)\) be a complete convex metric space and \(K\) be a non-empty closed subset of \(X\) and \(S\) and \(T\) be continuous mappings from \(X\) into \(X\) such that \(\partial K \subset S(K) \cap T(K)\). Suppose that, for every \(n \in \mathbb{N}\), \(A_n : K \to X\) is continuous mappings with \(A_{2n}(K) \cap K \subset T(K)\) and \(A_{2n-1}(K) \cap K \subset S(K)\) and the pairs \(\{A_{2n-1}, T\}\) and \(\{A_{2n}, S\}\) are relatively compatible of type \((P)\) such that for any \(n \in \mathbb{N}\),

\[
d(A_n x, A_{n+1} y) \leq \Phi(d(T x, S y))
\]

for all \(x, y \in K\), where \(\Phi(t)\) is the same as in Theorem 4.3.

If for every \(n \in \mathbb{N}\) and \(x \in K\), \(T x \in \partial K \Rightarrow A_n x \in K\) and \(S x \in \partial K \Rightarrow A_n x \in K\), then there exists a point \(z \in K\) such that \(z = Tz = Sz = A_n z\) for all \(n \in \mathbb{N}\). Further, if \(Tv = Sv = A_n v\) for every \(n \in \mathbb{N}\), then \(Tz = Tv\).

Remark 3. Theorem 4.4 is an extension of Theorem 1 in [8].

5. Applications.

Throughout this section, we assume that \(X, Y\) are Banach spaces, \(S \subset X\) is the state space and \(D \subset Y\) is the decision space. Let \(R = (-\infty, +\infty)\) and denote by \(B(S)\) the set of all bounded real-valued functions on \(S\).

Following Bellman and Lee [3], the basic form of the functional equation of dynamic programming is as follows:

\[
f(x) = \text{opt}_y H(x, y, f(T(x, y))),
\]

where \(x\) and \(y\) denote the state and decision vectors, respectively, \(T\) the transformation of the process and \(f(x)\) the optimal return with the initial state \(x\), where the \(\text{opt}\) denotes max or min.

In this section, we shall study the existence and uniqueness of common solution of the following functional equations arising in dynamic programming:

\[
f_i(x) = \sup_{y \in D} H_i(x, y, f_i(T(x, y))), \quad x \in S, \tag{5.1}
\]

\[
g_i(x) = \sup_{y \in D} F_i(x, y, g_i(T(x, y))), \quad x \in S, \tag{5.2}
\]

where \(T : S \times D \to S\) and \(H_i, F_i : S \times D \times R \to R\), \(i = 1, 2\).
Theorem 5.1. Suppose that the following conditions are satisfied:

(i) $H_i$ and $F_i$ are bounded for $i = 1, 2$,

(ii) $|H_1(x, y, h(t)) - H_2(x, y, k(t))| \leq \Phi(\max(|T_2 h(t) - T_2 k(t)|, |T_1 h(t) - A_1 h(t)|, |T_2 k(t) - A_2 k(t)|, \frac{1}{2}[|T_1 h(t) - A_2 k(t)| + |T_2 k(t) - A_1 h(t)|]))$

for all $(x, y) \in S \times D$, $h, k \in B(S)$ and $t \in S$, where $\Phi$ is the same as in Theorem 3.1 and mappings $A_i$ and $T_i$ are defined as follows:

$$A_i h(x) = \sup_{y \in D} H_i(x, y, h(T(x, y))), \quad x \in S, \ h \in B(S), \ i = 1, 2,$$

$$T_i k(x) = \sup_{y \in D} F_i(x, y, k(T(x, y))), \quad x \in S, \ k \in B(S), \ i = 1, 2.$$

(iii) for any $\{k_n\} \subset B(S)$ and $k \in B(S)$,

$$\lim_{n \to \infty} \sup_{x \in S} |k_n(x) - k(x)| = 0 \Rightarrow \lim_{n \to \infty} \sup_{x \in S} |T_i k_n(x) - T_i k(x)| = 0, \ i = 1, 2,$$

(iv) for any $h \in B(S)$, there exist $k_1, k_2 \in B(S)$ such that

$$A_1 h(x) = T_1 k_1(x), \quad A_2 h(x) = T_1 k_2(x), \quad x \in S,$$

(v) for any $\{k_n\} \subset B(S)$, if there exists $h \in B(S)$ such that

$$\lim_{n \to \infty} \sup_{x \in S} |A_i k_n(x) - h(x)| = \lim_{n \to \infty} \sup_{x \in S} |T_i k_n(x) - h(x)| = 0, \quad i = 1, 2,$$

then

$$\lim_{n \to \infty} \sup_{x \in S} |T_i T_i k_n(x) - A_i A_i k_n(x)| = 0, \quad i = 1, 2.$$

Then the system of functional equations (5.1) and (5.2) has a unique common solution in $B(S)$.

Proof. For any $h, k \in B(S)$, let

$$d(h, k) = \sup_{x \in S} |h(x) - k(x)|.$$

Then $(B(S), d)$ is a complete metric space. By virtue of (i) - (v), $A_i$ and $T_i$ are self mappings of $B(S)$, $T_i$ are continuous, $i = 1, 2$, $A_1(B(S)) \subset T_2(B(S))$, $A_2(B(S)) \subset T_1(B(S))$, and the pairs of mappings $A_i, T_i$ are compatible of type
(P), \( i = 1, 2 \). Let \( h_i \) (\( i = 1, 2 \)) be any two points of \( B(S) \), \( x \in S \) and \( \eta \) be any positive number. Suppose that there exists \( y_i \) (\( i = 1, 2 \)) in \( D \) such that

\[
A_i h_i(x) < H_i(x, y_i, h_i(x_i)) + \eta,
\]

where \( x_i = T(x, y_i), i = 1, 2. \) Also we have

\[
A_1 h_1(x) \geq H_1(x, y_2, h_1(x_2)),
\]

\[
A_2 h_2(x) \geq H_2(x, y_1, h_2(x_1)).
\]

From (5.3), (5.5) and (ii), we have

\[
A_1 h_1(x) - A_2 h_2(x)
\]

\[
< H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1)) + \eta
\]

\[
\leq |H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1))| + \eta
\]

\[
\leq \Phi(\max\{|T_1 h_1(x_1) - T_2 h_2(x_1)|, |T_1 h_1(x_1) - A_1 h_1(x_1)|, |T_2 h_2(x_1) - A_2 h_2(x_1)|, \frac{1}{2}|T_1 h_1(x_1) - A_2 h_2(x_1)| + + |T_2 h_2(x_1) - A_1 h_1(x_1)]|) + \eta
\]

\[
\leq \Phi(\max\{d(T_2 h_1, T_2 h_2), d(T_1 h_1, A_1 h_1), d(T_2 h_2, A_2 h_2), \frac{1}{2}[d(T_1 h_1, A_2 h_2) + d(T_2 h_2, A_1 h_1)]\}) + \eta.
\]

From (5.3), (5.4) and (ii), we have

\[
A_1 h_1(x) - A_2 h_2(x)
\]

\[
\geq -\Phi(\max\{d(T_2 h_1, T_2 h_2), d(T_1 h_1, A_1 h_1), d(T_2 h_2, A_2 h_2), \frac{1}{2}[d(T_1 h_1, A_2 h_2) + d(T_2 h_2, A_1 h_1)]\}) - \eta.
\]

Unification of (5.6) and (5.7) yields

\[
|A_1 h_1(x) - A_2 h_2(x)|
\]

\[
\leq \Phi(\max\{d(T_2 h_1, T_2 h_2), d(T_1 h_1, A_1 h_1), d(T_2 h_2, A_2 h_2), \frac{1}{2}[d(T_1 h_1, A_2 h_2) + d(T_2 h_2, A_1 h_1)]\}) + \eta.
\]

Since (5.8) is true for any \( x \in S \) and \( \eta \) is any positive number, we have, on taking supremum over all \( x \in S \),

\[
d(A_1 h_1, A_2 h_2) \leq \Phi(\max\{d(T_1 h_1, T_2 h_2), d(T_1 h_1, A_1 h_1), d(T_2 h_2, A_2 h_2), \frac{1}{2}[d(T_1 h_1, A_2 h_2) + d(T_2 h_2, A_1 h_1)]\}).
\]
Therefore, by Theorem 3.3, $A_1$, $A_2$, $T_1$ and $T_2$ have a unique common fixed point $h^* \in B(S)$, i.e., $h^*(x)$ is a unique solution of the functional equations (5.1) and (5.2). This completes the proof.

As an immediate consequence of Theorem 5.1 and Corollary 3.2, we can obtain the following:

**Theorem 5.2.** Suppose that the following conditions are satisfied:

(i) $H_i$ is bounded for $i = 1, 2$,

(ii) \[
|H_1(x, y, h(t)) - H_2(x, y, k(t))| \\
\leq \Phi(\max\{|h(t) - k(t)|, |h(t) - A_1h(t)|, |k(t) - A_2k(t)|, \\
\frac{1}{2}(|h(t) - A_2k(t)| + |k(t) - A_1h(t)|))
\]

for all $(x, y) \in S \times D$, $h, k \in B(S)$ and $t \in S$, where $\Phi$ is the same as in Theorem 3.1 and $A_i$ is defined by

$$A_i h(x) = \sup_{y \in D} H_i(x, y, h(T_i(x, y))), \quad x \in S, \; h \in B(S), \; i = 1, 2.$$ 

Then the functional equations (5.1) and (5.2) have a unique common solution in $B(S)$.

**Remark 4.** Theorem 5.2 is an extension of Theorem 2.1 in [4].

**Theorem 5.3.** Suppose that the following conditions are satisfied:

(i) $H_i$ and $F_i$ are bounded for $i = 1, 2$,

(ii) \[
|H_1(x, y, h(t)) - H_2(x, y, k(t))| \leq \Phi(|T_i h(t) - T_i k(t)|),
\]

for all $(x, y) \in S \times D$, $h, k \in B(S)$ and $t \in S$, where $\Phi$ is the same as in Theorem 4.3 and $T_i$ is defined as in Theorem 5.1 for $i = 1, 2$;

(iii) For any $\{k_n\} \in B(S)$ and $k \in B(S)$,

\[
\lim_{n \to \infty} \sup_{x \in S} |k_n(x) - k(x)| = 0 \Rightarrow \lim_{n \to \infty} \sup_{x \in S} |T_i k_n(x) - T_i k(x)| = 0, \; i = 1, 2,
\]

and

\[
\lim_{n \to \infty} \sup_{x \in S} |A_i k_n(x) - A_i k(x)| = 0, \; i = 1, 2,
\]

where $A_i$ is defined as in Theorem 5.1 for $i = 1, 2$,

(iv) for any $h \in B(S)$ with $\sup_{x \in S} |h(x)| = 1$, there exist $k_1, k_2 \in B(S)$ such that

\[
\sup_{x \in S} |h(x)| \leq 1 \quad \text{and} \quad T_i k_i(x) = h(x), \quad x \in S, \; i = 1, 2,
\]
(v) For any \( h \in B(S) \) with \( \sup_{x \in S} |h(x)| \leq 1 \), there exist \( k_1, k_2 \in B(S) \) such that

\[
\sup_{x \in S} |k_i(x)| \leq 1, \quad i = 1, 2, \quad A_1 h(x) = T_2 k_1(x), \quad A_2 h(x) = T_1 k_2(x), \quad x \in S,
\]

(vi) For any \( h \in B(S) \) with \( \sup_{x \in S} |h(x)| \leq 1 \),

\[
\sup_{x \in S} |T_i h(x)| = 1 \Rightarrow \sup_{x \in S} |A_j h(x)| \leq 1, \quad i, j = 1, 2,
\]

(vii) For any \( \{k_n\} \subset B(S) \), if there exists \( h \in B(S) \) such that \( \sup_{x \in S} |T_i k_n(x)| \leq 1 \) and

\[
\lim_{n \to \infty} \sup_{x \in S} |A_i k_n(x) - h(x)| = \lim_{n \to \infty} \sup_{x \in S} |T_i k_n(x) - h(x)| = 0, \quad i = 1, 2,
\]

then

\[
\lim_{n \to \infty} \sup_{x \in S} |A_i A_i k_n(x) - T_i T_i k_n(x)| = 0, \quad i = 1, 2.
\]

Then the system of the functional equations (5.1) and (5.2) have a unique common solution \( h^* \in B(S) \) and \( \sup_{x \in S} |h^*(x)| \leq 1 \).

Proof. Suppose that \( B(S) \) is a Banach space of all bounded real valued functions defined on \( S \) with supremum norm and \( K \) is the closed unit ball in \( B(S) \). By the conditions (i) – (vii), we know that \( A_i : K \to B(S) \) and \( T_i : B(S) \to B(S), \quad i = 1, 2, \) satisfy all the conditions of Theorem 4.3 and so they have a unique common fixed point \( h^* \in K \), i.e., \( h^*(x) \) is a unique common solution of the functional equations (5.1) and (5.2). This completes the proof.

Remark 5. Theorem 5.4 is an extension of Theorem 3.2 in [2].

Acknowledgements. The Present Studies were supported in part by the Basic Science Research Institute Program, Ministry of Education, Korea, 1994, Project No. BSRI-94-1405.
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