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ON KUMMER'S SECOND THEOREM INVOLVING PRODUCT OF GENERALIZED HYPERGEOMETRIC SERIES

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The aim of this paper is to obtain single series expression for

$$e^{-x/2}\,{}_1F_1(\alpha; 2\alpha + i; x)$$

for $i = 1$ and -1. For $i = 0$, we have the well known Kummer's second theorem. The results are derived with the help of generalized Gauss's second summation theorem recently obtained by Lavoie, Grondin and Rathie.

1. Introduction.

The well known Kummer's second theorem [2] is

$$e^{-x/2}\,{}_1F_1(\alpha; 2\alpha; x) = {}_0F_1\left(-; \alpha + \frac{1}{2}; x^2/16\right). \tag{1.1}$$

In 1928, Professor Bailey [1] has obtained the result (1.1) with the help of Gauss's second summation theorem viz.

$$ {}_2F_1\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+1) & \end{matrix} \middle| \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}b + \frac{1}{2}\right)}. \tag{1.2}$$

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Very recently Lavoie, Grondin and Rathie [3] have obtained explicit expression of

$$(1.3)\qquad {}_2F_1\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+i+1) & \end{matrix} \;\middle|\; \frac{1}{2}\right] \quad \text{for} \quad i = 0, \pm 1, \pm 2.$$

The aim of this paper is to derive two results contiguous to (1.1) by the same technique developed by Bailey.

2. Results required.

The following known results which are special cases of (1.3) will be required in our present investigations.

$$(2.1)\qquad {}_2F_1\left[\begin{matrix} -2n, & a+2n \\ \frac{1}{2}(a+2) & \end{matrix} \;\middle|\; \frac{1}{2}\right] = \frac{2\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+1\right)}{(a+4n)\,\Gamma\left(\frac{1}{2}a+n\right)\Gamma\left(-n+\frac{1}{2}\right)}$$

provided $a+2 \neq 0, -2, -4, \ldots.$

$$(2.2)\qquad {}_2F_1\left[\begin{matrix} -2n-1, & a+2n+1 \\ \frac{1}{2}(a+2) & \end{matrix} \;\middle|\; \frac{1}{2}\right] = \frac{-2\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+1\right)}{(a+4n+2)\,\Gamma\left(\frac{1}{2}a+1+n\right)\Gamma\left(-n-\frac{1}{2}\right)}$$

provided $a+2 \neq 0, -2, -4, \ldots.$

$$(2.3)\qquad {}_2F_1\left[\begin{matrix} -2n, & a+2n \\ \frac{1}{2}a & \end{matrix} \;\middle|\; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a\right)}{\Gamma\left(\frac{1}{2}a+n\right)\Gamma\left(-n+\frac{1}{2}\right)}$$

provided $a \neq 0, -2, -4, \ldots.$

$$(2.4)\qquad {}_2F_1\left[\begin{matrix} -2n-1, & a+2n+1 \\ \frac{1}{2}a & \end{matrix} \;\middle|\; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a\right)}{\Gamma\left(\frac{1}{2}a+n+1\right)\Gamma\left(-n-\frac{1}{2}\right)}$$

provided $a \neq 0, -2, -4, \ldots.$

3. Main results.

The results to be proved are

$$e^{-x/2}\,{}_1F_1\,(\alpha; 2\alpha+1; x) = {}_0F_1\left(-;\ \alpha+\frac{1}{2}; x^2/16\right) - \tag{3.1}$$

$$- \frac{x}{2(2\alpha+1)}\,{}_0F_1\left(-; \alpha+\left(\frac{3}{2}\right); x^2/16\right)$$

and

$$e^{-x/2}\,{}_1F_1\,(\alpha; 2\alpha-1; x) = {}_0F_1\left(-; \alpha-\frac{1}{2}; x^2/16\right) + \tag{3.2}$$

$$+ \frac{x}{2(2\alpha-1)}\,{}_0F_1\left(-; \alpha+\frac{1}{2}; x^2/16\right).$$

Proof. To prove (3.1), we proceed as follows. Let

$$e^{-x/2}\,{}_1F_1\,(\alpha; 2\alpha+1; x) = \sum_{n=0}^{\infty} a_{2n}\, x^{2n} + \sum_{n=0}^{\infty} a_{2n+1}\, x^{2n+1}. \tag{3.3}$$

Now in the product

$$e^{-x/2}\,{}_1F_1\,(\alpha; 2\alpha+1; x)$$

it is not difficult to see that the coefficient of x^n, after some simplification, is obtained as

$$a_n = \frac{(\alpha)_n}{(2\alpha+1)_n\, n!}\quad {}_2F_1\left[\begin{matrix} -n, & -2\alpha-n \\ & 1-\alpha-n \end{matrix} \,\middle|\, \frac{1}{2}\right]. \tag{3.4}$$

Now changing n to $2n$ and using the result (2.1), we get after some simplification

$$a_{2n} = \frac{1}{\left(\alpha+\frac{1}{2}\right)_n\ n!\, 2^{4n}}. \tag{3.5}$$

Similarly in (3.4), changing n to $2n+1$ and using the result (2.2), we get after some simplification

$$a_{2n+1} = -\frac{1}{2(2\alpha+1)(\alpha+\frac{3}{2})_n\, n!\, 2^{4n}}. \tag{3.6}$$

Substituting the values of a_{2n} and a_{2n+1} in (3.3), we arrive at the right hand side of (3.1).

In exactly the same manner, the result (3.2) can be established with the help of the result (2.3) and (2.4).

Clearly the results (3.1) and (3.2) are closely related to (1.1).

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