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FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS OF TYPE (P) AND APPLICATIONS TO DYNAMIC PROGRAMMING

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In this paper, we prove some common fixed point theorems for compatible mappings of type (P). As applications, the existence and uniqueness of common solutions for a class of the functional equations in dynamic programming are discussed.

1. Introduction.

In [18], the concept of compatible mappings of type (P) was introduced and compared with compatible mappings ([9]–[16]) and compatible mappings of type (A) ([13], [17]). The purpose of this paper is to prove some common fixed point theorems for compatible mappings of type (P), which extend and improve some recent results of [5], [8], [10] and [13]. As applications, we use our main results to study the existence and uniqueness problems of common solutions for a class of functional equations arising in dynamic programming. The main results extend and improve the corresponding results of [2], [4] and [5].

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2. Compatible Mapppings of Type (P).

Throughout this section, let (X, d) denote a metric space. We recall the following definitions and properties of compatible mappings, compatible mappings of type (A) and compatible mappings of type (P) ([9], [13], [18]).

Definition 2.1. Let $S,\ T : (X, d) \to (X, d)$ be mappings. The mappings S and T are said to be *compatible* if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X.

Definition 2.2. Let $S,\ T : (X, d) \to (X, d)$ be mappings. The mappings S and T are said to be *compatible of type* (A) if

$$\lim_{n\to\infty} d(TSx_n, SSx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(STx_n, TTx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X.

Definition 2.3. Let $S,\ T : (X, d) \to (X, d)$ be mappings. The mappings S and T are said to be *compatible of type* (P) if

$$\lim_{n\to\infty} d(SSx_n, TTx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X.

The following propositions show that Definitions 2.1 and 2.2 are equivalent under some conditions:

Proposition 2.1. *Let $S,\ T : (X, d) \to (X, d)$ be continuous mappings. If S and T are compatible, then they are compatible of type (A).*

Proposition 2.2. *Let $S,\ T : (X, d) \to (X, d)$ be compatible mappings of type (A). If one of S and T is continuous, then S and T are compatible.*

The following is a direct consequence of Propositions 2.1 and 2.2:

Proposition 2.3. *Let $S,\ T : (X, d) \to (X, d)$ be continuous mappings. Then S and T are compatible if and only if they are compatible of type (A).*

Remark 1. In [13], we can find two examples that Proposition 2.3 is not true if S and T are not continuous on a metric space.

We can show also that if S and T are continuous, then S and T are compatible if and only if they are compatible of type (P) as follows:

Proposition 2.4. *Let $S,\ T : (X, d) \to (X, d)$ be continuous mappings. Then S and T are compatible if and only if they are compatible of type (P).*

Proof. Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$$

for some $z \in X$. Since S and T are continuous,

$$\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} STx_n = Sz$$

and

$$\lim_{n\to\infty} TSx_n = \lim_{n\to\infty} TTx_n = Tz\,.$$

Suppose that S and T are compatible. Then we have

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0\,.$$

Now, since we have

$$\begin{aligned} d(SSx_n, TTx_n) &\le d(SSx_n, STx_n) + d(STx_n, TTx_n) \\ &\le d(SSx_n, STx_n) + d(STx_n, TSx_n) + d(TSx_n, TTx_n), \end{aligned}$$

it follows that $\lim_{n\to\infty} d(SSx_n, TTx_n) = 0$. Thus, the mappings S and T are compatible of type (P).

Conversely, suppose that S and T are compatible mappings of type (P), that is,

$$\lim_{n\to\infty} d(SSx_n, TTx_n) = 0\,.$$

We then have

$$\begin{aligned} d(STx_n, TSx_n) &\le d(STx_n, SSx_n) + d(SSx_n, TSx_n) \\ &\le d(STx_n, SSx_n) + d(SSx_n, TTx_n) + d(TTx_n, TSx_n)\,. \end{aligned}$$

Therefore, it follows that $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$. This completes the proof.

Proposition 2.5. *Let $S,\ T : (X, d) \to (X, d)$ be compatible mappings of type (A). If one of S and T is continuous, then S and T are compatible of type (P).*

Proof. Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$$

for some $z \in X$. Suppose that S and T are compatible mappings of type (A).

Assume, without loss of generality, that S is continuous. we then have

$$d(SSx_n, TTx_n) \leq d(SSx_n, STx_n) + d(STx_n, TTx_n)$$

and so, since S and T are compatible of type (A), we have

$$\lim_{n\to\infty} d(SSx_n, TSx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(STx_n, TTx_n) = 0 .$$

Therefore, it follows that

$$\lim_{n\to\infty} d(SSx_n, TTx_n) = 0 .$$

This completes the proof.

As a direct consequence of Propositions 2.3 – 2.5, we have the following:

Proposition 2.6. *Let $S,\ T : (X, d) \to (X, d)$ be continuous mappings. Then*

(1) *S and T are compatible if and only if they are compatible of type (P).*
(2) *S and T are compatible of type (A) if and only if they are compatible of type (P).*

Next, we give several properties of compatible mappings of type (P) for our main theorems:

Proposition 2.7. *Let $S,\ T : (X, d) \to (X, d)$ be mappings. If S and T are compatible of type (P) and $Sz = Tz$ for some $z \in X$, then $SSz = STz = TSz = TTz$.*

Proof. Let $\{x_n\}$ be a sequence in X defined by $x_n = z,\ n = 1, 2, \ldots,$ and $Sz = Tz$ for some $z \in X$. Then we have $Sx_n,\ Tx_n \to Sz$ as $n \to \infty$. Since S and T are compatible of type (P), we have

$$d(SSz, TTz) = \lim_{n\to\infty} d(SSx_n, TTx_n) = 0 .$$

Therefore, $SSz = TTz$. But $Sz = Tz$ implies $SSz = STz = TSz = TTz$. This completes the proof.

Proposition 2.8. *Let S, $T : (X, d) \to (X, d)$ be mappings. Let S and T are compatible mappings of type (P) and let Sx_n, $Tx_n \to z$ as $n \to \infty$ for some $z \in X$. Then we have the following:*

(1) $\lim_{n\to\infty} TTx_n = Sz$ *if S is continuous at z.*
(2) $\lim_{n\to\infty} SSx_n = Tz$ *if T is continuous at z.*
(3) $STz = TSz$ *and* $Sz = Tz$ *if S and T are continuous at z.*

Proof. (1) Suppose that S is continuous at z.
Since

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$$

for some $z \in X$, we have $SSx_n \to Sz$ as $n \to \infty$. Again, since S and T are compatible of type (P), we have $\lim_{n\to\infty} d(TTx_n, SSx_n) = 0$ and so, since we have

$$d(TTx_n, Sz) \le d(TTx_n, SSx_n) + d(SSx_n, Sz),$$

it follows that $TTx_n \to Sz$ as $n \to \infty$.

(2) The proof of $\lim_{n\to\infty} SSx_n = Tz$ follows on the similar lines as argued in (1).

(3) Suppose that S and T are continuous at z. Since $Tx_n \to z$ as $n \to \infty$ and S is continuous at z, by (1), $TTx_n \to Sz$ as $n \to \infty$. On the other hand, since $Tx_n \to z$ as $n \to \infty$ and T is also continuous at z, $TTx_n \to Tz$. Thus, we have $Sz = Tz$ by the uniqueness of the limit and so, by Proposition 2.7, $TSz = STz$. This completes the proof.

3. Common Fixed Point Theorems (I).

In this section, we prove some common fixed point theorems in metric spaces :

Theorem 3.1. *Let (X, d) be a complete metric space and A, B, S and T be mappings from X into itself. Suppose that S and T are continuous mappings satisfying the following conditions:*

(3.1) $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,
(3.2) *the pairs* $\{A, S\}$ *and* $\{B, T\}$ *are compatible of type* (P),

(3.3)
$$d(Ax, By) \le \Phi(\max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}[d(Sx, By) + d(Ty, Ax)]\})$$

for all $x, y \in X$, where $\Phi : [0, \infty) \to [0, \infty)$ is a nondecreasing and upper semicontinuous function and $\Phi(t) < t$ for all $t > 0$. Then A, B, S and T have a unique common fixed point in X.

Proof. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, we can choose a sequence $\{x_n\}$ in X such that $Sx_{2n} = Bx_{2n-1}$ and $Tx_{2n-1} = Ax_{2n-2}$ for $n = 1, 2, 3, \ldots$. Suppose that

$$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2} \quad \text{and} \quad y_{2n} = Sx_{2n} = Bx_{2n-1} \tag{3.4}$$

for $n = 1, 2, 3, \ldots$. By using the technique of Chang [5], we can prove that $\{y_n\}$ is a Cauchy sequence in X and so, since X is complete, it converges to a point z in X. On the other hand, the subsequences $\{Ax_{2n-2}\}$, $\{Bx_{2n-1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n-1}\}$ of $\{y_n\}$ also converge to the point z.

Since $\{A, S\}$ and $\{B, T\}$ are compatible of type (P), it follows from the continuity of S and T, (3.4) and Proposition 2.8 that

$$\begin{aligned} &Ty_{2n} \to Tz, \quad By_{2n} = BBx_{2n-1} \to Tz, \\ &Sy_{2n-1} \to Sz, \quad Ay_{2n-1} = AAx_{2n-2} \to Sz \end{aligned} \tag{3.5}$$

as $n \to \infty$. By (3.3) and (3.4), we have

$$\begin{aligned} &d(Ay_{2n-1}, By_{2n}) \\ &\leq \Phi(\max\{d(Sy_{2n-1}, Ty_{2n}), d(Sy_{2n-1}, Ay_{2n-1}), d(Ty_{2n}, By_{2n}), \\ &\quad \frac{1}{2}[d(Sy_{2n-1}, By_{2n-2}) + d(Ty_{2n}, Ay_{2n-1})]\}). \end{aligned}$$

By the upper semicontinuity of $\Phi(t)$, (3.4) and (3.5), if $Sz \neq Tz$, then we have

$$\begin{aligned} d(Sz, Tz) &\leq \Phi(\max\{d(Sz, Tz), 0, 0, d(Sz, Tz)\}) \\ &= \Phi(d(Sz, Tz)) < d(Sz, Tz), \end{aligned}$$

which is a contradiction. Thus it follows that $Sz = Tz$.

Similarly, from (3.3), (3.4), (3.5) and the upper semicontinuity of Φ, we can obtain $Sz = Bz$ and $Tz = Az$. Hence we have

$$Az = Bz = Sz = Tz. \tag{3.6}$$

From (3.3) and (3.4), we have also

$$\begin{aligned} d(Ax_{2n}, Bz) \leq \Phi(\max\{&d(Sx_{2n}, Tz), d(Sx_{2n}, Ax_{2n}), d(TzBz), \\ &\frac{1}{2}[d(Sx_{2n}, Bz) + d(Tz, Ax_{2n})]\}). \end{aligned}$$

This implies that, if $Bz \neq z$, then

$$d(z, Bz) \leq \Phi(d(z, Bz)) < d(z, Bz),$$

which is a contradiction. Therefore, we have $z = Az = Bz = Sz = Tz$. The uniqueness of the fixed point z is obvious from (3.2). This completes the proof.

From Theorem 3.1, we have the following:

Theorem 3.2. *Let (X, d) be a complete metric space and A and B be mappings from X into itself satisfying the following condition:*

$$d(Ax, By) \leq \Phi(\max\{d(x, y), d(x, Ax), d(y, By), \frac{1}{2}[d(x, By) + d(y, Ax)]\}) \tag{3.7}$$

for all x, y in X, where $\Phi(t)$ is the same as in Theorem 3.1. *Then A and B have a unique common fixed point in X.*

Proof. Define a sequence $\{x_n\}$ in X by

$$x_{2n-1} = Ax_{2n-2} \quad \text{and} \quad x_{2n} = Bx_{2n-2} \tag{3.8}$$

for $n = 1, 2, 3, \ldots$. Then it is easy to show that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, letting $x_n \to z \in X$ as $n \to \infty$, we know that $\{x_{2n-1}\}$ and $\{x_{2n}\}$ converge to z, too. By (3.7) and (3.8), we have

$$\begin{aligned} d(Az, x_{2n}) &\leq d(Az, Bx_{2n-2}) \\ &\leq \Phi(\max\{d(z, x_{2n-2}), d(z, Az), d(x_{2n-2}, x_{2n}), \\ &\qquad \frac{1}{2}[d(z, x_{2n}) + d(x_{2n-2}, Az)]\}). \end{aligned}$$

By the upper semicontinuity of $\Phi(t)$, if $Az \neq z$, then we have

$$d(Az, z) \leq \Phi(d(z, Az)) < d(z, Az),$$

which is contradiction and so $z = Az$. Similarly, we have $z = Bz$. This completes the proof.

The following result is an immediate consequence of Theorem 3.1:

Theorem 3.3. *Let (X, d) be a complete metric space and S, T and A_n be mappings from X into itself, $n = 1, 2, \ldots$. Suppose further that S and T are continuous and, for every $n \in N$, the pairs $\{A_{2n-1}, S\}$ and $\{A_{2n}, T\}$ are compatible of type (P), $A_{2n-1}(X) \subset T(X)$ and $A_{2n}(X) \subset S(X)$ and, for any $n \in N$, the set of positive integers, the following condition is satisfied:*

$$(3.9) \quad d(A_n x, A_{n+1} y) \leq \Phi(\max\{d(Sx, Ty), d(Sx, A_n x), d(Ty, A_{n+1} y), \frac{1}{2}[d(Sx, A_{n+1} y) + d(Ty, A_n x)]\})$$

for all $x, y \in X$, where $\Phi(t)$ is the same as in Theorem 3.1. Then S, T and $\{A_n\}$, $n \in N$, have a unique common fixed point in X.

Remark 2. Theorem 3.3 extends Theorem 3.1 in [10], Theorem 1 in [7] and the main results in [5] and [19].

4. Common Fixed Point Theorems (II).

In this section, we give some common fixed point theorems in convex metric spaces.

Definition 4.1. A metric space (X, d) is *convex* if for $x, y \in X$ with $x \neq y$, there exists a point $z \in X$ such that

$$d(x, z) + d(z, y) = d(x, y) .$$

Lemma 4.1. ([1]) *Let K be a closed subset of a complete convex metric space (X, d). If $x \in K$ and $y \in K$, then there exists a point $z \in K$ such that*

$$d(x, z) + d(z, y) = d(x, y) .$$

Definition 4.2. Let (X, d) be a metric space, K be a subset of X and $A, S : K \to X$ be mappings. The mappings A and S are said to be *relatively compatible of type* (P) if

$$\lim_{n\to\infty} d(AAx_n, SSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in K such that Ax_n, $Sx_n \in K$ and

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \in K .$$

Lemma 4.2. *Let (X, d) be a metric space, K be a subset of X and $A, S : K \to X$ be mappings. If the pair $\{A, S\}$ is relatively compatible of type (P), $Ax_n, Sx_n \in K$ and*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in K$, then $\lim_{n\to\infty} AAx_n = St$ if S is continuous at t.

Proof. From Definition 4.2, we have this lemma.

Theorem 4.3. *Let (X, d) be a complete convex metric space and K be a non-empty closed subset of X. Suppose that S and T are continuous mappings from X into itself with $\partial K \subset S(K) \cap T(K)$, where ∂K denotes the boundary of K, and $A, B : K \to X$ are continuous mappings with $A(K) \cap K \subset S(K)$, $B(K) \cap K \subset T(K)$. Suppose further that the pairs $\{A, T\}$ and $\{B, S\}$ are relatively compatible of type (P) satisfying*

$$d(Ax, By) \leq \Phi(d(Tx, Sy)) \tag{4.1}$$

for all x, y in K, where $\Phi : [0, \infty) \to [0, \infty)$ is a nondecreasing and upper semicontinuous function such that $\Phi(t) < t$ and $\sum \Phi^n(t) < \infty$ for all $t > 0$.

If for $x \in K$, $Tx \in \partial K \Rightarrow Ax, Bx \in K$ and $Sx \in \partial K \Rightarrow Ax, Bx \in K$, then there exists a point $z \in K$ such that $z = Az = Bz = Sz = Tz$. Further, if $Tv = Sv = Av = Bv$, then $Tz = Tv$.

Proof. Let $x \in \partial K$ and $p_0 \in K$ be such that $x = Tp_0$. Then $Ap_0 \in K$ and so $Ap_0 \in A(K) \cap K \subset S(K)$, which implies that there exists a point $p_1 \in K$ such that $Sp_1 = Ap_0 \in K$. Let $p_1' = Ap_0$ and $p_2' = Bp_1$. If $p_2' \in K$, then $p_2' \in B(K) \cap K \subset T(K)$ and so there exists a point $p_2 \in K$ such that $Tp_2 = Bp_1$, and if $p_2' \notin K$, since (M, d) is a convex metric space, by Lemma 4.1, there exists a point $p_2 \in K$ such that $Tp_2 \in \partial K$ and

$$d(Sp_1, Tp_2) + d(Tp_2, Bp_1) = d(Sp_1, Bp_1)\,.$$

If we continue this process, we obtain two sequences $\{p_n\}_{n\in N}$ and $\{p_n'\}_{n\in N}$ in K such that, for every $n \in N$, $p_n \in K$, $p_{2n-1}' = Ap_{2n}$, $p_{2n}' = Bp_{2n-1}$ and the following implications hold:

(1) $p_{2n}' \in K \Rightarrow p_{2n}' = Tp_{2n}$,
$p_{2n}' \notin K \Rightarrow p_{2n} \in \partial K$ and

$$d(Sp_{2n-1}, Tp_{2n}) + d(Tp_{2n}, Bp_{2n-1}) = d(Sp_{2n-1}, Bp_{2n-1}),$$

(2) $p_{2n+1}' \in K \Rightarrow p_{2n+1}' = Tp_{2n+1}$,

$p'_{2n+1} \notin K \Rightarrow p_{2n+1} \in \partial K$ and

$$d(Sp_{2n}, Tp_{2n+1}) + d(Tp_{2n+1}, Bp_{2n}) = d(Sp_{2n}, Bp_{2n}),$$

Now, we prove that there exists a point $z \in K$ such that

$$\lim_{n\to\infty} Tp_{2n} = \lim_{n\to\infty} Sp_{2n+1} = z.$$

In fact, we define the sets P_0, P_1, Q_0, Q_1 as follows:

$$P_0 = \{p_{2n} \in K : p'_{2n} = Tp_{2n},\ n \in N\},$$

$$P_1 = \{p_{2n} \in K : p'_{2n} \neq Tp_{2n},\ n \in N\},$$

$$Q_0 = \{p_{2n+1} \in K : p'_{2n+1} = Sp_{2n+1},\ n \in N\},$$

$$Q_1 = \{p_{2n+1} \in K : p'_{2n+1} \neq Sp_{2n+1},\ n \in N\}.$$

Then it is easy to show that

$$(p_{2n}, p_{2n+1}) \notin P_1 \times Q_1 \quad \text{and} \quad (p_{2n-1}, p_{2n}) \notin Q_1 \times P_1.$$

Thus we have

$$(p_{2n}, p_{2n+1}) \in P_0 \times Q_0, \quad (p_{2n}, p_{2n+1}) \in P_0 \times Q_1, \quad (p_{2n}, p_{2n+1}) \in P_1 \times Q_0,$$

and

$$(p_{2n-1}, p_{2n}) \in Q_0 \times P_0, \quad (p_{2n-1}, p_{2n}) \in Q_0 \times P_1, \quad (p_{2n-1}, p_{2n}) \in Q_1 \times P_0.$$

(i) $(p_{2n}, p_{2n+1}) \in P_0 \times Q_0$:

$$\begin{aligned} d(Tp_{2n}, Sp_{2n+1}) &= d(Bp_{2n-1}, Ap_{2n}) \\ &\leq \Phi(d(Tp_{2n}, Sp_{2n-1})). \end{aligned}$$

(ii) $(p_{2n}, p_{2n+1}) \in P_0 \times Q_1$:

$$\begin{aligned} d(Tp_{2n}, Sp_{2n+1}) &= d(Tp_{2n}, Ap_{2n}) - d(Sp_{2n+1}, Ap_{2n}) \\ &\leq d(Tp_{2n}, Ap_{2n}) \\ &= d(Bp_{2n-1}, Ap_{2n}) \\ &\leq \Phi(d(Sp_{2n-1}, Tp_{2n})). \end{aligned}$$

(iii) $(p_{2n}, p_{2n+1}) \in P_1 \times Q_0$:

$$\begin{aligned} d(Tp_{2n}, Sp_{2n+1}) &\le d(Tp_{2n}, Bp_{2n-1}) + d(Bp_{2n-1}, Sp_{2n+1}) \\ &= d(Tp_{2n}, Bp_{2n-1}) + d(Bp_{2n-1}, Ap_{2n}) \\ &\le d(Tp_{2n}, Bp_{2n-1}) + \Phi(d(Sp_{2n-1}, Tp_{2n})) \\ &\le d(Sp_{2n-1}, Tp_{2n}) + d(Tp_{2n}, Bp_{2n-1}) \\ &= d(Sp_{2n-1}, Bp_{2n-1}). \end{aligned}$$

Since $p_{2n} \in P_1$ implies that $p_{2n-1} \in Q_0$, we have $Sp_{2n_1} = Ap_{2n-2}$ and so

$$\begin{aligned} d(Tp_{2n}, Sp_{2n+1}) &\le d(Sp_{2n-1}, Bp_{2n-1}) \\ &= d(Ap_{2n-2}, Bp_{2n-1}) \\ &\le \Phi(d(Tp_{2n-2}, Sp_{2n-1})). \end{aligned}$$

Similarly, we have
(iv) $(p_{2n-1}, p_{2n}) \in Q_0 \times P_0$:

$$d(Sp_{2n-1}, Tp_{2n}) \le \Phi(d(Tp_{2n-2}, Sp_{2n-1})),$$

(v) $(p_{2n-1}, p_{2n}) \in Q_0 \times P_1$:

$$d(Sp_{2n-1}, Tp_{2n}) \le \Phi(d(Tp_{2n-2}, Sp_{2n-1})),$$

(vi) $(p_{2n-1}, p_{2n}) \in Q_1 \times P_0$:

$$d(Sp_{2n-1}, Tp_{2n}) \le \Phi(d(Tp_{2n-2}, Sp_{2n-3})).$$

Therefore, it follows that

$$(4.2) \qquad d(Tp_{2n}, Sp_{2n+1}) \le \Phi^{n-1}(r), \quad d(Sp_{2n+1}, Tp_{2n+2}) \le \Phi^n(r)$$

for every $n \in N$, where $r = \max\{d(Tp_2, Sp_3, d(Tp_2, Sp_1)\}$. This implies that for every $n \in N$,

$$d(Tp_{2n}, Tp_{2n+2}) \le \Phi^{n-1}(r) + \Phi^n(r).$$

Hence $\sum \Phi^n(r)$ is finite, the sequence $\{Tp_{2n}\}_{n\in N}$ is a Cauchy sequence in K. Since X is complete and K is closed, it follows that there exists a point $z \in K$ such that $z = \lim_{n\to\infty} Tp_{2n}$. Then from (4.2), we have

$$z = \lim_{n\to\infty} Tp_{2n} = \lim_{n\to\infty} Sp_{2n+1}.$$

By hypothesis, there exists a sequence $\{n_k\}$ in N such that $Tp_{2n_k} = Bp_{2n_k-1}$ for all $k \in N$ or $Sp_{2n_k-1} = Ap_{2n_k-2}$ for all $k \in N$. Without loss of generality, we can suppose that $Tp_{2n_k} = Bp_{2n_k-1}$ for all $k \in N$. From (4.1), we have

$$\begin{aligned} d(SSp_{2n_k-1}, Az) &\le d(SSp_{2n_k-1}, BBp_{2n_k-1}) + d(BBp_{2n_k-1}, Az) \\ &\le d(SSp_{2n_k-1}, BBp_{2n_k-1}) + \Phi(d(SBp_{2n_k-1}, Tz)). \end{aligned}$$

Since the pair $\{B, S\}$ is relatively compatible of type (P) and S is continuous, we have

$$d(Sz, Az) \le \Phi(d(Sz, Tz)). \tag{4.3}$$

From (4.1), we have

$$d(Ap_{2n_k}, Tp_{2n_k}) = d(Ap_{2n_k}, Bp_{2n_k-1}) \le \Phi(d(Sp_{2n_k-1}, Tp_{2n_k})).$$

By the upper semi-continuity of $\Phi(t)$, it follows that

$$\lim_{k\to\infty} Ap_{2n_k} = z. \tag{4.4}$$

Again, using (4.1), we have

$$d(Ap_{2n_k}, BBp_{2n_k-1}) \le \Phi(d(Tp_{2n_k}, SBp_{2n_k-1})).$$

Since the pair $\{B, S\}$ are relatively compatible of type (P) and S is continuous, it follows from (4.4) and Lemma 4.2 that

$$d(z, Sz) \le \Phi(d(z, Sz)).$$

This implies that $d(z, Sz) = 0$, i.e., $z = Sz$.

Since the pair $\{A, T\}$ is relatively compatible of type (P) and A and T are continuous, from (4.4) and Lemma 4.2, we have

$$Az = \lim_{k\to\infty} AAp_{2n_k} = Tz.$$

In view of (4.3), we have $d(Sz, Tz) \le \Phi(d(Sz, Tz))$. Hence $z = Sz = Tz = Az$. Besides, from (4.1), we have

$$d(Az, Bz) \le \Phi(d(Sz, Tz)) = \Phi(0) = 0.$$

Thus $z \in K$ and $z = Az = Bz = Sz = Tz$.

Finally, if $Tv = Sv = Av = Bv$, then $d(Tv, Sz) = d(Av, Bz) \le \Phi(d(Tv, Sz))$. Therefore, $Tv = Sz = Tz$. This completes the proof.

The following result is an immediate consequence of Theorem 4.3:

Theorem 4.4. *Let (X, d) be a complete convex metric space and K be a non-empty closed subset of X and S and T be continuous mappings from X into X such that $\partial K \subset S(K) \cap T(K)$. Suppose that, for every $n \in N$, $A_n : K \to X$ is continuous mappings with $A_{2n}(K) \cap K \subset T(K)$ and $A_{2n-1}(K) \cap K \subset S(K)$ and the pairs $\{A_{2n-1}, T\}$ and $\{A_{2n}, S\}$ are relatively compatible of type (P) such that for any $n \in N$,*

$$d(A_n x, A_{n+1} y) \leq \Phi(d(Tx, Sy))$$

for all $x, y \in K$, where $\Phi(t)$ is the same as in Theorem 4.3.

If for every $n \in N$ and $x \in K$, $Tx \in \partial K \Rightarrow A_n x \in K$ and $Sx \in \partial K \Rightarrow A_n x \in K$, then there exists a point $z \in K$ such that $z = Tz = Sz = A_n z$ for all $n \in N$. Further, if $Tv = Sv = A_n v$ for every $n \in N$, then $Tz = Tv$.

Remark 3. Theorem 4.4 is an extension of Theorem 1 in [8].

5. Applications.

Throughout this section, we assume that X, Y are Banach spaces, $S \subset X$ is the state space and $D \subset Y$ is the decision space. Let $R = (-\infty, +\infty)$ and denote by $B(S)$ the set of all bounded real-valued functions on S.

Following Bellman and Lee [3], the basic form of the functional equation of dynamic programming is as follows:

$$f(x) = \text{opt}_y\, H(x, y, f(T(x, y))),$$

where x and y denote the state and decision vectors, respectively, T the transformation of the process and $f(x)$ the optimal return with the initial state x, where the opt denotes max or min.

In this section, we shall study the existence and uniqueness of common solution of the following functional equations arising in dynamic programming:

$$f_i(x) = \sup_{y \in D} H_i(x, y, f_i(T(x, y))), \quad x \in S, \tag{5.1}$$

$$g_i(x) = \sup_{y \in D} F_i(x, y, g_i(T(x, y))), \quad x \in S, \tag{5.2}$$

where $T : S \times D \to S$ and $H_i, F_i : S \times D \times R \to R$, $i = 1, 2$.

Theorem 5.1. *Suppose that the following conditions are satisfied:*

(i) *H_i and F_i are bounded for $i = 1, 2$,*

(ii)
$$|H_1(x, y, h(t)) - H_2(x, y, k(t))| \le \Phi(\max\{|T_2h(t) - T_2k(t)|, |T_1h(t) - A_1h(t)|, |T_2k(t) - A_2k(t)|, \frac{1}{2}[|T_1h(t) - A_2k(t)| + |T_2k(t) - A_1h(t)|]\})$$

for all $(x, y) \in S \times D$, $h, k \in B(S)$ and $t \in S$, where Φ is the same as in Theorem 3.1 and mappings A_i and T_i are defined as follows:

$$A_ih(x) = \sup_{y \in D} H_i(x, y, h(T(x, y))), \quad x \in S,\ h \in B(S),\ i = 1, 2,$$

$$T_ik(x) = \sup_{y \in D} F_i(x, y, k(T(x, y))), \quad x \in S,\ k \in B(S),\ i = 1, 2.$$

(iii) *for any $\{k_n\} \subset B(S)$ and $k \in B(S)$,*

$$\lim_{n\to\infty} \sup_{x\in S} |k_n(x) - k(x)| = 0 \Rightarrow \lim_{n\to\infty} \sup_{x\in S} |T_ik_n(x) - T_ik(x)| = 0,\ i = 1, 2,$$

(iv) *for any $h \in B(S)$, there exist $k_1, k_2 \in B(S)$ such that*

$$A_1h(x) = T_1k_1(x), \quad A_2h(x) = T_1k_2(x), \quad x \in S,$$

(v) *for any $\{k_n\} \subset B(S)$, if there exists $h \in B(S)$ such that*

$$\lim_{n\to\infty} \sup_{x\in S} |A_ik_n(x) - h(x)| = \lim_{n\to\infty} \sup_{x\in S} |T_ik_n(x) - h(x)| = 0, \quad i = 1, 2,$$

then

$$\lim_{n\to\infty} \sup_{x\in S} |T_iT_ik_n(x) - A_iA_ik_n(x)| = 0, \quad i = 1, 2.$$

Then the system of functional equations (5.1) *and* (5.2) *has a unique common solution in $B(S)$.*

Proof. For any $h, k \in B(S)$, let

$$d(h, k) = \sup\{|h(x) - k(x)| : x \in S\}.$$

Then $(B(S), d)$ is a complete metric space. By virtue of (i) – (v), A_i and T_i are self mappings of $B(S)$, T_i are continuous, $i = 1, 2$, $A_1(B(S)) \subset T_2(B(S))$, $A_2(B(S)) \subset T_1(B(S))$, and the pairs of mappings A_i, T_i are compatible of type

(P), $i = 1, 2$. Let h_i $(i = 1, 2)$ be any two points of $B(S)$, $x \in S$ and η be any positive number. Suppose that there exists y_i $(i = 1, 2)$ in D such that

$$A_i h_i(x) < H_i(x, y_i, h_i(x_i)) + \eta, \tag{5.3}$$

where $x_i = T(x, y_i)$, $i = 1, 2$. Also we have

$$A_1 h_1(x) \geq H_1(x, y_2, h_1(x_2)), \tag{5.4}$$

$$A_2 h_2(x) \geq H_2(x, y_1, h_2(x_1)). \tag{5.5}$$

From (5.3), (5.5) and (ii), we have

$$\begin{aligned} &A_1 h_1(x) - A_2 h_2(x) \\ &< H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1)) + \eta \\ &\leq |H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1))| + \eta \\ &\leq \Phi(\max\{|T_1 h_1(x_1) - T_2 h_2(x_1)|, |T_1 h_1(x_1) - A_1 h_1(x_1)|, \\ &\qquad |T_2 h_2(x_1) - A_2 h_2(x_1)|, \frac{1}{2}[|T_1 h_1(x_1) - A_2 h_2(x_1)| + \\ &\qquad + |T_2 h_2(x_1) - A_1 h_1(x_1)|]\}) + \eta \\ &\leq \Phi(\max\{d(T_2 h_1, T_2 h_2), d(T_1 h_1, A_1 h_1), d(T_2 h_2, A_2 h_2), \\ &\qquad \frac{1}{2}[d(T_1 h_1, A_2 h_2) + d(T_2 h_2, A_1 h_1)]\}) + \eta. \end{aligned} \tag{5.6}$$

From (5.3), (5.4) and (ii), we have

$$\begin{aligned} &A_1 h_1(x) - A_2 h_2(x) \\ &\geq -\Phi(\max\{d(T_1 h_1, T_2 h_2), d(T_1 h_1, A_1 h_1), d(T_2 h_2, A_2 h_2), \\ &\qquad \frac{1}{2}[d(T_1 h_1, A_2 h_2) + d(T_2 h_2, A_1 h_1)]\}) - \eta. \end{aligned} \tag{5.7}$$

Unification of (5.6) and (5.7) yields

$$\begin{aligned} &|A_1 h_1(x) - A_2 h_2(x)| \\ &\leq \Phi(\max\{d(T_1 h_1, T_2 h_2), d(T_1 h_1, A_1 h_1), d(T_2 h_2, A_2 h_2), \\ &\qquad \frac{1}{2}[d(T_1 h_1, A_2 h_2) + d(T_2 h_2, A_1 h_1)]\}) + \eta. \end{aligned} \tag{5.8}$$

Since (5.8) is true for any $x \in S$ and η is any positive number, we have, on taking supremum over all $x \in S$,

$$\begin{aligned} d(A_1 h_1, A_2 h_2) \leq \Phi(\max\{&d(T_1 h_1, T_2 h_2), d(T_1 h_1, A_1 h_1), d(T_2 h_2, A_2 h_2), \\ &\frac{1}{2}[d(T_1 h_1, A_2 h_2) + d(T_2 h_2, A_1 h_1)]\}). \end{aligned}$$

Therefore, by Theorem 3.3, A_1, A_2, T_1 and T_2 have a unique common fixed point $h^* \in B(S)$, i.e., $h^*(x)$ is a unique solution of the functional equations (5.1) and (5.2). This completes the proof.

As an immediate consequence of Theorem 5.1 and Corollary 3.2, we can obtain the following:

Theorem 5.2. *Suppose that the following conditions are satisfied:*

(i) *H_i is bounded for $i = 1, 2$,*

(ii)
$$|H_1(x, y, h(t)) - H_2(x, y, k(t))| \le \Phi(\max\{|h(t) - k(t)|, |h(t) - A_1h(t)|, |k(t) - A_2k(t)|, \frac{1}{2}[|h(t) - A_2k(t)| + |k(t) - A_1h(t)|]\})$$

for all $(x, y) \in S \times D$, $h, k \in B(S)$ and $t \in S$, where Φ is the same as in Theorem 3.1 and A_i is defined by

$$A_ih(x) = \sup_{y\in D} H_i(x, y, h(T(x, y))), \quad x \in S,\ h \in B(S),\ i = 1, 2.$$

Then the functional equations (5.1) and (5.2) have a unique common solution in $B(S)$.

Remark 4. Theorem 5.2 is an extension of Theorem 2.1 in [4].

Theorem 5.3. *Suppose that the following conditions are satisfied:*

(i) *H_i and F_i are bounded for $i = 1, 2$,*

(ii) $$|H_1(x, y, h(t)) - H_2(x, y, k(t))| \le \Phi(|T_1h(t) - T_2k(t)|),$$
for all $(x, y) \in S \times D$, $h, k \in B(S)$ and $t \in S$, where Φ is the same as in Theorem 4.3 and T_i is defined as in Theorem 5.1 for $i = 1, 2$;

(iii) *For any $\{k_n\} \in B(S)$ and $k \in B(S)$,*

$$\lim_{n\to\infty} \sup_{x\in S} |k_n(x) - k(x)| = 0 \Rightarrow \lim_{n\to\infty} \sup_{x\in S} |T_ik_n(x) - T_ik(x)| = 0,\ i = 1, 2,$$

and

$$\lim_{n\to\infty} \sup_{x\in S} |A_ik_n(x) - A_ik(x)| = 0, \quad i = 1, 2,$$

where A_i is defined as in Theorem 5.1 for $i = 1, 2$,

(iv) *for any $h \in B(S)$ with $\sup_{x\in S} |h(x)| = 1$, there exist $k_1, k_2 \in B(S)$ such that*

$$\sup_{x\in S} |h(x)| \le 1 \quad \text{and} \quad T_ik_i(x) = h(x), \quad x \in S, i = 1, 2,$$

(v) *for any* $h \in B(S)$ *with* $\sup_{x\in S} |h(x)| \leq 1$, *there exist* $k_1, k_2 \in B(S)$ *such that*

$$\sup_{x\in S} |k_i(x)| \leq 1,\ i = 1, 2,\ A_1h(x) = T_2k_1(x),\ A_2h(x) = T_1k_2(x),\ x \in S,$$

(vi) *for any* $h \in B(S)$ *with* $\sup_{x\in S} |h(x)| \leq 1$,

$$\sup_{x\in S} |T_ih(x)| = 1 \Rightarrow \sup_{x\in S} |A_jh(x)| \leq 1, \quad i, j = 1, 2,$$

(vii) *for any* $\{k_n\} \subset B(S)$, *if there exists* $h \in B(S)$ *such that* $\sup_{x\in S} |T_ik_n(x)| \leq 1$ *and*

$$\lim_{n\to\infty} \sup_{x\in S} |A_ik_n(x) - h(x)| = \lim_{n\to\infty} \sup_{x\in S} |T_ik_n(x) - h(x)| = 0, \quad i = 1, 2,$$

then

$$\lim_{n\to\infty} \sup_{x\in S} |A_iA_ik_n(x) - T_iT_ik_n(x)| = 0, \quad i = 1, 2.$$

Then the system of the functional equations (5.1) *and* (5.2) *have a unique common solution* $h^* \in B(S)$ *and* $\sup_{x\in S} |h^*(x)| \leq 1$.

Proof. Suppose that $B(S)$ is a Banach space of all bounded real valued functions defined on S with supremum norm and K is the closed unit ball in $B(S)$. By the conditions (i) – (vii), we know that $A_i : K \to B(S)$ and $T_i : B(S) \to B(S)$, $i = 1, 2$, satisfy all the conditions of Theorem 4.3 and so they have a unique common fixed point $h^* \in K$, i.e., $h^*(x)$ is a unique common solution of the functional equations (5.1) and (5.2). This completes the proof.

Remark 5. Theorem 5.4 is an extension of Theorem 3.2 in [2].

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